UNIT ONE.

Q.1. 1. $x^2/2 + y^2 = 1$ Here $a = \sqrt{2}$ and $b = 1$

(i) Since $b^2 = a^2(1-e^2)$ the eccentricity

$$e = \frac{\sqrt{a^2-b^2}}{a}, \quad \text{So } e = \frac{\sqrt{2-1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(ii) foci are; $(\pm ae, 0) = (\pm 1, 0)$

(iii) directrixes; $x = \pm \frac{a}{e}$ i.e. $x = \pm \frac{\sqrt{2}}{1/\sqrt{2}}$ so $x = \pm 2$

Q.1. 2. $x^2/_6 + y^2/_4 = 1$ Here $a = \sqrt{6}$, $b = 2$

(i) $e = \frac{\sqrt{a^2 - b^2}}{a}$ So $e = \frac{\sqrt{6-4}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$

(ii) foci are; $(\pm ae, 0) = (\pm \frac{\sqrt{6}}{\sqrt{3}}, 0) = (\pm \sqrt{2}, 0)$

(iii) directrixes; $x = \pm \frac{a}{e}$ i.e. $x = \pm \frac{\sqrt{6}}{1/\sqrt{3}}$ so $x = \pm 3\sqrt{2}$

Q.1. 3. $2x^2 + y^2 = 8$ i.e. $x^2/_4 + y^2/_8 = 1$

Here $a = \sqrt{8} = 2\sqrt{2}$ and $b = 2$. (Note: This ellipse has its major axis on the y axis!)

(i) $e = \frac{\sqrt{8-4}}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

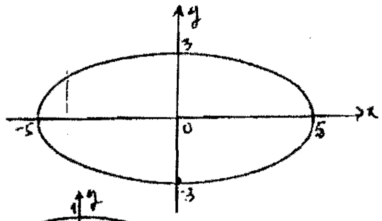
(ii) foci are; $(0, \pm ae) = (0, \pm 2\sqrt{2}/_{\sqrt{2}}) = (0, \pm 2)$

(iii) directrixes; $y = \pm \frac{a}{e}$ i.e. $y = \pm 2\sqrt{2} \times \sqrt{2}$ so $y = \pm 4$

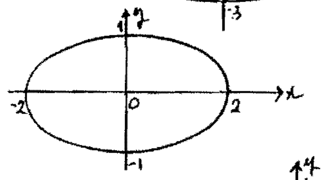
Q.1.4. $\frac{x^2}{4} + \frac{y^2}{16/_9} = 1.$ Here $a = 2$ $b = 4/_3$

Q.1.5. $\frac{x^2}{25} + \frac{y^2}{25/_{16}} = 1.$ Here $a = 5$ and $b = 5/_4$

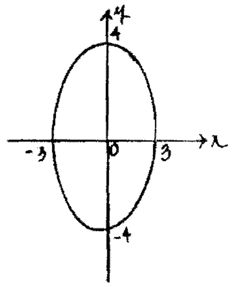
Q.2. 1. $(5 \cos \theta,\ 3 \sin \theta)$, $a = 5$, $b = 3$



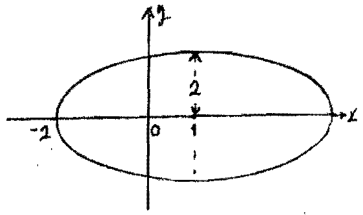
Q.2. 2. $(2 \cos \theta,\ \sin \theta)$, $a = 2$, $b = 1$



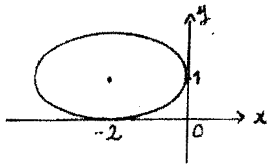
Q.2. 3. $(3 \sin \theta,\ 4 \cos \theta)$, $\left.\begin{aligned}\frac{x}{3} &= \sin \theta \\ \frac{y}{4} &= \cos \theta\end{aligned}\right\}$ $\therefore \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$



Q.2. 4. $(1 + 3 \cos \theta,\ 2 \sin \theta)$, $a = 3$, $b = 2$
centre $(1,0)$



Q.2. 5. $(2 \cos \theta - 2,\ \sin \theta + 1)$, $a = 2$, $b = 1$
centre $(-2,1)$



Q.2. 6. $\left.\begin{aligned}x &= 6 \cos \theta \\ y &= 2 \sin \theta\end{aligned}\right\}\begin{aligned}a &= 6 \\ b &= 2\end{aligned}$

(i) $\dfrac{b^2}{a^2} = 1 - e^2$

$\therefore e^2 = \dfrac{a^2 - b^2}{a^2}$ so $e = \dfrac{\sqrt{a^2 - b^2}}{a}$

$= \dfrac{\sqrt{36 - 4}}{6} = \dfrac{4\sqrt{2}}{6}$

eccentricity $= \underline{\dfrac{2\sqrt{2}}{3}}$

(ii) focus $(ae,0)$, $(-ae,0)$

$ae = \pm\sqrt{a^2 - b^2} = \pm 4\sqrt{2}$

foci $(4\sqrt{2},0)$ and $(-4\sqrt{2},0)$

(iii) directrix $x = \pm \frac{a}{e}$

$$x = \pm \frac{6}{2\sqrt{2}} \cdot \frac{3}{1} = \pm \frac{9}{\sqrt{2}}$$

$$x = \pm \frac{9}{\sqrt{2}}$$

Q.2.8. $\left.\begin{aligned} x &= \sqrt{2}\cos\theta \\ y &= \sin\theta \end{aligned}\right\} \begin{aligned} a &= \sqrt{2} \\ b &= 1 \end{aligned}$

(i) $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{2 - 1}}{\sqrt{2}}$

eccentricity $= \frac{1}{\sqrt{2}}$

(ii) foci; $ae = \pm\sqrt{2} \times \frac{1}{\sqrt{2}} = 1$

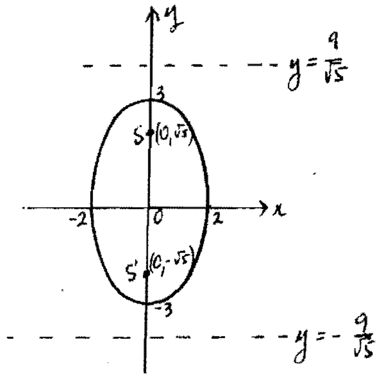
foci are $(-1,0)$ and $(1,0)$

(iii) $\frac{a}{e} = \frac{\pm\sqrt{2}}{\frac{1}{\sqrt{2}}}$

directrix are $x = \pm (\sqrt{2})^2$
$= \pm 2$

Q.2.9. $\left.\begin{aligned} x &= 2\sin\theta \\ y &= 3\cos\theta \end{aligned}\right\} \begin{aligned} a &= 3 \\ b &= 2 \end{aligned}$

b sin θ
a cos θ



(i) $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{9 - 4}}{3} = \frac{\sqrt{5}}{3}$

eccentricity is $\sqrt{5}/3$

(ii) $ae = \pm 3 \times \frac{\sqrt{5}}{3}$

foci $(0, \pm\sqrt{5})$

(iii) $\frac{a}{e} = \frac{3}{\sqrt{5}} \cdot \frac{3}{1}$

directrix $y = \pm \frac{9}{\sqrt{5}}$

Q.2.10. $x = 2 + 3\cos\theta \Rightarrow \cos\theta = \frac{x - 2}{3}$

$y = 2\sin\theta - 3 \Rightarrow \sin\theta = \frac{y + 3}{2}$

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$$

The equations represent an ellipse with centre (2,-3)

Area $= \pi ab$

$= \pi \times 3 \times 2$

$= 6\pi$ sq. units.

2.11. $\left.\begin{aligned} x &= 1 + 4\cos\theta \\ y &= 1 + \sin\theta \end{aligned}\right\}$ $\quad SS^1 = 2ae$ $\quad a = 4$

$$= 2 \times 4 \times \frac{\sqrt{15}}{4} \qquad e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{4^2 - 1}}{4}$$

$$= 2\sqrt{15}$$

The distance between the foci is $2\sqrt{15}$ units

2.14. $4x^2 + 9y^2 = 9 \iff \dfrac{x^2}{9/4} + \dfrac{y^2}{1} = 1$

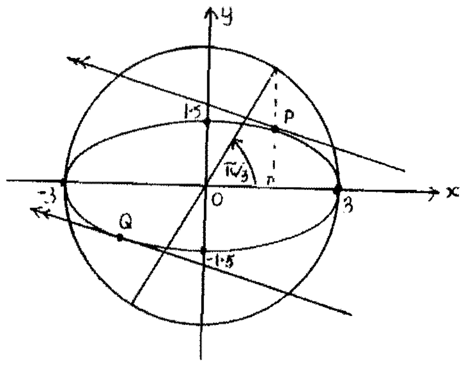
$$\therefore a = \frac{3}{2},\ b = 1$$

$$x = \frac{3}{2}\cos\theta,\ y = \sin\theta$$

2.16. $\dfrac{(x+2)^2}{16} + \dfrac{(y-1)^2}{9} = 1 \iff \left.\begin{aligned} x &= -2 + 4\cos\theta \\ y &= 1 + 3\sin\theta \end{aligned}\right\}$

$a = 4,\quad b = 3$

2.17.



$x^2 + 4y^2 = 9 \iff \dfrac{x^2}{9} + \dfrac{y^2}{9/4} = 1$

$\left.\begin{aligned} x &= 3\cos\theta \\ y &= \frac{3}{2}\sin\theta \end{aligned}\right\}$ $\therefore$ The coordinates of the point

Q are $\left(3\cos\dfrac{4\pi}{3},\ \dfrac{3}{2}\sin\dfrac{4\pi}{3}\right)$

$Q\left(-\dfrac{3}{2}, -\dfrac{3\sqrt{3}}{4}\right)$ $P\left(\dfrac{3}{2}, \dfrac{3\sqrt{3}}{4}\right)$

$\therefore$ eccentric angle of Q is $\dfrac{4\pi}{3}$ or $-\dfrac{2\pi}{3}$

$$m \equiv 2x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}$$

m at $P\left(\dfrac{3}{2}, \dfrac{3\sqrt{3}}{4}\right)$ is $-\dfrac{3/2}{4 \times 3\sqrt{3}/4} = -\dfrac{\cancel{3}^{1}}{2} \cdot \dfrac{\cancel{4}^{1}}{\cancel{4}_{1} \times \cancel{3}_{1}\sqrt{3}} = -\dfrac{1}{2\sqrt{3}}$

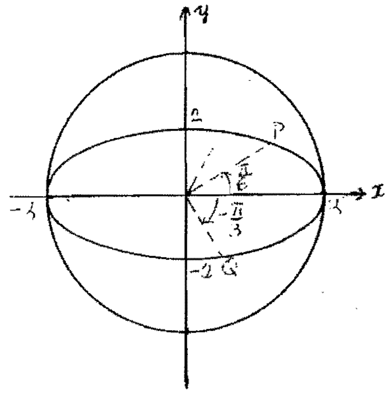
m at $Q\left(-\dfrac{3}{2}, -\dfrac{3\sqrt{3}}{4}\right)$ is also $-\dfrac{1}{2\sqrt{3}}$

The tangents at the extremities of a diameter are //
This is true for the other conics also (except parabola).

Q.2.18. $4x^2 + 9y^2 = 36 \iff \frac{x^2}{9} + \frac{y^2}{4} = 1$

$a = 3$

$b = 2$



$x = 3 \cos \theta$

$y = 2 \sin \theta$

(i) The point at $\pi/6$

$x = 3 \cos \pi/6 = \frac{3\sqrt{3}}{2}$

$y = 2 \sin \pi/6 = 3 \times \frac{1}{2}$

$P(\frac{3\sqrt{3}}{2}, \frac{3}{2})$

(ii) The point at $-\frac{\pi}{3}$

$x = 3 \times \cos(-\frac{\pi}{3}) = \frac{3}{2}$

$y = 2 \times \sin(-\frac{\pi}{3} = -\sqrt{3}$

$Q(\frac{3}{2}, -\sqrt{3})$

Q.2.19. $\frac{\pi ab}{\pi a^2} = \frac{5}{9} \quad \therefore \quad \frac{a}{b} = \frac{9}{5}$

$e^2 = 1 - \frac{b^2}{a^2} \iff e^2 = 1 - \frac{25}{81}$

$e = 0.83$ or $\quad e = \sqrt{\frac{81 - 25}{81}} = \frac{\sqrt{56}}{9} = \frac{2\sqrt{14}}{9}$

Q.2.20. $2ae = 8, \quad e = \frac{3}{4} \quad \therefore \frac{3}{2}a = 8$

$a = 8 \times \frac{2}{3}$

$$b^2 = a^2(1 - e^2)$$
$$= \frac{16^2}{3^2}(1 - \frac{9}{16})$$
$$= \frac{\overset{16}{\cancel{256}}}{9} \cdot \frac{7}{\underset{1}{\cancel{16}}}$$
$$= \frac{112}{9} \qquad \therefore b = \frac{4\sqrt{7}}{3}$$

Area of ellipse $= \pi ab$

$$= \pi \times \frac{16}{3} \times \frac{4\sqrt{7}}{3}$$
$$= \frac{64\pi\sqrt{7}}{9} \text{ sq. units.}$$

Q.2.21. $b^2x^2 + a^2y^2 = a^2b^2 \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cos\theta$ is the x coord. of the point on the auxilliary circle $x^2 + y^2 = a^2$

$\therefore\ y = a\sin\theta$

$\therefore\ P(a\cos\theta,\ a\sin\theta)$

Q.3.1. $x^2 + 4y^2 = 9 \iff \frac{x^2}{9} + \frac{y^2}{9/4} = 1$

$\therefore\ a = 3$

$b = \frac{3}{2}$

The equation of the tangent in terms of its gradient is
$(y = mx \pm\sqrt{a^2m^2 + b^2})$

$$y = mx \pm\sqrt{9m^2 + \frac{9}{4}}$$

If $\parallel$ to line $2x + 3y = 0$ then $m = -\frac{2}{3}$

$$\therefore\ y = -\frac{2}{3}x \pm\sqrt{\frac{\cancel{9}^{1}}{1}\cdot\frac{4}{\cancel{9}_{1}} + \frac{9}{4}} \implies \sqrt{\frac{16+9}{4}}$$

$$y = -\frac{2}{3}x + \frac{5}{2} \quad\text{or}\quad y = -\frac{2}{3}x - \frac{5}{2}$$

$$4x + 6y = 15 \quad\text{or}\quad 4x + 6y = -15$$

or using $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$, i.e. $\frac{x}{3}\cos\theta + \frac{2y\sin\theta}{3} = 1$

$$m = -\frac{\cos\theta}{\cancel{3}_{1}}\cdot\frac{\cancel{3}^{1}}{2\sin\theta} = \frac{1}{2}\cot\theta = -\frac{2}{3}$$

$$\cot\theta = \frac{4}{3}\ \therefore\ \tan\theta = \frac{3}{4}$$

and $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ (when θ is in Quad 3. $\sin\theta$, $\cos\theta$ are neg.)

$$\frac{x \times \frac{4}{5}}{3} + \frac{y \times \frac{3}{5}}{3/2} = \pm 1 \iff 4x + 6y = \pm 15$$

Q.3.2. $16x^2 + y^2 = 169$

$$\frac{x^2}{169/16} + \frac{y^2}{169} = 1 \qquad \left.\begin{array}{l} a = 13 \\ b = \frac{13}{4} \end{array}\right\} \text{ at } (3,5)$$

Equ. of tangent; $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ however this ellipse is the in the form of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. Therefore the tangent will take the form; $\frac{xx_1}{b^2} + \frac{yy_1}{a^2} = 1$

$$\frac{3x}{169/16} + \frac{5y}{169} = 1 \iff 48x + 5y = 169$$

Equ. of the normal; $\frac{xa^2}{x_1} - \frac{yb^2}{y_1} = a^2 - b^2$

so we have $\frac{xb^2}{x_1} - \frac{ya^2}{y_1} = \qquad a^2 - b^2$

$$\frac{169}{16} \times \frac{x}{3} - \frac{169y}{5} = -169 + \frac{169}{16}$$

$$5 \times 169x - 169y \times 48 = -169 \times 16 \times 15 + 169 \times 15$$

$$5x - 48y = -\frac{\overset{1}{\cancel{169}} \times 15(16-1)}{\underset{1}{\cancel{169}}}$$

$$5x - 48Y + 225 = 0$$

Equ. of diameter three $(3,5)$ $y = \frac{5}{3}x$

$$5x - 3y = 0.$$

Q.3. 3. $2x - 3y + 1 = 0 \cap x^2 + 3y^2 = 1$

$$x = \frac{3y - 1}{2} \longrightarrow x^2 + 3y^2 = 1$$

$$\frac{(3y - 1)^2}{4} + 3y^2 = 1$$

$$9y^2 - 6y + 1 + 12y^2 = 4$$

$$21y^2 - 6y - 3 = 0$$

$$7y^2 - 2y - 1 = 0$$

$$y = \frac{2 \pm \sqrt{4 + 28}}{14} = \frac{\overset{1}{\cancel{2}} \pm \overset{2}{\cancel{4}}\sqrt{2}}{\underset{7}{\cancel{14}}} = \frac{1 \pm 2\sqrt{2}}{7}$$

$$x = \frac{\frac{3 \pm 6\sqrt{2}}{7} - 1}{2} = \frac{3 \pm 6\sqrt{2} - 7}{7 \times 2}$$

$$x = \frac{-4 \pm 6\sqrt{6}}{14} = \frac{-2 \pm 3\sqrt{6}}{7}$$

(continued on next page)

Q.3.3 (cont'd)

The points are $\left(\frac{-2 + 3\sqrt{6}}{7}, \frac{1 + 2\sqrt{2}}{7}\right)$ and

$\left(\frac{-2 - 3\sqrt{6}}{7}, \frac{1 - 2\sqrt{2}}{7}\right)$

Midpoint of the chord $(-\frac{2}{7}, \frac{1}{7})$

$$\left.\begin{array}{l} x = -\frac{2}{7} \\ y = \frac{1}{7} \end{array}\right\} \longrightarrow x + 2y = 0 \quad \text{True} \quad \therefore \text{ diameter bisects the chord } 2x - 3y + 1 = 0.$$

Q.3.4. Putting $(2 \cos \theta, 3 \sin \theta) \longrightarrow 3x \cos \theta + 2y \sin \theta = 6$ is insufficient as proof!! (may be a secant!)

Rather; Equ. through $(2 \cos \theta, 3 \sin \theta)$..(2) to the ellipse $\frac{x^2}{\underset{\uparrow b^2}{4}} + \frac{y^2}{\underset{\uparrow a^2}{9}} = 1$ is;

$$\frac{x \cos \theta}{2} + \frac{y \sin \theta}{3} = 1 \text{ .....(1)}$$

$a = 9$

$b = 2$

$$(2) \longrightarrow (1) \quad \frac{2 \cos^2\theta}{2} + \frac{3 \sin^2\theta}{3} = 1$$

LHS = RHS.

$\therefore$ $(2 \cos \theta, 3 \sin \theta)$ is on the tangent, and is an ellipse.

Hence $\frac{x \cos \theta}{2} + \frac{y \sin \theta}{3} = 1 \iff 3x \cos \theta + 2y \sin \theta = 6$

is the tangent to $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at $(2 \cos \theta, 3 \sin \theta)$.

———————— 0 ————————

(diameter) $y = 2x \cap \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{4} + \frac{4x^2}{9} = 1$

$$25x^2 = 36$$

$$x = \pm \frac{6}{5} \text{ and } y = \pm \frac{12}{5}$$

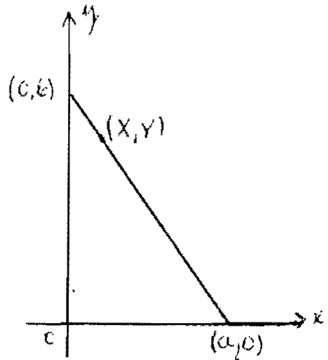
Tangents at $(\frac{6}{5}, \frac{12}{5})$ and at $(-\frac{6}{5}, -\frac{12}{5})$

$$\frac{xx_1}{4} + \frac{yy_1}{9} = 1 \iff \frac{\cancel{6}^{3}x}{\cancel{20}_{10}} + \frac{\cancel{12}^{4}y}{5 \times \cancel{9}_{3}} = \pm 1$$

$$45x + 40y = 150$$

$$9x + 8y = \pm 30$$

Q.3.5.



$X = \frac{1 \times a + 3 \times 0}{1 + 3}$ $\quad Y = \frac{1 \times 0 + 3 \times b}{1 + 3}$

$X = \frac{a}{b} \therefore a = 4X$ $\quad Y = \frac{3b}{4} \therefore b = \frac{4Y}{3}$

By Pythagoras $a^2 + b^2 = \text{constant}$

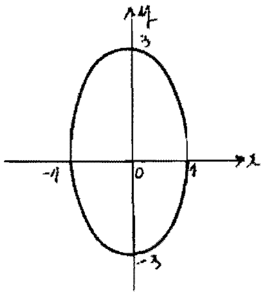
$$a^2 + b^2 = 4^2$$

$$\therefore 16X^2 + \frac{16Y^2}{9} = 16$$

(continued on next page)

Q.3.5 (cont'd)

The locus has equation $X^2 + \frac{Y^2}{9} = 1 \iff 9X^2 + Y^2 = 9$



OR

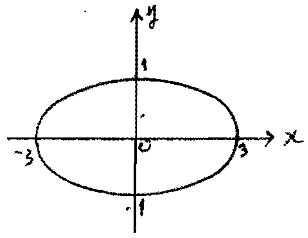
Instead of 3 : 1, divide the 4 unit interval in the ratio 1 : 3 then

$X = \frac{3a}{4}$ and $Y = \frac{b}{4} \iff a = \frac{4X}{3}, \quad b = 4Y.$

$a^2 + b^2 = 16 \iff \frac{16X^2}{9} + 16Y^2 = 16$

$\frac{X^2}{9} + Y^2 = 1$ or

we have $X^2 + 9Y^2 = 9 \longrightarrow$



Q.3.6. $x^2 + 16y^2 = 25 \iff \frac{x^2}{25} + \frac{y^2}{\frac{25}{16}} = 1$

Semi major axis is $a = 5$

Semi minor axis is $b = \frac{5}{4}$

Tangent at (3,1) $\frac{3x}{25} + \frac{y}{\frac{25}{16}} = 1$

$\therefore 3x + 16y = 25$

Normal at (3,1) $16(x-3) - 3(y-1) = 0$

$16x - 3y = 45$

Q.3.7. $9x^2 + 16y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9/4} = 1 \qquad a = 2, \; b = \frac{3}{2}$

$\perp$ to $x + y = 4 \quad \therefore m = 1$

Tangent is $y = m \pm \sqrt{a^2m^2 + b^2}$

$y = x \pm \sqrt{4 + \frac{9}{4}}$

$y = x \pm \frac{5}{2} \iff 2x - 2y = \pm 5$

(cont. on next p.)

Equ. of tangent $\perp$ to $x + y = 4$ is

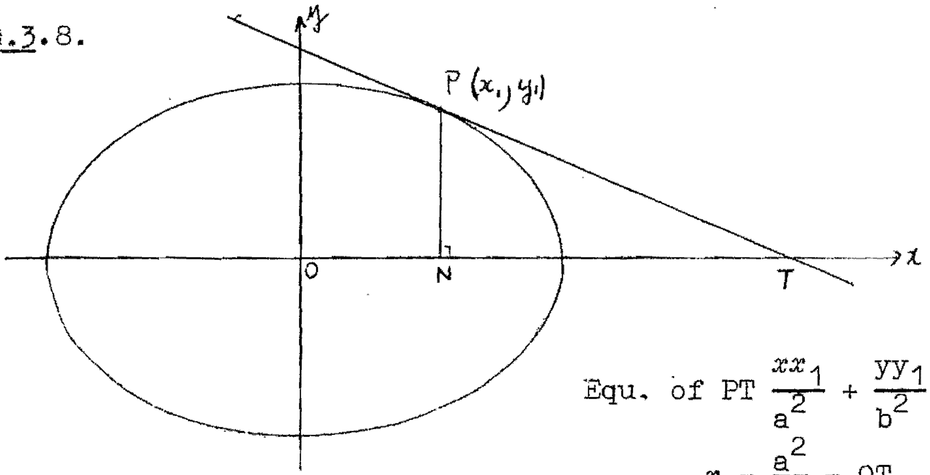
$y = x \pm 5/2 \rightarrow 9x^2 + 16y^2 = 36 \Leftrightarrow 9x^2 + 16x^2 \pm 80x + 100 - 36 = 0$

$25x^2 \pm 80x + 64 = 0 \Leftrightarrow (8 \pm 5x)(8 \pm 5x) = 0$

$$\therefore x = -\frac{8}{5} \quad y = \frac{9}{10}$$

$$\text{or} \quad x = \frac{8}{5} \quad y = -\frac{9}{10}$$

Q.3.8.



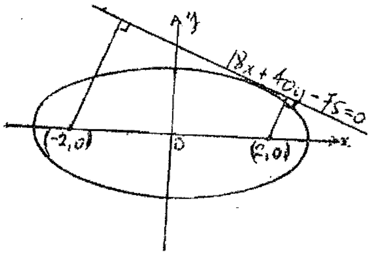
Equ. of PT $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ at $y = 0$

$$x = \frac{a^2}{x_1} = OT$$

$$ON = x_1$$

$$\therefore ON.OT = x_1 . \frac{a^2}{x_1} = a^2$$

Q.3.9.



$$36x^2 + 100y = 225$$

$$\frac{x^2}{\frac{225}{36}} + \frac{y^2}{\frac{9}{4}} = 1 \qquad \therefore a = \frac{\cancel{15}^{5}}{\cancel{6}_{2}}; \; b = \frac{3}{2}$$

$$\frac{x^2}{25/4} + \frac{y^2}{9/4} = 1$$

Equ. of tangent at $(\frac{3}{2}, \frac{6}{5})$ is

$$\frac{3x}{\frac{2 \times 25}{4}} + \frac{6y}{\frac{5 \times 9}{4}} = 1$$

$$\frac{\cancel{12x}^{6}}{\cancel{50}_{25}} + \frac{\cancel{24y}^{8}}{\cancel{45}_{15}} = 1 \Leftrightarrow 18x + 40y = 75 \text{ is the}$$

equation of the tangent.

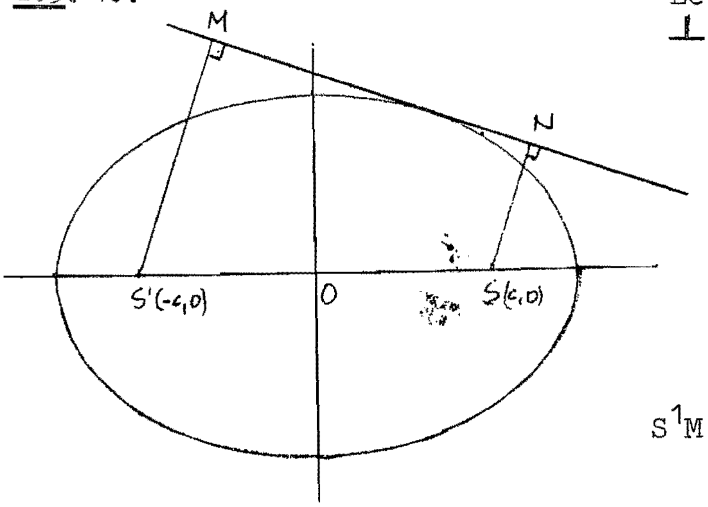
Product of $\perp$ distances $= \left|\dfrac{18 \times 2 - 75}{\sqrt{18^2 + 40^2}}\right| . \left|\dfrac{18 \times -2 - 75}{\sqrt{18^2 + 40^2}}\right|$

$$= \frac{39 \times 111}{\sqrt{1924^2}} = \frac{4329}{1924} = \frac{13 \times 333}{13 \times 148} = \frac{37 \times 9}{37 \times 4}$$

$$= \frac{9}{4}$$

$\therefore$ The product of the $\perp$ dist. from $(\pm 2,0)$ to the tangent $= \frac{9}{4}$

Q.3. 10.



Let M and N be the foot of the $\perp$ from S^1 and S to the tangent

$$S^1M = \left| \frac{-b^2cx_1 - a^2b^2}{\sqrt{(b^2x_1)^2 + (a^2y_1)^2}} \right|$$

$$SN = \left| \frac{b^2cx_1 - a^2b^2}{\sqrt{(b^2x_1)^2 + (a^2y_1)^2}} \right|$$

$$S'M \cdot SN = \frac{-b^2(x_1c + a^2)b^2(x_1c - a^2)}{(b^4x_1^2 + a^4y_1^2)}$$

$$= \frac{-b^4(x_1^2c^2 - a^4)}{b^4x_1^2 + a^4y_1^2}$$

$$= \frac{-b^4(x_1^2a^2 - x_1^2b^2 - a^4)}{b^4x_1^2 + a^{\cancel{4}^2}\frac{b^2}{\cancel{a^2}_1}(a^2 - x_1^2)}$$

$$c^2 = a^2 - b^2$$

$$y_1^2 = \frac{a^2b^2 - b^2x^2}{a^2} = \frac{b^2}{a^2}(a^2 - x_1^2)$$

$$= \frac{b^4(a^4 + x_1^2b^2 - x_1^2a^2)}{b^4x_1^2 + a^4b^2 - a^2b^2x_1^2}$$

$$= \frac{b^4\cancel{(a^4 + x_1^2b^2 - x_1^2a^2)}}{b^2\cancel{(a^4 + x_1^2b^2 - x_1^2a^2}}$$

$$= b^2$$

The product of the $\perp$ distances from (c,0) and (-c,0) (i.e. from the foci) to the tangent of the ellipse is b^2.

Q.3.11. $9x^2 + 25y^2 = 225$

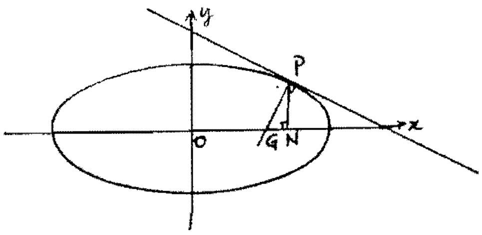
$\frac{x^2}{25} + \frac{y^2}{9} = 1$ $\therefore$ the semi major axis a = 5
the semi minor axis b = 3

The equ. of the normal at $(3, 12/5)$ is $\left(\frac{xa^2}{x_1} - \frac{yb^2}{y_1} = a^2 - b^2\right)$

$$\frac{25}{3}x - \frac{\cancel{9}^{3} \times 5}{\cancel{12}_{4}}y = 25 - 9$$

$100x - 45y = 192$ is the normal at $(3, \frac{12}{5})$

At G y = 0 $\therefore$ $\frac{\cancel{192}^{48}}{\cancel{100}_{25}} = x$



GN = ON − OG

$= 3 - 1\frac{23}{25}$

$= 1\frac{2}{25}$ #.

Q.3.12. $5x^2 + 9y^2 = 45 \Leftrightarrow \frac{x^2}{9} + \frac{y^2}{5_1} = 1$

Tangent at $(2, \frac{5}{3})$ is $\frac{2x}{9} + \frac{\cancel{5y}}{\cancel{15}_{3}} = 1$

$$2x + 3y = 9$$

m of $\perp = \frac{3}{2}$. Thru $(-2,0)$	Thru $(2,0)$ the equ. is
$y + 0 = \frac{3}{2}(x + 2)$	$y = \frac{3x}{2} - 3$
$6y = 3x + 6$	
$y = \frac{3x}{2} + 3$	

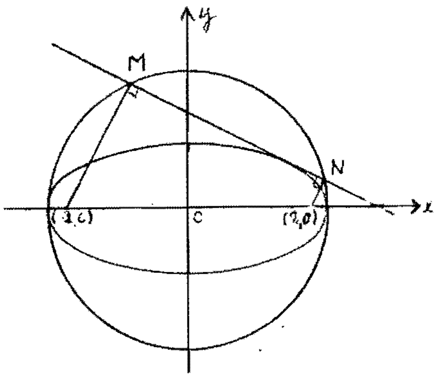
$3x - 2y = 6 \cap 2x + 3y = 9$

$9x - 6y = 18$
$4x + 6y = 18$
$13x = 36$

$x = \frac{36}{13}$ $y = \frac{3}{\cancel{2}_{1}} \times \frac{\cancel{36}^{18}}{13} - 3$

$= \frac{54}{13} - \frac{39}{13}$

$= \frac{15}{13}$



$M(\frac{36}{13}, \frac{15}{13}) \longrightarrow x^2 + y^2 = 9 \Leftrightarrow \frac{36^2}{13^2} + \frac{15^2}{13^2} = \frac{1521}{169}$

$= 9$

LHS = RHS

(continued on next page)

Q.3. 12 (continued)

$$3x - 2y = -6 \cap 2x + 3y = 9$$

$$\begin{aligned} 9x - 6y &= -18 \\ 4x + 6y &= 18 \\ \hline 13x &= 0 \end{aligned}$$

$$x = 0 \quad \therefore y = 3$$

$$N(0,3) \longrightarrow x^2 + y^2 = 9 \qquad 0 + 9 = 9$$

LHS = RHS.

The feet of the $\perp$ from the points $(-2,0)$ and $(2,0)$ (i.e., from the foci) lie on the circle $x^2 + y^2 = 9$ (i.e. on the auxilliary circle).

Q.3. 13. $a^2 = \frac{1}{4}$ $\quad b^2 = \frac{1}{9}$ $\quad m = -\frac{1}{2}$

Tangents with $m = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}x \pm \sqrt{\frac{1}{4} \cdot \frac{1}{4} + \frac{1}{9}}$

$$y = -\frac{1}{2}x \pm \frac{5}{12}$$

$$6x + 12y = \pm 5$$

Q.3. 14. $x^2 + 4y^2 = 65 \Leftrightarrow \dfrac{x^2}{65} + \dfrac{y^2}{\frac{65}{4}} = 1$

Normal at $(1,4)$ $\quad 65x - \dfrac{65y}{16} = 65 - \dfrac{65}{4}$

$$1040x - 65y = 1040 - 260$$

$$1040x - 65y = 780$$

$$208x - 13y = 156$$

$$16x - y = 12 \quad \text{——— (1)}$$

Normal at $(7,2)$ $\quad \dfrac{65x}{7} - \dfrac{65y}{8} = \dfrac{195}{4}$

$$520x - 455y = 2730$$

$$104x - 91y = 546$$

$$8x - 7y = 42 \quad \text{——— (2)}$$

$$\begin{aligned} 16x - y &= 12 \\ 16x - 14y &= 84 \\ \hline 13y &= -72 \end{aligned} \qquad y = -\frac{72}{13}$$

Normals $\cap$ at $\left(\frac{21}{52}, \frac{-72}{13}\right)$ equ. thru 0

$$\frac{y}{x} = \frac{\cancel{72}^{24}}{\cancel{13}_1} \cdot \frac{\cancel{52}^{4}}{\cancel{21}_7} \qquad 96x + 7y = 0$$

$$x = \left(42 + \frac{7 \times 72}{13}\right)\frac{1}{8} = \frac{\cancel{84}^{21}}{13} \cdot \frac{1}{\cancel{8}_4} = \frac{21}{52}$$

Q.3. 15. $x = 2y - 3 \cap x^2 + 2y^2 = 9$ $\quad a^2 = 9$

$b^2 = \frac{9}{2}$

$4y^2 - 12y + 9 + 2y^2 - 9 = 0$

$6y^2 - 12y = 0$

$y(y-2) = 0$ $\quad x = 2 \times 0 - 3 = -3$

$\therefore y = 0$ or $y = 2$ $\quad x = 2 \times 2 - 3 = 1$

The line cuts the ellipse at $(-3,0)$ and at $(1,2)$

(i) Tangent at $(-3,0)$ $\quad \frac{-3x}{9} = 1 \quad x = -3$

(ii) Tangent at $(1,2)$ $\quad \frac{x}{9} + \frac{22y}{9} = 1$

$x + 4y = 9$

The tangents $\cap$ when $-3 + 4y = 9$

$y = 3$

i.e. at $(-3,3)$

Q.3. 16. $9x^2 + 25y^2 = 169$

$\frac{x^2}{\frac{169}{9}} + \frac{y^2}{\frac{169}{25}} = 1$ $\quad a = \frac{13}{3}, \quad b = \frac{13}{5}$

Tangent at $(4,-1)$ $\quad \frac{4x}{\frac{169}{9}} + \frac{-y}{\frac{169}{25}} = 1$

$36x + -25y = 169$

$36x - 25y = 169$

Circle $x^2 + y^2 + 28x - 23y = 152 \cap 9x^2 + 25y^2 = 169 = \{(4,-1)\}$

$\underline{(4,-1)} \rightarrow$ Circle LHS $= 16 + 1 + 112 + 23$

$= 152$

$\therefore \quad = \text{RHS}$

$\therefore$ circle cuts ellipse at $(4,-1)$

If this $(4,-1)$ is the only $\cap$ pt between the curves then the tangent must be a tangent to the circle also. $\therefore \quad \Delta = 0$

$x = \frac{169 + 25y}{36} \longrightarrow$ circle

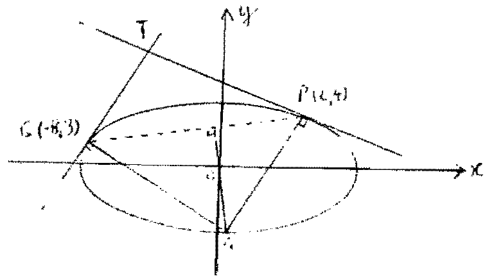
$\frac{(169 + 25y)^2}{36} + y^2 + \cancel{28}^{7}\frac{(169 + 25y)}{\cancel{36}_{9}} - 23y - 152 = 0$

$1921y^2 + 3842y + 1921 = y^2 + 2y + 1 = 0$ So $\Delta = 0$ $\therefore$ tangent is common to both ellipse and circle. So circle touches ellipse at $(4, 1)$.

Q.3. 17. P(6,4)

Q(−8,3) are pts. on $x^2 + 4y^2 = 100$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$



Tangent at P $\frac{\overset{3}{\cancel{6}x}}{\underset{50}{\cancel{100}}} + \frac{4y}{25} = 1 \iff 3x + 8y = 50$

Tangent at Q $\frac{\overset{2}{-\cancel{8}x}}{\underset{25}{\cancel{100}}} + \frac{3y}{25} = 1 \iff 2x - 3y = -25$

$$\begin{aligned} 6x + 16y &= 100 \\ \underline{6x - 9y} &\underline{= -75} \\ 25y &= 175 \\ y &= 7 \\ x &= -2 \end{aligned}$$

$\therefore$ T (−2,7)

Normal at P $8(x-6) - 3(y-4) = 0 \Rightarrow 8x - 3y = 36$ ——— (1)

Normal at Q $3(x+8) + 2(y-3) = 0 \Rightarrow 3x + 2y = -18$ ——— (2)

$$\begin{aligned} 16x - 6y &= 72 \quad \text{——— (1) x 2} \\ \underline{9x + 6y} &\underline{= -54} \quad \text{——— (2) x 3} \\ 25x &= 18 \end{aligned}$$

$x = \frac{18}{25}$ $\quad 2y = -18 - \frac{54}{25}$

$y = \frac{-504}{100}$

$y = \frac{-252}{25}$

G $\left(\frac{18}{25}, \frac{-252}{25}\right)$

m_1 of diameter OG $= \frac{-252}{25} \cdot \frac{25}{18}$

$= -14$

m_2 of chord PQ $= \frac{4 - 3}{6 + 8}$

$= \frac{1}{14}$

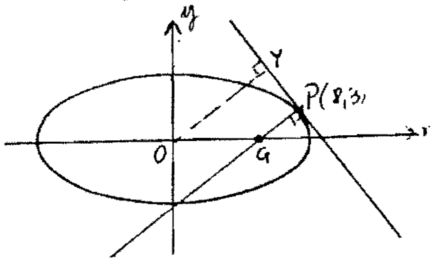
$m_1 \times m_2 = -1 \quad \therefore$ PQ is $\perp$ OG

3.18. Normal to $x^2 + 4y^2 = 100$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1 \quad \text{at } (8,3)P$$

$$\frac{\overset{25}{\cancel{100}}}{\underset{2}{\cancel{8}}} - \frac{25y}{3} = 100 - 25$$

$$\frac{x}{2} - \frac{y}{3} = 3 \Rightarrow 3x - 2y = 18$$



at G, y = 0 $\therefore$ $3x = 18$

$x = 6$

$\therefore$ Point G is (6,0)

$$PG = \sqrt{(8-6)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$

Equ. of OY. $m = \frac{3}{2}$

$$y = \frac{3}{2}x$$

Tangent at P $\frac{\overset{2}{\cancel{8x}}}{\underset{25}{\cancel{100}}} + \frac{3y}{25} = 1 \Leftrightarrow 2x + 3y = 25$

$\{Y\} = 2x + 3y = 25 \cap y = \frac{3}{2}x \Rightarrow 2x + \frac{9}{2}x = 25$

$$4x + 9x = 0$$

$$13x = 50$$

$$x = \frac{50}{13} \qquad y = \frac{\overset{75}{\cancel{150}}}{\underset{1}{\cancel{2}}\times 13}$$

$$Y\left(\frac{50}{13}, \frac{75}{13}\right)$$

$$OY = \sqrt{\left(\frac{50}{13}\right)^2 + \left(\frac{75}{13}\right)^2}$$

$$= \sqrt{\frac{2500 + 5625}{169}}$$

$$= \frac{25}{13}$$

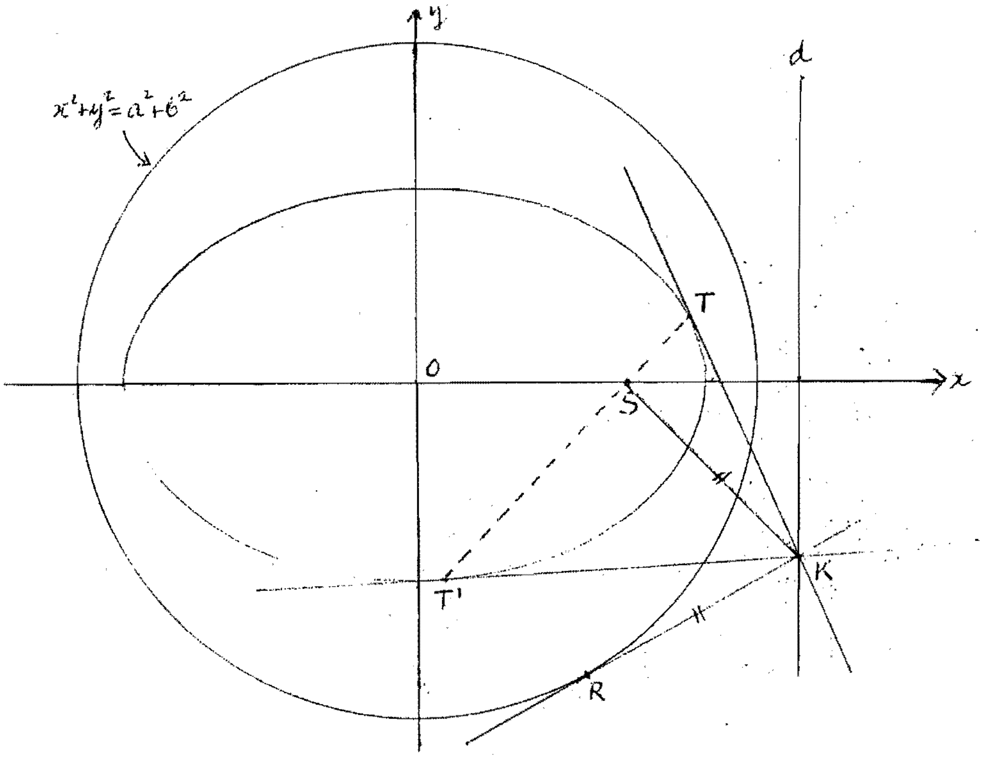
$$PG \cdot OY = \sqrt{13} \times \frac{25}{\sqrt{13}} = 25$$

$b^2 = 25$

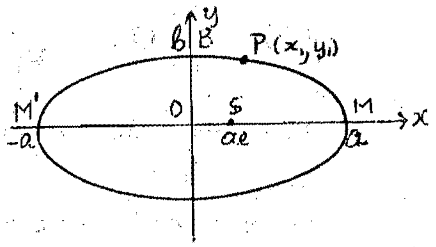
$$\therefore PG \cdot OY = b^2$$

PG $\cdot$ OY is equal to the square on the minor semi axis.

(See diagram on next page)



Q.4. 1. Let S be the earth, SM and SM^1 be the least and greatest distance of the earth from the moon respectively.



$$\text{Average (mean) dist.} = \frac{SM + SM^1}{2}$$

$$384\,000 = \frac{(a-ae)+(a+ae)}{2}$$

$$a = 384\,000$$

So $SM = a - ae = 384\,000 - 384\,000 \times 0.055$

$SM \doteqdot 363\,000$ km

and $SM^1 = a + ae = 384\,000 + 384\,000 \times 0.055$
$\doteqdot 405\,000$ km as required

Note: Diagram is not drawn to scale and it may seem that other points of the ellipse are closer to S than M is.
(i.e. $SM > SP$ where P is any point on it)
This can easily be disproved,

$SM = a - ae$ $SP = a - ex_1$ (see the proof of this in Q.4. 5.) Now for all x_1 : $0 \leqslant x_1 \leqslant a$ $ex_1 \leqslant ea$

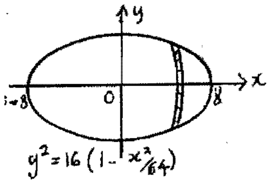
$\therefore SM \angle SP$ i.e. SM is the least distance of the earth from the moon. Also note that as $e \longrightarrow 0$ ellipse becomes a circle so for $e = 0.055$ we have "almost" a circle.

Q.4. 2. $4x^2 + 25y^2 = 100 \iff x^2/_{25} + y^2/_4 = 1$

Here $a = 5$ so external circle is $x^2 + y^2 = 25$

$b = 2$ so internally touching circle is $x^2 + y^2 = 4$.

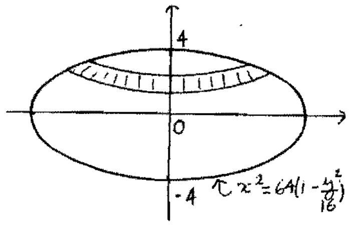
Q.4. 3. (i) $x^2/_{64} + y^2/_{16} = 1$ so $y^2 = 16(1 - x^2/_{64})$



$$V = 16\pi \int_{-8}^{8} (1 - x^2/_{64})dx = 16\pi \left[x - x^3/_{192} \right]_{-8}^{+8}$$

$= 512\pi/_3$ cu.units

(ii)



$x^2/_{64} + y^2/_{16} = 1$ so $x^2 = 64\,(1 - y^2/_{16})$

$$V = 64\pi \int_{-4}^{4} (1 - y^2/_{16})dy = 64\pi.\left[y - y^3/_{48} \right]_{-4}^{4}$$

$= 1024\,\pi/_3$ cu. units.

Q4. 4. Method (i) Differentiate $x^2/_{25} + y^2/_9 = 1$ -----(1) implicitly

so $2x/_{25} + 2y/_9\ dy/_{dx} = 0$ i.e. $dy/_{dx} = -9x/_{25y}$

If parallel to $y = 2x$ then $-9x/_{25y} = 2$

and $x = 50y/_9$ -----(2) (condition for tangent to be parallel to diameter).

(2) $\longrightarrow$ (1) $\dfrac{2500y^2}{81 \times 25} + \dfrac{y^2}{9} = 1$ so $y = \pm 9/\sqrt{109}$ -----(3)

(3) $\longrightarrow$ (2) so $x = \dfrac{50}{9} \times \pm \dfrac{9}{\sqrt{109}} = \pm \dfrac{50}{\sqrt{109}}$

So tangent touches ellipse at $\left(\pm \dfrac{50}{\sqrt{109}}\ \pm \dfrac{9}{\sqrt{109}} \right)$

Deduce that the equation of the tangent to the ellipse is

$$\frac{xx_1}{25} + \frac{yy_1}{9} = 1$$

$$\text{i.e.} \quad \pm \frac{2x}{\sqrt{109}} \pm \frac{y}{\sqrt{109}} = 1$$

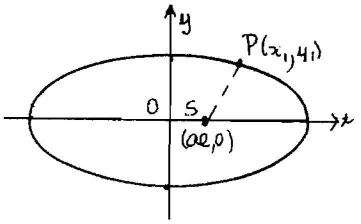
which is $y = 2x \pm \sqrt{109}$ as rqd.

Method (ii) using $y = mx \pm \sqrt{a^2m^2 + b^2}$

$m = 2,\ a^2 = 25,\ b^2 = 9$ we have $y = 2x \pm \sqrt{25 \times 4 + 9}$

which simplifies to $y = 2x \pm \sqrt{109}$ as rqd.

Q.4. 5.



Let $P(x_1, y_1)$ be any point on $x^2/a^2 + y^2/b^2 = 1$.

$$(*PS)^2 = (ae - x_1)^2 + (0 - y_1)^2$$

$$= a^2e^2 - 2aex_1 + x_1^2 + y_1^2$$

$$= a^2e^2 - 2aex_1 + x_1^2 + \frac{b^2}{a^2}(a^2 - x^2)$$

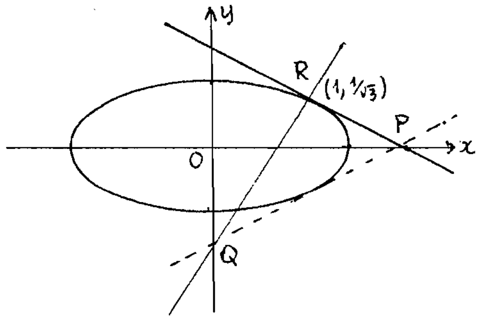
(aim: get rid of b since required required result has no b in it)

$$= a^2e^2 - 2aex_1 + x_1^2 + (1-e^2)(a^2 - x_1^2) \qquad \text{Since } \frac{b^2}{a^2} = 1 - e^2$$

$$= a^2 - 2aex_1 + e^2x_1^2$$

$$= (a - ex_1)^2 \qquad \text{so } PS = a - ex_1 \text{ as required.}$$

Q4. 6.



On differentiating $x^2+3y^2 = 2$ implicitly, $2x + 6y\, dy/_{dx} = 0$

$dy/_{dx} = -x/_{3y}$,

so at $x = 1$ the gradient of the tangent $m_1 = -\sqrt{3}/_3$. Gradient of PR is $-\sqrt{3}/_3 = \dfrac{1/\sqrt{3}}{1 - x}$, so

$x = 2$ and $P = (2,0)$

Similarly at $x = 1$ the gradient of the normal $m_2 = 3/\sqrt{3}$. Gradient of RQ is $3/\sqrt{3} = \dfrac{1/\sqrt{3} - y}{1}$, so $y = -2/\sqrt{3}$ and $Q = (0, -2/\sqrt{3})$.

Deduce that the equation of the tangent to the ellipse is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$. i.e. $\dfrac{xx_1}{2} + \dfrac{yy_1}{2/_3} = 1$. Show that both $P(2,0)$ and $Q(0, -2/\sqrt{3})$ satisfies this equation.