nth Roots Of Unity $\left(z^n = \pm 1\right)$

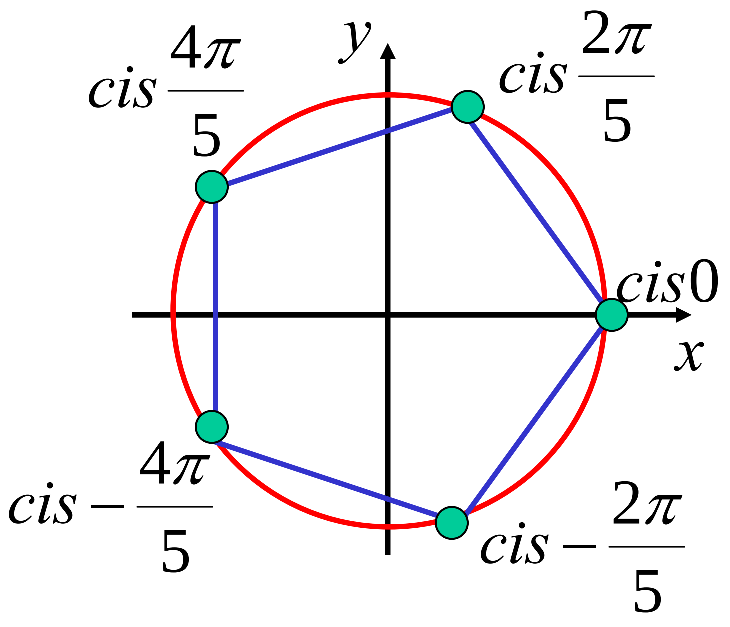
The solutions of equations of the form, $z^n = \pm 1$, are the nth roots of unity

When placed on an Argand Diagram, they form a regular n sided polygon, with vertices on the unit circle.

$$e.g.\ \ z^5 = 1$$

$$z = cis\left[\frac{2\pi k + 0}{5}\right] \quad k = 0, \pm 1, \pm 2$$

$$z = cis0, cis\frac{2\pi}{5}, cis-\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis-\frac{4\pi}{5}$$



b) (i) If ω is a complex root of $z^5 - 1 = 0$, show that $\omega^2, \omega^3, \omega^4$ and ω^5 are the other roots.

$z^5 = 1$ If ω is a solution then $\omega^5 = 1$

$$\therefore \left(\omega^5\right)^5 = 1^5$$

$$= 1 \quad \therefore \omega^5 \text{ is a solution}$$

$$\left(\omega^2\right)^5 = \left(\omega^5\right)^2 = 1^2 = 1$$

$\therefore \omega^2$ is a solution

$$\left(\omega^3\right)^5 = \left(\omega^5\right)^3 = 1^3 = 1$$

$\therefore \omega^3$ is a solution

$$\left(\omega^4\right)^5 = \left(\omega^5\right)^4 = 1^4 = 1$$

$\therefore \omega^4$ is a solution

Thus if ω is a root then $\omega^2, \omega^3, \omega^4$ and ω^5 are also roots

(ii) Prove that $1+\omega+\omega^2+\omega^3+\omega^4=0$

$$\omega+\omega^2+\omega^3+\omega^4+\omega^5=-\frac{b}{a} \qquad \text{(sum of the roots)}$$

$$1+\omega+\omega^2+\omega^3+\omega^4=0 \qquad (\omega^5=1)$$

OR $\omega^5-1=0$

$$(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4)=0$$

$$\therefore 1+\omega+\omega^2+\omega^3+\omega^4=0 \qquad (\omega\neq 1)$$

$$\left[\begin{array}{l} NOTE: \\ \omega^n-1=0 \\ (\omega-1)(1+\omega+\omega^2+\ldots+\omega^{n-1})=0 \end{array}\right]$$

OR

$$1+\omega+\omega^2+\omega^3+\omega^4=\frac{a(r^n-1)}{r-1} \qquad \text{GP: } a=1, r=\omega, n=5$$

$$=\frac{1(\omega^5-1)}{\omega-1} \quad =0 \qquad (\omega^5-1=0)$$

(iii) Find the quadratic equation whose roots are $\omega+\omega^4$ and $\omega^2+\omega^3$

$$\alpha+\beta=\omega+\omega^4+\omega^2+\omega^3$$
$$=-1$$

$$\alpha\beta=(\omega+\omega^4)(\omega^2+\omega^3)$$
$$=\omega^3+\omega^4+\omega^6+\omega^7$$
$$=\omega^3+\omega^4+\omega+\omega^2$$
$$=-1$$

$\therefore$ equation is $x^2+x-1=0$

c) If ω is a complex cube root of unity, use the fact that $1+\omega+\omega^2=0$ to;

(i) Evaluate $\left(1+\omega^2\right)^3$

$$=(-\omega)^3$$

$$=-\omega^3$$

$$=-1$$

(ii) Evaluate $\dfrac{1}{1+\omega}+\dfrac{1}{1+\omega^2}$

$$=-\frac{1}{\omega^2}-\frac{1}{\omega}$$

$$=\frac{-1-\omega}{\omega^2}$$

$$=\frac{\omega^2}{\omega^2}$$

$$=1$$

(iii) Form the cubic equation with roots $1,\ 1+\omega,\ 1+\omega^2$

$$(z-1)\left\{z^2-\left(2+\omega+\omega^2\right)z+\left(1+\omega+\omega^2+\omega^3\right)\right\}=0$$

$$(z-1)\left(z^2-z+1\right)=0$$

$$z^3-z^2+z-z^2+z-1=0$$

$$z^3-2z^2+2z-1=0$$

Exercise 4I; 1, 3, 5 (need 4b), 7