

UNIT TWO.

$$Q1. 1. \quad \frac{x^2}{4} - y^2 = 1$$

$$\left. \begin{array}{l} a = 2 \\ b = 1 \end{array} \right\} \begin{array}{l} b^2 = a^2(e^2 - 1) \\ \frac{b^2}{a^2} = e^2 - 1 \end{array} \therefore e^2 = 1 + \frac{b^2}{a^2}$$

$$e = \sqrt{\frac{a^2 + b^2}{a^2}}$$

$$(i) \text{ eccentricity } e = \frac{\sqrt{4+1}}{2} = \frac{\sqrt{5}}{2}$$

$$(ii) \text{ foci } ae = \frac{2}{1} \times \frac{\sqrt{5}}{2} \quad S(\sqrt{5}, 0) \quad S'(-\sqrt{5}, 0)$$

$$(iii) \text{ directrices } x = \pm \frac{a}{e} \quad x = \pm \frac{4}{\sqrt{5}}$$

$$(iv) \text{ asymptotes } y = \pm \frac{b}{a} \quad y = \pm \frac{1}{2}x$$

$$Q1. 2. \quad x^2 - y^2 = 4 \quad a = b = 2 \quad e = \frac{\sqrt{a^2 + a^2}}{a} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$(i) \text{ eccentricity } e = \sqrt{2}$$

$$(ii) \text{ foci, } ae = 2\sqrt{2} \quad (\pm 2\sqrt{2}, 0)$$

$$(iii) \text{ directrices } x = \pm \frac{a}{e} \quad \therefore x = \pm \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm \sqrt{2}$$

$$(iv) \text{ asymptotes } y = \pm x$$

$$Q1. 3. \quad \frac{x^2}{12} - \frac{y^2}{4} = 1 \quad \begin{array}{l} a = 2\sqrt{3} \\ b = 2 \end{array} \quad e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{12+4}}{\sqrt{12}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$(i) \text{ eccentricity } e = \frac{2}{\sqrt{3}}$$

$$(ii) \text{ foci, } ae = \frac{2\sqrt{3}}{3} \quad (\pm 4, 0)$$

$$(iii) \text{ directrices } x = \pm 3$$

$$(iv) \text{ asymptotes } y = \pm \frac{1}{\sqrt{3}} x$$

$$Q1. 4. \quad \frac{x^2}{144} - \frac{y^2}{25} = 1 \quad \begin{array}{l} a = 12 \\ b = 5 \end{array}$$

$$(i) e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{169}}{12} = \frac{13}{12}$$

$$(ii) ae = \frac{12}{1} \cdot \frac{13}{12} \quad (\pm 13, 0)$$

$$(iii) x = \pm \frac{a}{e} \quad x = \pm \frac{144}{13}$$

$$(iv) y = \pm \frac{5}{12} x$$

$$Q1. 5. \quad x^2 - 4y^2 = 36 \iff \frac{x^2}{36} - \frac{y^2}{9} = 1 \quad \left. \begin{array}{l} a = 6 \\ b = 3 \end{array} \right\}$$

$$(i) \quad e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{3\sqrt{5}}{6} = \frac{\sqrt{5}}{2}$$

$$(ii) \quad ae = 6^3 \cdot \frac{\sqrt{5}}{2} = 3\sqrt{5} \quad (\mp 3\sqrt{5}, 0)$$

$$(iii) \quad x = \pm \frac{a}{e} \quad x = \pm \frac{12}{\sqrt{5}}$$

$$(iv) \quad y = \pm \frac{1}{2}x$$

$$Q1. 6. \quad 4x^2 - 4y^2 = 9 \quad a = \frac{3}{2} = b$$

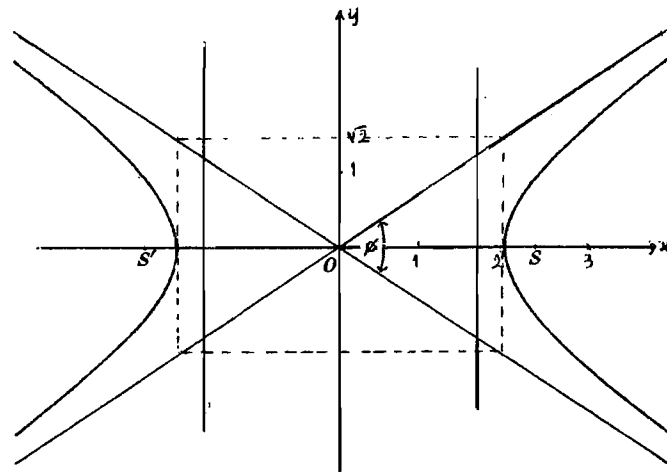
$$(i) \quad e = \sqrt{2}$$

$$(ii) \quad ae = \frac{3\sqrt{2}}{2} \quad \text{foci } (\mp \frac{3\sqrt{2}}{2}, 0)$$

$$(iii) \quad x = \pm \frac{3}{2\sqrt{2}}$$

$$(iv) \quad y = \pm x$$

$$Q1. 9. \quad \frac{x^2}{4} - \frac{y^2}{2} = 1 \quad \left\{ \begin{array}{l} a = 2 \\ b = \sqrt{2} \end{array} \right.$$



ϕ = angle between asymptotes
= $70^{\circ}32'$

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{6}}{2}; \quad ae = \frac{\sqrt{6}}{2} \cdot 2 = \sqrt{6}; \quad \frac{a}{e} = \frac{4}{\sqrt{6}} = 1.6$$

Q1. 10. Asymptotes of $\frac{(x-3)^2}{8} - \frac{(y+1)^2}{8} = 1$ $a = b = 2\sqrt{2}$

The hyperbola is rectangular. $\therefore m = 1$

Has center at $(3, -1)$

\therefore Equ. is $y + 1 = \pm 1(x - 3)$

$y = -x + 2$ and

$y = x - 4$ are the asymptotes.

Q1. 11. $x^2 - y^2 + 2x + 4y = 0$

$x^2 + 2x + 1 - (y^2 - 4y + 4) = 1 - 4$

$(x+1)^2 - (y-2)^2 = -3$

$\frac{(y-2)^2}{3} - \frac{(x+1)^2}{3} = 1$

Hence the hyperbola is rectangular ($a = b$)

$\therefore e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{2}$

and the asymptotes have equs.

$y = \pm x$

i.e. are \perp .

Q1. 12. $S(5,0)$, $e = \frac{5}{4}$

\therefore equ. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Working

$ae = 5$

$a = \frac{5}{e} = \frac{5 \times 4}{5} = 4$

$b^2 = a^2(e^2 - 1)$

$= 16\left(\frac{25}{16} - 1\right)$

$b^2 = 9$

$a^2 = 16$

Q2. 1. $x^2 - 2y^2 = 2 \Leftrightarrow \frac{x^2}{2} - y^2 = 1$ $\begin{cases} a = \sqrt{2} \\ b = 1 \end{cases}$

Tangent at $(2, 1)$

$\frac{2x}{2} - y = 1 \Rightarrow x - y = 1$

Focus is $S(ae, 0)$ $\because (2, 1)$ is on the positive branch

$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{2 + 1}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}$

$\therefore ae = \sqrt{3}$

$\therefore S(\sqrt{3}, 0)$ and $S^1(-\sqrt{3}, 0)$

distance of S from $x - y - 1 = 0$ $d = \left| \frac{\sqrt{3} - 1}{\sqrt{2}} \right|$

distance of S^1 from $x - y - 1 = 0$ $d^1 = \left| \frac{-\sqrt{3} - 1}{\sqrt{2}} \right|$

$dd^1 = \frac{(\sqrt{3} + 1)(\sqrt{3} - 1)}{2}$

$dd^1 = 1$ as required.

$$\begin{aligned} \text{Q2. 2. } 4x^2 - 9y^2 &= 36 \cap 2y = x + 1 \\ 4(2y - 1)^2 - 9y^2 &= 36 \\ 16y^2 - 16y + 4 - 9y^2 - 36 &= 0 \\ 7y^2 - 16y - 32 &= 0 \end{aligned}$$

$$y = \frac{8 + 12\sqrt{2}}{7} \text{ or } \frac{8 - 12\sqrt{2}}{7}$$

$$\begin{aligned} x &= 2y - 1 \\ &= \frac{16 + 24\sqrt{2}}{7} - 1 \\ &= \frac{9 + 24\sqrt{2}}{7} \text{ or } \frac{9 - 24\sqrt{2}}{7} \end{aligned}$$

$$\begin{aligned} \text{Midpt. } M &= \left[\left(\frac{9 + 24\sqrt{2}}{7} + \frac{9 - 24\sqrt{2}}{7} \right) \div 2, \left(\frac{8 + 12\sqrt{2}}{7} + \frac{8 - 12\sqrt{2}}{7} \right) \div 2 \right] \\ &= \left(\frac{18}{14}, \frac{16}{14} \right) \\ &= \left(\frac{9}{7}, \frac{8}{7} \right) \end{aligned}$$

A more "elegant" method.

Let M be (x, y) then $x_1 + x_2 = 2x$

$y_1 + y_2 = 2y$, then y_1 and y_2

are the roots of $7y^2 - 16y - 32 = 0$

$$y_1 + y_2 = -\frac{b}{a} = \frac{16}{7}$$

$$\frac{y_1 + y_2}{2} = y = \frac{8}{7} \therefore x = 2 \times \frac{8}{7} - 1 = \frac{9}{7}$$

So midpt. is $M \left(\frac{9}{7}, \frac{8}{7} \right)$

$$\text{Q2. 3. } x^2 - 2y^2 = 1 \iff x^2 - \frac{-y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$a = 1$$

$$b = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Equ. of tangent: $y = mx \pm \sqrt{a^2 m^2 - b^2}$

\parallel to $y = \frac{3}{4}x$.

$$y = \frac{3}{4}x \pm \sqrt{\frac{3^2}{4^2} - \frac{1}{2}}$$

$$\sqrt{\frac{9}{16} - \frac{8}{16}}$$

$$y = \frac{3}{4}x \pm \frac{1}{4}$$

$$4y = 3x \pm 1.$$

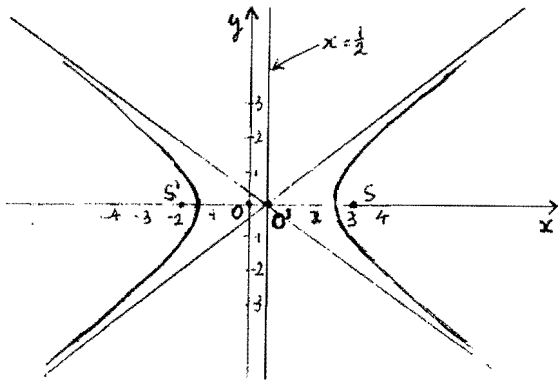
Q2. 4. $S(3,0) S'(-2,0) \therefore$ equ. is $\frac{(x - \frac{1}{2})^2}{a^2} - \frac{y^2}{b^2} = 1$

$a = 2$. Centre $(\frac{1}{2}, 0)$

Translate centre to $(\frac{1}{2}, 0) \therefore$ equ. is $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$

Then $S(2\frac{1}{2}, 0), S'(-2\frac{1}{2}, 0)$

$$\begin{aligned} ae &= \frac{5}{2} \quad \therefore e = \frac{5}{4} \quad \text{then } b = a\sqrt{e^2 - 1} \\ &= 2\sqrt{\frac{25}{16} - 1} \\ &= \frac{3}{2} \end{aligned}$$



Q2. 5. $\frac{x^2}{4} - \frac{y^2}{2} = 1$ Equation of tangent; (at any pt. (x_1, y_1))

$$\frac{xx_1}{4} - \frac{yy_1}{2} = 1$$

Equation of asymptotes; $a = 2, b = \sqrt{2}$

$$y = \pm \frac{1}{\sqrt{2}} x \iff y = \pm \frac{\sqrt{2}x}{2}$$

$$\frac{xx_1}{4} - \frac{yy_1}{2} = 1 \cap y = \frac{\sqrt{2}x}{2} \quad \left| \quad \frac{xx_1}{4} - \frac{yy_1}{2} = 1 \cap y = -\frac{\sqrt{2}x}{2}$$

$$\frac{xx_1}{4} - \frac{\sqrt{2}xy_1}{4} = 1$$

$$\frac{xx_1}{4} + \frac{\sqrt{2}xy_1}{4} = 1$$

$$x(x_1 - \sqrt{2}y_1) = 4$$

$$x = \frac{4}{x_1 + \sqrt{2}y_1}$$

$$x = \frac{4}{x_1 - \sqrt{2}y_1}$$

$$y = -\frac{2\sqrt{2}}{x_1 + \sqrt{2}y_1}$$

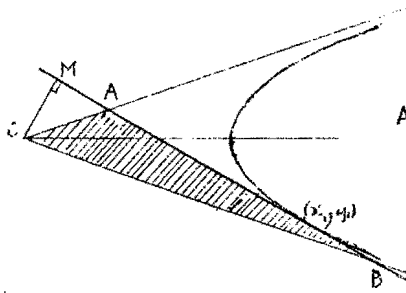
$$y = \frac{2\sqrt{2}}{x_1 - \sqrt{2}y_1}$$

$$OM = \left| \frac{-4}{\sqrt{x_1^2 + 4y_1^2}} \right|$$

dist. of $(0,0)$ from
 $xx_1 - 2yy_1 - 4 = 0$

$$AB^2 = \frac{(4(x_1 + \sqrt{2}y_1) - x_1 + \sqrt{2}y_1)^2}{x_1^2 - 2y_1^2} + \frac{(2\sqrt{2}(x_1 + \sqrt{2}y_1) + x_1 - \sqrt{2}y_1)^2}{x_1^2 - 2y_1^2}$$

(cont'd)



$$AB = \frac{\sqrt{128y_1^2 + 32x_1^2}}{x_1^2 - 2y_1^2}$$

$$= \frac{4\sqrt{2}}{x_1^2 - 2y_1^2} \sqrt{4y_1^2 + x_1^2} \quad x^2 - 2y^2 = 4$$

$$AB = \sqrt{2} \sqrt{4y_1^2 + x_1^2}$$

$$\text{Area of } \triangle OAB = \frac{1}{2} \cdot OM \cdot AB$$

$$= \frac{1}{2} \cdot \frac{4}{\sqrt{4y_1^2 + x_1^2}} \cdot \frac{\sqrt{2} \sqrt{4y_1^2 + x_1^2}}{1}$$

$$= 2\sqrt{2} \text{ sq units which is constant.}$$

(i.e., independent of x and y .)

$$\text{Q2. 6. } x^2 - 9y^2 = 9 \Leftrightarrow \frac{x^2}{9} - y^2 = 1 \quad \left. \begin{array}{l} a = 3 \\ b = 1 \end{array} \right\}$$

$$\text{Gradient of asymptotes } m_1 = \frac{1}{3} \quad m_2 = -\frac{1}{3}$$

Equation of tangents \perp to these asymptotes have $m_1 = -3$, $m_2 = 3$.

$$\text{General equ. of tang. } y = mx \pm \sqrt{a^2 m^2 - b^2}$$

$$y = -3x \pm \sqrt{9 \cdot 9 - 1} \Leftrightarrow y = -3x \pm 4\sqrt{5} \quad \text{---(1)}$$

$$\text{or } y = 3x \pm \sqrt{9 \cdot 9 - 1} \Leftrightarrow y = 3x \pm 4\sqrt{5} \quad \text{---(2)}$$

$$(i) y = -3x + 4\sqrt{5} \cap x^2 - 9y^2 = 9 \Rightarrow x^2 - 9(9x^2 - 24\sqrt{5}x + 80) - 9 = 0$$

$$80x^2 - 216\sqrt{5}x + 729 = 0$$

$$(4\sqrt{5}x - 27)^2 = 0 \therefore x = \frac{27}{4\sqrt{5}} \left. \begin{array}{l} \therefore A\left(\frac{27}{4\sqrt{5}}, \frac{-1}{4\sqrt{5}}\right) \\ y = -\frac{1}{4\sqrt{5}} \end{array} \right\}$$

$$(ii) y = -3x - 4\sqrt{5} \cap x^2 - 9y^2 = 9 \Rightarrow x^2 - 9(9x^2 + 24\sqrt{5}x + 80) - 9 = 0 \Rightarrow 80x^2 + 216\sqrt{5}x + 729 = 0$$

$$\Rightarrow (4\sqrt{5}x + 27)^2 = 0 \therefore x = -\frac{27}{4\sqrt{5}}, y = \frac{1}{4\sqrt{5}} \therefore B\left(\frac{-27}{4\sqrt{5}}, \frac{1}{4\sqrt{5}}\right)$$

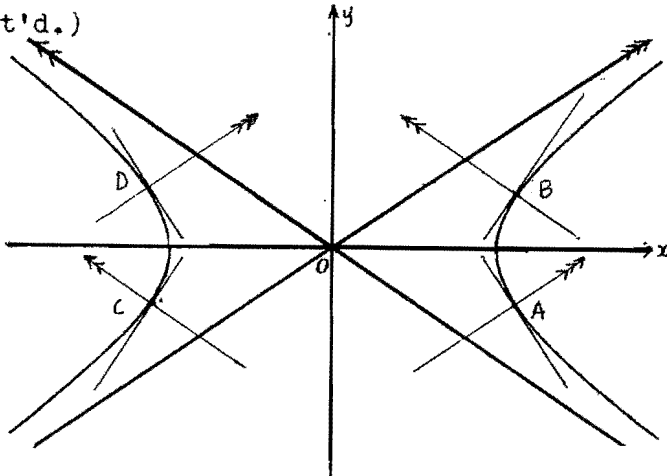
$$(iii) y = 3x + 4\sqrt{5} \cap x^2 - 9y^2 = 9 \Rightarrow x^2 - 9(9x^2 + 24\sqrt{5}x + 80) - 9 = 0 \Rightarrow 80x^2 + 216\sqrt{5}x + 729 = 0$$

$$\Rightarrow (4\sqrt{5}x + 27)^2 = 0 \therefore x = \frac{-27}{4\sqrt{5}}, y = \frac{-1}{4\sqrt{5}} \therefore C\left(\frac{-27}{4\sqrt{5}}, \frac{-1}{4\sqrt{5}}\right)$$

$$(iv) y = 3x - 4\sqrt{5} \cap x^2 - 9y^2 = 9 \Rightarrow D\left(\frac{-27}{4\sqrt{5}}, \frac{1}{4\sqrt{5}}\right)$$

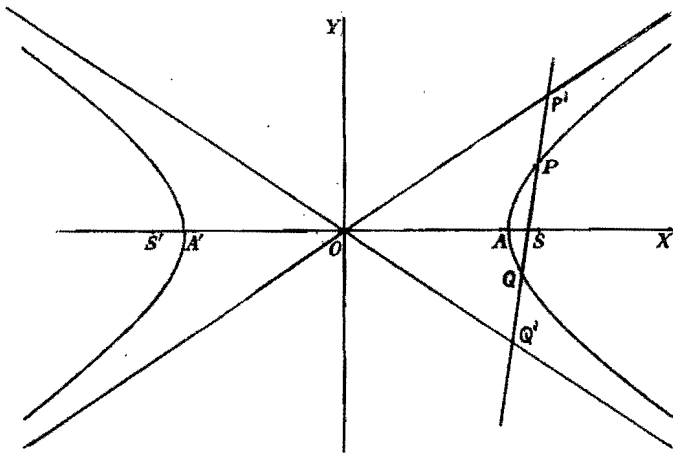
(diagram on following page)

Q2. 6. (cont'd.)



Q2. 7.
$$\left. \begin{aligned} 4x^2 - 8y^2 &= 64 \\ \frac{x^2}{16} - \frac{y^2}{8} &= 1 \end{aligned} \right\} \quad a = 4, b = 2\sqrt{2}$$

Equation of asymptotes $y = \pm \frac{\sqrt{2}}{2}x$



Let the equation of the straight line be $y = mx + c$ intersecting the curve $\frac{x^2}{16} - \frac{y^2}{8} = 1$ at P, Q and intersecting the asymptotes at P^1, Q^1 .

If P and Q are the points $(x_1, y_1), (x_2, y_2)$ then x_1 and x_2 are the roots of the equation;

$$4x^2 - 8(mx + c)^2 = 64$$

$$x^2 - 2(m^2x^2 + 2mxc + c^2) - 16 = 0$$

$$x^2(1-2m^2) - 4mcx - 2c^2 - 16 = 0$$

∴ Sum of roots,

$$x_1 + x_2 = \frac{4mc}{1 - 2m^2}$$

(continued on next page)

Let R be the midpoint of PQ, where R is (x_3, y_3)

$$\text{Then } x_3 = \frac{x_1 + x_2}{2} = \frac{2mc}{1 - 2m^2}$$

(Solving for y_1 it can be shown that $y^2(1-2m^2) - 2cy + c^2 - 8m^2 = 0$

$$\therefore \frac{y_1 + y_2}{2} = \frac{c}{1 - 2m^2}$$

If P^1 and Q^1 are the points (x'_1, y'_1) and (x'_2, y'_2) then x'_1 is the root of

$$mx + c = \frac{\sqrt{2}}{2}x$$

$$\text{so } x'_1 = \frac{-2c}{2m - \sqrt{2}} \quad \text{and } y'_1 = \frac{\sqrt{2}c}{\sqrt{2} - 2m}$$

x'_2 is the root of $mx + c = -\frac{\sqrt{2}}{2}x$

$$\text{so } x'_2 = \frac{-2c}{2m + \sqrt{2}} \quad \text{and } y'_2 = \frac{\sqrt{2}c}{\sqrt{2} + 2m}$$

Let S be the midpoint of P^1Q^1 , where S is (x_4, y_4)

$$\text{then } x_4 = \frac{x'_1 + x'_2}{2} = \frac{\frac{-2c}{2m - \sqrt{2}} + \frac{-2c}{2m + \sqrt{2}}}{2} = \frac{-2mc}{2m^2 - 1} = \frac{2mc}{1 - 2m^2}$$

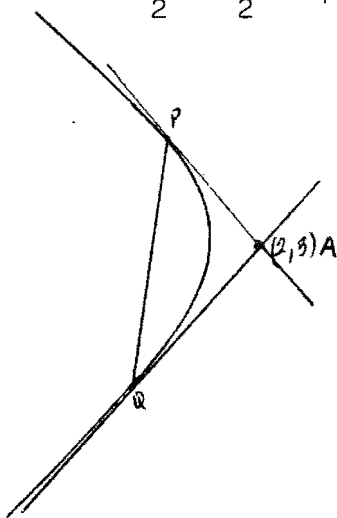
$$y_4 = \frac{y'_1 + y'_2}{2} = \frac{c}{1 - 2m^2}$$

Now we find that R and S are the same point, i.e. PQ and P^1Q^1 have the same midpoint.

$\therefore PP^1 = QQ^1$ as required.

Q2. 8. Find Q between tangents from $(2, 3)$ to

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (a = b = \sqrt{2})$$



$$\text{Equ. of PQ} \Rightarrow \frac{2x}{2} - \frac{3y}{2} = 1$$

$$2x - 3y = 2 \Rightarrow x = 1 + \frac{3y}{2}$$

$$\text{Chord of contact PQ} \cap \frac{x^2}{2} - \frac{y^2}{2} = 1;$$

$$\left(1 + \frac{3y}{2}\right)^2 - y^2 - 2 = 0$$

$$1 + \frac{9y^2}{4} + 3y - y^2 - 2 = 0$$

$$9y^2 + 12y - 4y^2 - 4 = 0$$

$$5y^2 + 12y - 4 = 0$$

$$y = \frac{12 + \sqrt{224}}{10} \text{ or } \frac{-12 - \sqrt{224}}{10} \Rightarrow y = \frac{-6 + 2\sqrt{14}}{5} \text{ or } y = \frac{-6 - 2\sqrt{14}}{5}$$

$$x = \frac{-8 + 6\sqrt{14}}{10} \text{ or } x = \frac{-8 - 6\sqrt{14}}{10}$$

(Cont'd on next page)

$$P\left(\frac{-8 + 6\sqrt{14}}{10}, \frac{-6 + 2\sqrt{14}}{5}\right) \doteq (1.44, 0.296)$$

$$Q\left(\frac{-8 - 6\sqrt{14}}{10}, \frac{-6 - 2\sqrt{14}}{5}\right) \doteq (-3.04, -2.7)$$

$$A(2, 3)$$

$$m_1 = \frac{3 - 0.296}{2 - 1.44} = 4.829$$

$$m_2 = \frac{3 + 2.7}{2 + 3.04} = 1.131$$

$$\begin{aligned} \tan \psi &= \frac{4.829 - 1.131}{1 + 4.829 \times 1.131} \\ &= \frac{3.696}{6.462} \\ &= 0.5723 \end{aligned}$$

$$\psi \doteq 29^\circ 47'$$

Q2. 9. $2x^2 - 8y^2 = 3 \Leftrightarrow \frac{x^2}{3/2} - \frac{y^2}{3/8} = 1$ $a = \frac{\sqrt{3}}{\sqrt{2}}, b = \frac{\sqrt{3}}{2\sqrt{2}}$

Equation of asymptote OM; $y = \frac{1}{2}x$

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{2} \frac{\sqrt{3/2 + 3/8}}{\sqrt{3}} = \sqrt{2} \frac{\sqrt{3 + 3/4}}{\sqrt{3}} = \sqrt{\frac{5}{2}}$$

$$S \Leftrightarrow (ae, 0) \Rightarrow \left(\frac{\sqrt{15}}{2\sqrt{2}}, 0\right) \quad ae = \sqrt{\frac{5}{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}}$$

$$\frac{a}{e} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{2}{\sqrt{5}} = \frac{2\sqrt{3}}{\sqrt{10}}, \text{ directrix; } x = \frac{2\sqrt{3}}{\sqrt{10}}$$

$$x = \frac{2\sqrt{3}}{\sqrt{10}} \cap y = \frac{1}{2}x \therefore y = \frac{\sqrt{3}}{\sqrt{10}}$$

i.e. $M\left(\frac{2\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{3}}{\sqrt{10}}\right)$

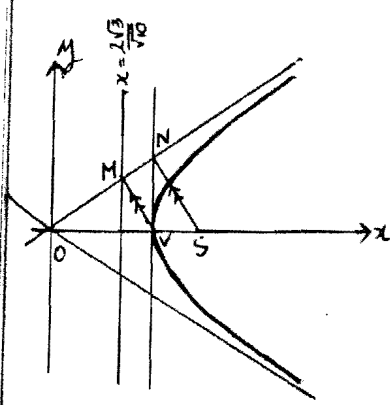
$$V\left(\frac{\sqrt{3}}{\sqrt{2}}, 0\right)$$

$$N\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$$

$$\text{Gradient of MV} = \frac{\frac{\sqrt{3}}{\sqrt{10}} - 0}{\frac{2\sqrt{3}}{\sqrt{10}} - \frac{\sqrt{3}}{\sqrt{2}}} = \frac{\sqrt{3}}{2\sqrt{3} - \sqrt{15}}$$

$$\text{Gradient of NS} = \frac{\frac{\sqrt{3}}{2\sqrt{2}} - 0}{\frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{15}}{2\sqrt{2}}} = \frac{\sqrt{3}}{2\sqrt{3} - \sqrt{15}}$$

$\therefore MV \parallel NS$



$$Q2. 10. \left. \begin{aligned} \frac{x^2}{16} - \frac{y^2}{9} = 1 \quad a = 4 \\ b = 3 \end{aligned} \right\} \text{Equation of asymptote } y = \pm \frac{3}{4}x$$

Equation of tangent at (x_1, y_1) ;

$$\frac{xx_1}{16} - \frac{yy_1}{9} = 1$$

$$9xx_1 - 16yy_1 = 144 \quad \text{--- (1)}$$

$$A \Rightarrow y = \frac{3}{4}x \quad \text{--- (1)} \quad 9xx_1 - 16x \cdot \frac{3}{4}xy_1 = 144$$

$$3xx_1 - 4xy_1 = 48$$

$$x(3x_1 - 4y_1) = 48 \quad \therefore x = \frac{48}{3x_1 - 4y_1}$$

$$y = \frac{3}{4} \cdot \frac{48}{3x_1 - 4y_1}$$

$$y = \frac{36}{3x_1 - 4y_1}$$

$$\text{i.e. } A\left(\frac{48}{3x_1 - 4y_1}, \frac{36}{3x_1 - 4y_1}\right)$$

$$B \Rightarrow y = -\frac{3}{4}x \quad \text{--- (2)} \quad 3xx_1 + 4xy_1 = 48 \quad \therefore x = \frac{48}{3x_1 + 4y_1}$$

$$y = -\frac{3}{4} \cdot \frac{48}{3x_1 + 4y_1} \quad \therefore y = \frac{-36}{3x_1 + 4y_1}$$

$$\text{i.e. } B\left(\frac{48}{3x_1 + 4y_1}, \frac{-36}{3x_1 + 4y_1}\right)$$

Midpt. of AB;

$$\left(\frac{48}{3x_1 - 4y_1} + \frac{48}{3x_1 + 4y_1}\right) \div 2 = \frac{48(3x_1 + 4y_1 + 3x_1 - 4y_1)}{2(9x_1^2 - 16y_1^2)} = \frac{144x_1}{9x_1^2 - 16y_1^2}$$

$$= x_1 \quad (\text{Since } 9x_1^2 - 16y_1^2 = 144)$$

$$\left(\frac{36}{3x_1 - 4y_1} - \frac{36}{3x_1 + 4y_1}\right) \div 2 = \frac{36(3x_1 + 4y_1 - 3x_1 + 4y_1)}{2(9x_1^2 - 16y_1^2)} = \frac{144y_1}{9x_1^2 - 16y_1^2}$$

$$= y_1$$

$\therefore P(x_1, y_1)$ is the midpt. of AB.

$$Q2. 11. b^2x^2 - a^2y^2 = a^2b^2 \quad \text{if } x = a \sec \phi$$

$$\text{Then } b^2a^2\sec^2\phi - a^2y^2 = a^2b^2$$

$$a^2y^2 = a^2b^2(\sec^2\phi - 1)$$

$$y^2 = b^2 \tan^2\phi \quad (\text{for all } \phi)$$

$$\therefore y = b \tan \phi$$

$x = a \sec \phi$, $y = b \tan \phi$ is the parametric equation of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

Q2. 11. (cont'd)

$$\frac{dy}{dx} = \frac{dy}{d\phi} \cdot \frac{d\phi}{dx}$$

$$= \frac{b \sec^2 \phi}{a \tan \phi \sec \phi}$$

$$= \frac{b \sec \phi}{a \tan \phi} = \frac{b}{a} \frac{\frac{1}{\cos \phi}}{\frac{\sin \phi}{\cos \phi}} = \frac{b}{a \sin \phi}$$

$$\begin{aligned} \frac{d}{d\phi} (\cos \phi)^{-1} &= -(\cos \phi)^{-2} (-\sin \phi) \\ &= \frac{\sin \phi}{\cos^2 \phi} \\ &= \tan \phi \sec \phi \end{aligned}$$

tangent; $y - b \tan \phi = \frac{b}{a \sin \phi} (x - a \sec \phi)$

$$ay \sin \phi - ab \frac{\sin^2 \phi}{\cos \phi} = bx - ab \frac{1}{\cos \phi}$$

$$\begin{aligned} bx - ay \sin \phi &= \frac{ab}{\cos \phi} - \frac{ab \sin^2 \phi}{\cos \phi} \\ &= \frac{ab}{\cos \phi} (1 - \sin^2 \phi) \end{aligned}$$

$bx - ay \sin \phi = ab \cos \phi$ is the reqd. equ. of the tangent
at $x = a \sec \phi$, $y = b \tan \phi$

Q2. 12. $x = \frac{at}{2} + \frac{a}{2t} \rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$y^2 = b^2 \left(\frac{x^2}{a^2} - 1 \right)$$

$$= b^2 \left[\frac{1}{a^2} - \frac{a}{4} \left(t + \frac{1}{t} \right)^2 - \frac{a^2}{a^2} \right]$$

$$= \frac{b^2}{4a^2} \left[a^2 \left(t^2 + 2 + \frac{1}{t^2} \right) - 4a^2 \right]$$

$$= \frac{b^2}{4} \left[\left(t + \frac{1}{t} \right)^2 - 4 \right]$$

$$= \frac{b^2}{4} \left(t^2 + 2 + \frac{1}{t^2} - 4 \right)$$

$$= \frac{b^2}{4} \left(t^2 - 2 + \frac{1}{t^2} \right)$$

$$= \frac{b^2}{4} \left(t - \frac{1}{t} \right)^2$$

$$\therefore y = \frac{b}{2} \left(t - \frac{1}{t} \right)$$

$$y = \frac{b}{2} \overline{t - \frac{1}{t}}$$

i.e. the pt. $\left(\frac{a}{2} \left[t + \frac{1}{t} \right], \frac{b}{2} \left[t - \frac{1}{t} \right] \right)$

lies on the hyperbola for varying values of t .

(continued on next page)

Q2. 12 (cont'd)

The equ. of the tangent;

$$\frac{dy}{dx} \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{a^2} \frac{b^2}{y}$$

$$= \frac{b^2}{a^2} \cdot \frac{1}{t} \left(t + \frac{1}{t} \right) \frac{1}{t - \frac{1}{t}}$$

$$= \frac{b}{a} \frac{t^2 + 1}{t} \cdot \frac{t}{t^2 - 1}$$

$$= \frac{b(t^2 + 1)}{a(t^2 - 1)}$$

$$y - \frac{b(t^2 - 1)}{2} = \frac{b}{a} \frac{t^2 + 1}{t^2 - 1} \left(x - \frac{a}{2} \frac{t^2 + 1}{t} \right)$$

$$ay(t^2 - 1) - \frac{ab(t^2 - 1)^2}{2} = bx(t^2 + 1) - \frac{ab(t^2 + 1)^2}{2}$$

$$\begin{aligned} bx(t^2 + 1) - ay(t^2 - 1) &= \frac{ab(t^2 + 1)^2}{2} - \frac{ab(t^2 - 1)^2}{2} \\ &= \frac{ab}{2t} (t^4 + 2t^2 + 1 - t^4 + 2t^2 - 1) \\ &= \frac{ab4t^2}{2t} \end{aligned}$$

$$bx(t^2 + 1) - ay(t^2 - 1) = 2abt$$

Q3. 1. $x^2 - 4y^2 = 9 \Leftrightarrow \frac{x^2}{9} - \frac{y^2}{9/4} = 1$

$$a = 3, \quad b = \frac{3}{2}.$$

Tang. // to $2x + 3y = 0$ i.e. has $m = -\frac{2}{3}$.

$$y = -\frac{2}{3}x \pm \sqrt{\frac{4}{1} \cdot \frac{4}{9} - \frac{9}{4}}$$

$$= -\frac{2}{3}x \pm \sqrt{\frac{7}{4}}$$

$$3y = -2x \pm \frac{3\sqrt{7}}{2} \Rightarrow 6y + 4x = \pm 3\sqrt{7}$$

Q3. 2. $16x^2 - y^2 = 12$ P(1, -2) $\frac{x^2}{3/4} - \frac{y^2}{12} = 1$

$$\text{Tangent; } \frac{4x}{3} + \frac{y}{6} = 1$$

$$8x + y = 6$$

(continued on next page)

Q3. 2. (continued)

$$\text{Normal; } \frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$$

$$\frac{3x}{4} - \frac{24y}{18} = 12\frac{3}{4}$$

$$3x - 24y = 51$$

$$x - 8y = 17$$

Diameter $y = -2x$

$$\text{Q3. 3. } 9x^2 - y^2 = 32 \Leftrightarrow \frac{x^2}{32/9} - \frac{y^2}{32} = 1$$

$$a = \frac{4\sqrt{2}}{3}, \quad b = 4\sqrt{2}$$

P(2, -2) Q(-3, 7)

$$\text{Normal at P. } \frac{32x}{9 \times 8} - \frac{32y}{8} = 32 + \frac{32}{9}$$

$$\frac{16x}{9} - 16y = \frac{320}{9}$$

$$x - 9y = 20 \quad \text{————— (1)}$$

$$\text{Normal at Q - } \frac{32x}{9 \times 3} + \frac{32y}{7} = \frac{320}{9}$$

$$-7x + 27y = 210$$

$$7x - 27y = -210 \quad \text{————— (2)}$$

$$(1) \times 7 \quad 7x - 63y = 140$$

$$(2) \quad -7x + 27y = +210$$

$$-36y = 350$$

$$y = -\frac{175}{18}$$

$$x = -\frac{135}{2}$$

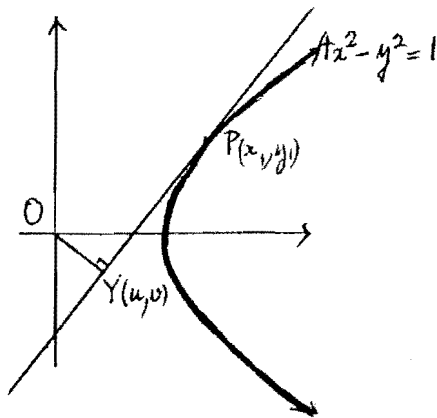
Normals intersect at T. $T(-\frac{135}{2}, -\frac{175}{18})$

Equation of the chord of contact from T is

$$-\frac{135x}{2 \times 32} + \frac{175y}{18 \times 32} = 1$$

$$10935x - 175y + 576 = 0$$

Q. 4.



Equation of tangent at $P(x_1, y_1)$ is $4xx_1 - yy_1 = 1$.

Gradient of perpendicular through O is $-\frac{y_1}{4x_1}$

Equation of OY; $y = \frac{-y_1}{4x_1}x$

$$OY \cap PY = \{Y\}$$

$$y = \frac{-y_1}{4x_1}x \cap 4xx_1 - yy_1 = 1$$

$$x(4x_1 + \frac{y_1^2}{4x_1}) = 1 \Leftrightarrow u = \frac{4x_1}{16x_1^2 + y_1^2} \text{ --- (1) and } v = \frac{-y_1}{16x_1^2 + y_1^2} \text{ --- (2)}$$

$$\text{i.e., } Y(\frac{4x_1}{16x_1^2 + y_1^2}, \frac{-y_1}{16x_1^2 + y_1^2})$$

Aim: 1. Find x_1 and y_1 in terms of u and v .

2. Express $4x^2 - y^2 = 1$ in terms of u, v in order to find equ. of locus.

$$OY \perp PY \therefore \frac{v}{u} \times \frac{4x_1}{y_1} = -1 \text{ i.e. } \frac{u}{v} = \frac{-4x_1}{y_1} \text{ --- (3)}$$

$$\text{From (1) } 16x_1^2u + uy_1^2 = 4x_1 \quad \text{From (2) } 16x_1^2v + vy_1^2 = -y_1$$

$$\text{Subst (3)} \rightarrow (1) \ 4x_1(u + \frac{v^2}{u}) = 1. \text{ Subst (3)} \rightarrow (2) \ y_1^2(\frac{u^2}{v} + v) = -y_1$$

$$\therefore x_1 = \frac{u}{4(u^2 + v^2)} \text{ --- (4) } \quad \therefore y_1 = \frac{-v}{u^2 + v^2} \text{ --- (5)}$$

Subst. (4) and (5) into $4x^2 - y^2 = 1$ since (x_1, y_1) lies on it.

$$\frac{u^2}{4(u^2 + v^2)^2} - \frac{v^2}{(u^2 + v^2)^2} = 1$$

$$4(u^2 + v^2)^2 = u^2 - 4v^2 \text{ which is the required equation of the locus of } Y.$$

$$Q3. 5. \quad 9x^2 - 4y^2 = 2 \Leftrightarrow \frac{x^2}{\frac{2}{9}} - \frac{y^2}{\frac{2}{4}} = 1 \quad \begin{cases} a = \frac{\sqrt{20}}{3} \\ b = \sqrt{5} \end{cases}$$

Tangent at $P(-2, 2)$

$$-\frac{18x}{20} - \frac{8y}{20} = 1$$

$$9x + 4y = -10$$

$$\text{Equation of asymptotes } y = \pm \frac{3\sqrt{5}}{\sqrt{20}}x \Leftrightarrow y = \pm \frac{3}{2}x$$

(continued on next page)

Q3. 5. (cont'd)

$$L \Rightarrow 9x + 4y = -10 \cap y = \frac{3}{2}x$$

$$9x + 6x = -10$$

$$15x = -10$$

$$x = -\frac{2}{3} \quad y = \frac{3}{2} \cdot -\frac{2}{3} = -1$$

$$\therefore L \left(-\frac{2}{3}, -1\right)$$

$$M \Rightarrow 9x + 4y = -10 \cap y = -\frac{3}{2}x$$

$$9x - 6x = -10$$

$$3x = -10$$

$$x = -\frac{10}{3} \quad y = -\frac{10}{3} - \frac{3}{2} \cdot \frac{10}{3}$$

$$y = 5$$

$$M \left(-\frac{10}{3}, 5\right)$$

$$\text{Midpoint of LM} \left(\frac{-\frac{2}{3} - \frac{10}{3}}{2}, \frac{-1 + 5}{2}\right) \Rightarrow (-2, 2)P$$

i.e., Midpt. of LM is P.

$$Q3. 6. (1) \quad 4x^2 - y^2 = 35 \quad \Leftrightarrow \frac{x^2}{\frac{35}{4}} - \frac{y^2}{35} = 1 \quad \begin{cases} a = \frac{\sqrt{35}}{2} \\ b = \sqrt{35} \end{cases}$$

Equation of the asymptotes $y = \pm \frac{1}{2}x$

$P\left(3\frac{1}{2}, 7\right)$ lies on the asymptote $y = \frac{1}{2}x$

Equation of tangent to hyperbola at (x_1, y_1)

$$4xx_1 - yy_1 = 35; \text{ through } \left(3\frac{1}{2}, 7\right) \text{ is } 14x_1 - yy_1 = 35$$

$$\text{i.e. } y_1 = 2x_1 - 5 \quad (2)$$

To find pt. of intersection of (1) and (2)

$$4x_1^2 - (4x_1^2 - 20x_1 + 25) - 35 = 0$$

$$\therefore \begin{cases} x_1 = 3 \\ y_1 = 1 \end{cases} \text{ i.e. the "line" through } \left(3\frac{1}{2}, 7\right) \text{ is touching the hyperbola at } Q(3, 1).$$

$$\text{The equation of PQ is } y - 1 = \frac{7 - 1}{\frac{7}{2} - 3}(x - 3)$$

$$12x - y - 35 = 0$$

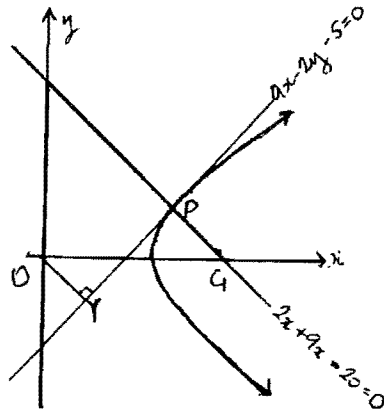
$$\begin{aligned} \text{Q3. 7. } 9x^2 - y^2 &= 5 & \left. \begin{aligned} a &= \sqrt{5}/3 \\ b &= \sqrt{5} \end{aligned} \right\} \\ \frac{x^2}{5/9} - \frac{y^2}{5} &= 1 \end{aligned}$$

$$\text{Normal } \frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + bb^2 \text{ at } P(1,2)$$

$$\frac{1}{9}x + \frac{1}{2}y = \frac{10}{9}$$

$$2x + 9y = 20 \text{ ————— (1)}$$

(1) meets x axis at $G(10,0)$



Equation of tangent at P

$$9x - 2y = 5 \text{ ————— (2)}$$

Equation of OY $y = -\frac{2}{9}x$ — (3)

$$(3) \rightarrow (2) \quad 9x + \frac{4}{9}x = 5$$

$$x = \frac{9}{17}y = -\frac{2}{17} \therefore Y\left(\frac{9}{17}, -\frac{2}{17}\right)$$

$$\begin{aligned} OY \cdot PG &= \left| \frac{5}{\sqrt{9^2 + 2^2}} \right| \cdot \left| \frac{10 \times 9 - 5}{\sqrt{9^2 + 2^2}} \right| \\ &= \left| \frac{-5 \times 85}{85} \right| \\ &= 5 \end{aligned}$$