# De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all positive integers $n$

this extends to;

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$e.g.(i)\ (1-i)^5$

$$|z| = \sqrt{1^2 + (-1)^2}$$
$$= \sqrt{2}$$

$$\arg z = \tan^{-1}\left(\frac{-1}{1}\right)$$
$$= -\frac{\pi}{4}$$

$$= \left[\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right]^5$$

$$= \left(\sqrt{2}\right)^5 cis\left(-\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}cis\left(\frac{3\pi}{4}\right)$$

$$= 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$\underline{= -4 + 4i}$$

$(ii)$ Express $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\end{aligned}$$

By equating real and imaginary parts

$$\underline{\cos 2\theta = \cos^2\theta - \sin^2\theta} \qquad \underline{\sin 2\theta = 2\sin\theta\cos\theta}$$

# Finding Roots

$$\text{If } z^n = x + iy$$

$$z^n = r\,cis\,\theta$$

$$z = \sqrt[n]{r}\,cis\left[\frac{2\pi k + \theta}{n}\right] \quad k = 0,1,\ldots,n-1$$

$$e.g.(i) \quad z^2 = 4i$$

$$z^2 = 4cis\frac{\pi}{2}$$

$$z = 2cis\left[\frac{2\pi k + \frac{\pi}{2}}{2}\right] \quad k = 0,1$$

$$z = 2cis\frac{\pi}{4}, 2cis\frac{5\pi}{4}$$

$$z = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right), 2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$z = \sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i$$

$(ii)\ x^4 - 8 = 0$

$$x^4 = 8$$

$$x^4 = 8cis0$$

$$x = \sqrt[4]{8}cis\left[\frac{2\pi k}{4}\right] \qquad k = 0,1,2,3$$

$$x = \sqrt[4]{8}cis0, \sqrt[4]{8}cis\frac{\pi}{2}, \sqrt[4]{8}cis\pi, \sqrt[4]{8}cis\frac{3\pi}{2}$$

$$x = \sqrt[4]{8}, \sqrt[4]{8}i, -\sqrt[4]{8}, -\sqrt[4]{8}i$$

$(iii)$ If $z = \cos\theta + i\sin\theta$, find $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$

$$z^n = \cos n\theta + i\sin n\theta$$

$$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta \qquad \left\{\begin{array}{l}\text{cos is even function} \Rightarrow \cos(-x) = \cos x \\ \text{sin is odd function} \Rightarrow \sin(-x) = -\sin x\end{array}\right\}$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$(iv)$ Express $\cos^3\theta$ in terms of $\cos n\theta$

$$(2\cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$$

$$8\cos^3\theta = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

$$= \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

$$= 2\cos 3\theta + 6\cos\theta$$

$$\therefore \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

## ***Alternative for finding*** $\sqrt{x+iy}$

If $\sqrt{x+iy} = a + ib$

then $a = \sqrt{\dfrac{x+|x+iy|}{2}}$ $\qquad b = \dfrac{y}{2a}$

$e.g.$ Find $\sqrt{-12+16i}$

$$|-12+16i| = \sqrt{12^2+16^2}$$
$$= 20$$

$$a = \sqrt{\frac{-12+20}{2}} \qquad b = \frac{16}{4}$$
$$= \sqrt{\frac{8}{2}} \qquad\qquad = 4$$
$$= 2$$

$$\therefore \sqrt{-12+16i} = \pm(2+4i)$$

**Exercise 4D; evens**

**Exercise 4E; 1 to 4 ac**

**Exercise 4F; 1 to 4, 5ab, 10**