UNIT THREE.

Q1. 1. $xy = 9 \quad c^2 = 9 \quad \therefore c = 3 \qquad x = 3t,\ y = \frac{3}{t}$

Q1. 2. $xy = 16 \quad c^2 = 16 \quad c = 4 \qquad x = 4t \quad y = \frac{4}{t}$

Q1. 3. $xy = \frac{25}{4} \quad c = \frac{5}{2} \qquad x = \frac{5t}{2} \quad y = \frac{5}{2t}$

Q1. 4. $xy = \frac{1}{9} \quad c = \frac{1}{3} \qquad x = \frac{t}{3} \quad y = \frac{1}{3t}$

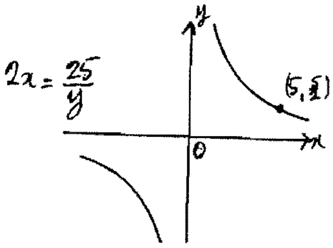
Q1. 5. $xy = 2 \quad c = \sqrt{2} \qquad x = \sqrt{2}t \quad y = \frac{\sqrt{2}}{t}$

Q1. 6. $-xy = +4 \quad c = 2 \qquad x = 2t \quad y = -\frac{2}{t}$

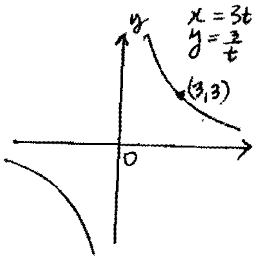
Q1. 7. $x = 5t \ y = \frac{5}{t} \Longleftrightarrow \quad xy = 25$

Q1. 10. $(t, -\frac{1}{t}) \Rightarrow xy = -1$

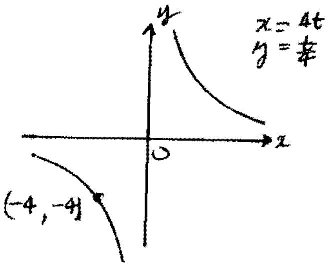
Q1. 11.



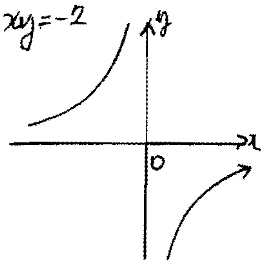
Q1. 12.



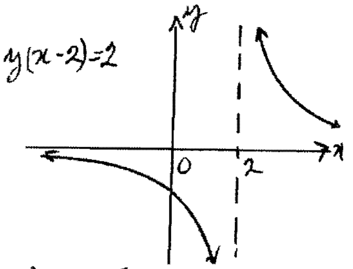
Q1. 13.



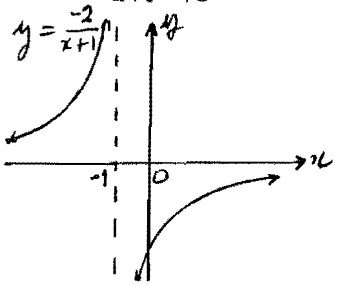
Q1. 14.



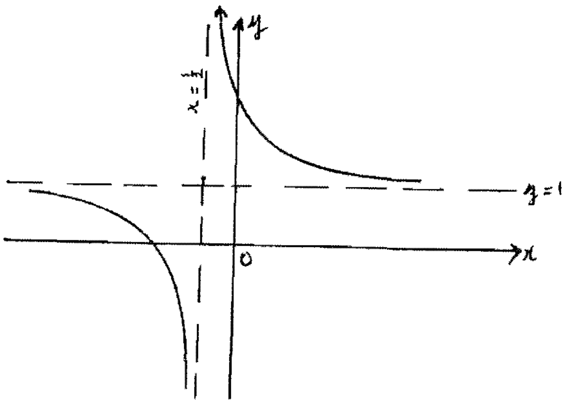
Q1. 15



Q1. 16



Q1. 17. $(y-1)(2x+1) = 16$



Q1. 18. $xy = 18$ $\quad c^2 = 18$ $\quad \therefore 18 = \frac{a^2}{2}$ i.e. $a = 6$, $\underline{e = \sqrt{2}}$

(i) length of transv. axis = 12 units

$ae = 6\sqrt{2}$ $\qquad 2x^2 = (6\sqrt{2})^2$

$x = 6$

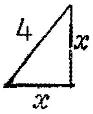
(ii) focus; $S(6,6)$ $\quad S^1(-6,6)$

Q1. 19. $xy = 4$ $\quad c^2 = \frac{a^2}{2} = 4$ $\quad \therefore a = 2\sqrt{2}$

(i) transv. axis is $4\sqrt{2}$ units

(ii) $ae = 2\sqrt{2}\sqrt{2}$

$= 4$ foci $S(2\sqrt{2}, 2\sqrt{2})$ $S^1(-2\sqrt{2}, -2\sqrt{2})$



$x^2 = 8$

$x = 2\sqrt{2}$

focus $(ae \cdot \cos 45^\circ,\ ae \cos 45^\circ)$

$= (a, a)$

Q1. 20. $\left.\begin{aligned} x &= 8t \\ y &= \frac{8}{t} \end{aligned}\right\}$ $xy = 64$ $\quad c^2 = \frac{a^2}{2} = 64$ $\quad a = 8\sqrt{2}$

(i) transverse axis; $8\sqrt{2} \times 2 = 16\sqrt{2}$ units.

(ii) foci; $S(8\sqrt{2}, 8\sqrt{2})$

$S^1(-8\sqrt{2}, -8\sqrt{2})$

Q1. 22. $xy = 8$ $\quad \therefore a^2 = 16$ $\quad a = 4$ $\qquad c = 2\sqrt{2}$

$a^1 = 4 \cos 45^\circ$

$= \frac{4}{\sqrt{2}}$

$\therefore V(4/\sqrt{2}, 4/\sqrt{2})$ $\qquad V^1(-4/\sqrt{2}, -4/\sqrt{2})$

Equ. of tangent;

$x + t^2y = 2ct$

$\left\{\begin{aligned} c &= 2\sqrt{2} \\ x &= 4/\sqrt{2} \\ y &= 4/\sqrt{2} \end{aligned}\right\}$

$x = ct$

$4/\sqrt{2} = 2\sqrt{2}t$

$t = \frac{4}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = 1$ $\qquad$ where $t = \pm 1$

tangents; $x + y = \pm 4\sqrt{2}$

Q1. 23. Tang. and normal at $(4t, \frac{4}{t})$ on $xy = 16$ $\therefore$ c = 4.

tangent; $x + t^2y = 8t$

normal; gradient $= t^2$

$$y - \frac{c}{t} = t^2(x - ct)$$

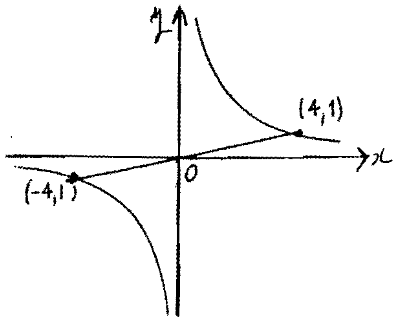
$$(y - \frac{4}{t}) = t^2(x - 4t)$$

$$t^3(x - 4t) = ty - 4$$

$$ty - t^3x = 4 - 4t^4$$

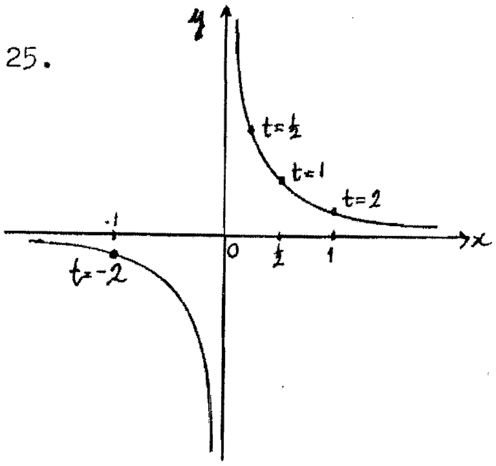
$$ty - t^3x = 4(1 - t^4)$$

Q1. 24. Length of diameter. $xy = 4$, through (4,1)



$d = 2\sqrt{17}$

Q1. 25.



$$\left.\begin{aligned} x &= \frac{t}{2} \\ y &= \frac{1}{2t} \end{aligned}\right\}$$

Q1. 26. $xy = 3$; tangents // $y + 3x = 0$

Equ. of tang; $x + t^2y = 2ct$. $m = -\frac{1}{t^2}$

$$\therefore -\frac{1}{t^2} = -3$$

hence $t = \pm\frac{1}{\sqrt{3}}$ $c = \sqrt{3}$

$\therefore$ equation of tangent // $y + 3x = 0$ is

$3x + y = \pm 6$ $d = 2\left|\frac{6}{\sqrt{10}}\right| = \frac{12}{\sqrt{10}}$

Q1. 27. Pt. of contact of 2 tangents from $(-5,1)$ to $xy = 4$

Method 1: (non parametric)

Let $y = mx + c$ be the equation of the tangent.

$x(mx + c) = 4 \iff mx^2 + cx - 4 = 0$

if tangent, then $b^2 - 4ac = 0$ i.e. $c^2 + 16m = 0$ i.e. $m = -\frac{c^2}{16}$

Now we have $y = -\frac{c^2}{16}x + c$ as tangent. But it is through

$(-5,1)$ $\therefore$ $1 = \frac{5}{16}c^2 + c \iff (5c - 4)(c + 4)$ $\therefore c = -4$ or $\frac{4}{5}$

and the tangents are $y = -x - 4$ and $y = \frac{x + 20}{25}$.

$xy = 4 \cap y = -x - 4 \implies (x+2)(x+2) = 0$ $\therefore x = -2$ and $y = -2$

$xy = 4 \cap y = \frac{-x + 20}{25} \implies x^2 + 20x + 100 = 0$ $\therefore x = +10$ and $y = \frac{2}{5}$

So the pt. of contacts are $(10,\frac{2}{5})$ and $-2,-2)$.

Method 2:

Let $x + t^2y = 2ct$ be the tangent, which is a quadratic in t. ($\therefore$ it has 2 roots.)

Since $yt^2 - 2ct + x = 0$ is through $(-5,1)$ and $c = 2$ we have

$t^2 - 4t - 5 = 0 \iff (t-5)(t+1) = 0$

$\therefore$ $t = 5$ or $t = -1$.

if $t = 5$; $x = ct \iff x = 10$, $y = \frac{c}{t} \iff y = \frac{2}{5}$ $\}$ $(10, 2/5)$

if $t = -1$; $x = ct \iff x = -2$, $y = \frac{c}{t} \iff y = 2$ $\}$ $(-2,-2)$

$\therefore$ the points of contact are $(10,\frac{2}{5})$, $(-2,-2)$.

Q1. 28. Normal at P(8,2) cuts $(4t, \frac{4}{t})$ at Q.

$x = 8 = 4t \therefore t = 2$

Gradient of normal; $t^2 = 4$

Equation of normal; $y - 2 = 4(x - 8)$

$$4x - y = 30$$

Normal $\cap$ $xy = 16$ when $x(4x - 30) = 16$

$$2x^2 - 15x - 8 = 0$$

$$(2x + 1)(x - 8) = 0$$

$\therefore x = -\frac{1}{2}$ is the other abscissa. $y = -32$.

$Q(-\frac{1}{2}, -32)$

$$PQ^2 = (8 + \tfrac{1}{2})^2 + (2 + 32)^2$$

$$= (\tfrac{17}{4})^2 + 1156$$

$$= \frac{4913}{4}$$

$$PQ = \frac{17\sqrt{17}}{2}$$

Q1. 29. $xy = 9 \quad \therefore \quad c = 3 \quad$ Tangent through $(-9,3)$

Let the equation of the tangent be $y = mx + b$.

Condition; $mx^2 + bx - 9 = 0$ has $b^2 - 4ac = 0$

$$b^2 + 36m = 0$$

$$m = \frac{-b^2}{36}$$

$\therefore$ The tangent is $y = \frac{-b^2}{36}x + b$

Through $(-9,3)$ $\therefore$ we have $3 = \frac{b^2}{4} + b$

$$12 = b^2 + 4b$$

$b^2 + 4b - 12 = 0 \iff (b + 6)(b - 2) = 0$

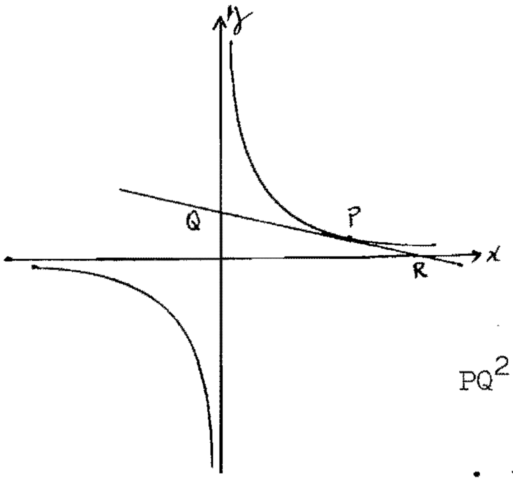
$\therefore b = 2$ or -6

Hence the tangents are

$y = -\frac{1}{9}x + 2$ or $y = -x - 6$

$x + 9y = 18$ or $x + y + 6 = 0$

Q2. 1. Tangent at P(6,2) of $xy = 12$



$x = ct$ $\qquad c = 2\sqrt{3}$

$6 = 2\sqrt{3}t$

$\therefore t = \sqrt{3}$

Tangent; $x + 3y = 2 \times 2\sqrt{3} \times \sqrt{3}$

$x + 3y = 12$

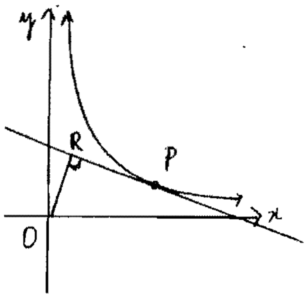
Q(12,0) R(0,4)

$PQ^2 = (6-12)^2 + (2-0)^2 \qquad PR^2 = (6-0)^2 + (2-4)^2$

$= 40 \qquad = 40$

$\therefore PQ = PR$

Q2. 2.



$x = t,\ y = \frac{1}{t}$

Equation of tangent at P

$t^2y + x = 2ct \qquad c = 1$

$t^2y + x = 2t$ ——— (A)

Equation of OR, where R(X,Y) is the foot of the ⊥ through O to the tangent; $y = t^2x$ ——— (B)

(B) — (A) $t^4x - 2t + x = 0 \iff X = \frac{2t}{t^4 + 1}$ and $Y = \frac{2t^3}{t^4 + 1}$

——— o ———

OR ⊥ PR $\therefore \frac{Y}{X} = t^2$ ——— (1) $\quad (m_1m_2 = -1)$

$X = \frac{2t}{t^4 + 1} \iff Xt^4 + X = 2t$ ——— (2)

(1) ⟶ (2) $t = \frac{1}{2}\left(\frac{X^2 + Y^2}{X}\right)$

$\therefore t^2 = \frac{1}{4X^2}(X^2 + Y^2)^2$ ——— (3)

$Y = t^2X$ ——— (1)

(3) ⟶ (1) $Y = \frac{1}{4X^2}(X^2 + Y^2)^2\ X$

$\therefore 4XY = (X^2 + Y^2)^2$ is the locus of R.

(Showing that $t^3 = \frac{1}{8X^3}(X^2 + Y^2)^3$ ——— (4) and substituting (1) and (4) into $Y = \frac{2t^3}{t^4 + 1}$ would also yield the same result)

Q2. 3. $A(m,0) \quad B(0,\frac{4}{m})$ $\qquad xy = 1$ ——— (2)

$$AB \Rightarrow \frac{x}{m} + \frac{y}{4/m} = 1$$

$$4x + m^2y = 4m$$

$$y = \frac{4}{m^2}(m - x) \text{ ——— (1)}$$

Then (1) $\longrightarrow$ (2) gives $x\frac{4}{m^2}(m - x) = 1$

$$4x^2 - 4mx + m^2 = 0 \text{ ——— (3)}$$

If (1) is a tangent, than $\Delta = b^2 - 4ac = 0$

i.e. $\Delta = 16m^2 - 16m^2$

$= 0$ for all m.

$\therefore$ AB touches the hyperbola $xy = 1$ for all values of m.

$4x^2 - 4mx + m^2 = 0$ ——— (3)

$(2x - m)^2 = 0$

$\therefore \frac{m}{2} = x, \ y = \frac{2}{m}$

Hence the point of contact is;

$$(\frac{m}{2}, \frac{2}{m})$$

Q2. 4. Tangent at $(2t, \frac{1}{t})$ to $xy = 2$

$y + x\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

m at $(2t, \frac{1}{t})$ is $= -\frac{1}{2t^2}$

Equation of tangent;

$$y - \frac{1}{t} = -\frac{1}{2t^2}(x - 2t)$$

$$x + 2t^2y - 4t = 0$$

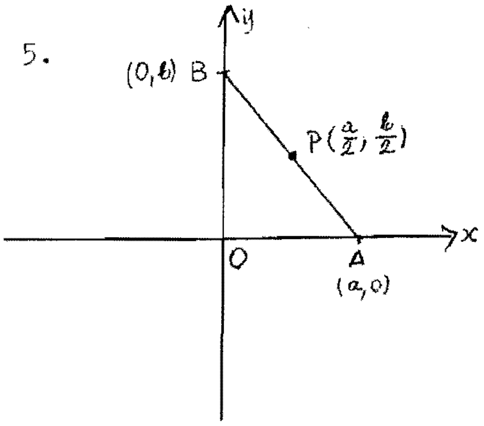
Distance of (2,2) from tangent $= \left|\frac{2 + 4t^2 - 4t}{\sqrt{1 + 4t^4}}\right| = d_1$

Distance of (-2,2) from tangent $= \left|\frac{-2 - 4t^2 - 4t}{\sqrt{1 + 4t^4}}\right| = d_2$

$$d_1d_2 = \frac{4|(2t^2 - 2t + 1)(2t^2 + 2t + 1)|}{1 + 4t^4}$$

$= 4$ as required.

Q2. 5.



A(a,0)
B(0,b)

Midpt. P is $(\frac{a}{2}, \frac{b}{2})$

$\triangle BOA = 2c^2$

i.e. $\frac{ab}{2} = 2c^2$

$\therefore = ab = 4c^2$ ——(1)

Let $X = \frac{a}{2}$, $Y = \frac{b}{2}$ i.e. $a = 2X$

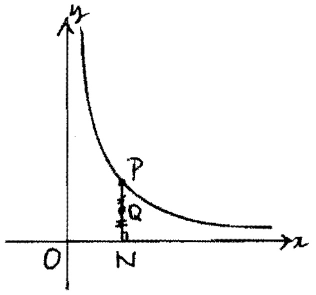
$b = 2Y$

$\therefore \quad ab = 4XY$ —— (2)

(1)⟶(2) $4c^2 = 4XY$

$\therefore XY = c^2$ is the locus of P.

Q2. 6.



Let P be the point $(ct, \frac{c}{t})$ and Q(X,Y) is the midpoint of PN

$\therefore N(ct, 0)$

$\therefore \quad X = \frac{ct}{2} \implies t = \frac{2X}{c}$ —(1)

$Y = \frac{c}{t}$ —— (2)

(1) ⟶ (2) $\quad Y = \frac{c^2}{2X}$ i.e., $XY = \frac{c^2}{2}$ is the ,locus of Q which is a rectangular hyperbola with the same asymptotes as $x = ct$, $y = \frac{c}{t}$.

Q2. 7. Normal to $xy = 6$ at A(2,3)

$c = \sqrt{6} \qquad x = ct$

$\therefore \quad 2 = \sqrt{6}t \qquad t = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

Normal; $t^3x - ty = c\,(t^4 - 1)$

$\frac{2\sqrt{6}}{9}x - \frac{\sqrt{6}}{3}y = \sqrt{6}(\frac{36}{81} - 1)$

$2x - 3y = -5$

B(1,6) $\quad m_{AB} = \frac{6 - 3}{1 - 2} = -3 \quad \therefore m_{AC} = \frac{1}{3}$

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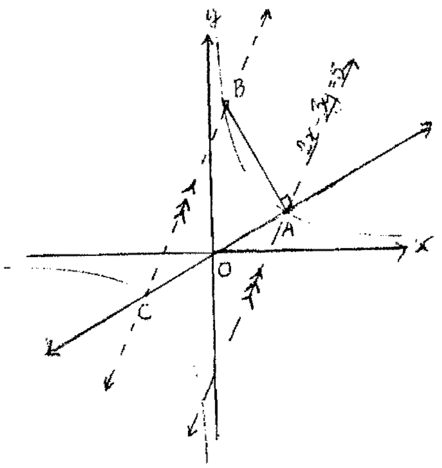
Q2. 7. (cont'd)



Equ. of AC $x - 3y + 7 = 0$

to find C;

$y(3y - 7) = 6$

$(3y + 2)(y - 3) = 0$

$\therefore\ y = -2/3$ or $y = 3$

and $x = -9$ or $x = 2$

$\therefore\ C\ (-9, -\frac{2}{3})$

Q2. 8. $xy = 4$ Normal // to $4x - y = 2$

Gradient of normal $= t^2$
Gradient of line $= 4$
$\therefore\ t^2 = 4$, $t = \pm 2$

Equation of normal;

$t^3x - ty = c(t^4 - 1)$ $c = 2$

$t^3x - t2y = 2(t^4 - 1)$

when $t = 2$ $8x - 2y = 30$

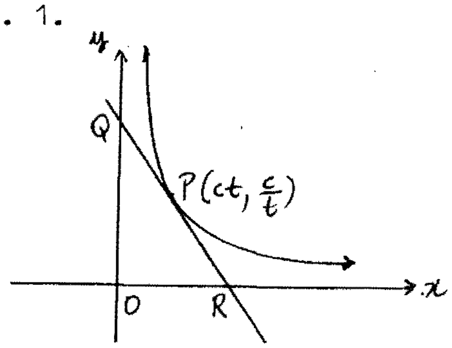
$4x - y = 15$

when $t = -2$ $-8x + 2y = 30$

$4x - y = -15$

$\therefore$ Equation of normals // to $4x - y = 2$ are $4x - y = \pm 15$.

Q3. 1.



Equation of tangent at P

$x + t^2y = 2ct$

$\therefore\ Q \Rightarrow (0, \frac{2c}{t})$

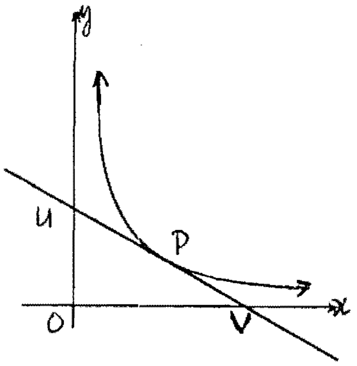
$R \Rightarrow (2ct, 0)$

$QR^2 = 4c^2t^2 + \frac{4c^2}{t^2} = 4(c^2t^2 + \frac{c^2}{t^2})$

$OP^2 = c^2t^2 + \frac{c^2}{t^2}$

$\therefore\ 2OP = QR$

Q3. 2.



Let P be the point $(ct, \frac{c}{t})$

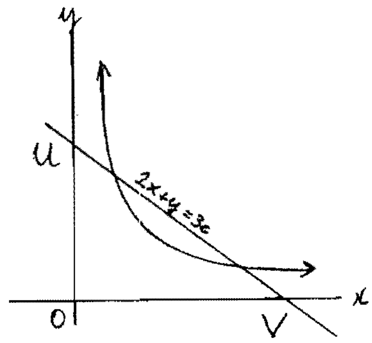
$\therefore$ $U(0, \frac{2c}{t})$

$V(2ct, 0)$

$\Delta\, UOV = \frac{1}{2} \cdot 2c\not{t}^{1} \cdot \frac{2c}{\not{t}_1}$

$= 2c^2$ which is constant.

Q3. 3.



$U(0, 3c)$

$V(\frac{3c}{2}, 0)$

Let $P(x,y)$ be the pt. of trisection of UV.

$x = \dfrac{2 \cdot \frac{3c}{2} + 1.0}{2 + 1}$ $\qquad$ $y = \dfrac{2.0 + 1.3c}{2 + 1}$

$x = c$ $\qquad$ $y = c \longrightarrow (c, c)$

$\therefore x \Rightarrow ct = c$ or $y \Rightarrow \frac{c}{t} = c$

$t = 1$ $\qquad$ $t = 1$

$\therefore$ the point (c,c) is on $xy = c^2$

OR $x = \dfrac{1 \cdot \frac{3c}{2} + 2.0}{2 + 1}$ $\qquad$ $y = \dfrac{2.3c + 1.0}{2 + 1}$

$x = \frac{c}{2}$ $\qquad$ $y = 2c \longrightarrow (\frac{c}{2}, 2c)$

$\frac{c}{2} \cdot 2c = c^2$ $\qquad$ $\therefore$ the point $(\frac{c}{2}, 2c)$ is

also on $xy = c^2$.

$\therefore$ Both pts. of trisection of UV lies on $xy = c^2$.

Q3. 4. The following are to be proven:

(i) The tangent at P is the midpt. of UV, where U and V are the intercepts on the asymptotes.

(ii) UV = 2OP (See Q3. 1.)

$\therefore$ PV = PU = OP.

i.e. P is the centre of the circle through O,U and V.

Answer: draw circle radius 3c centre P.

Proof for (i) See diagram for Q3. 1.)

$U(2ct, 0)$ $v(0, \frac{2c}{t})$ $P(ct, \frac{c}{t})$

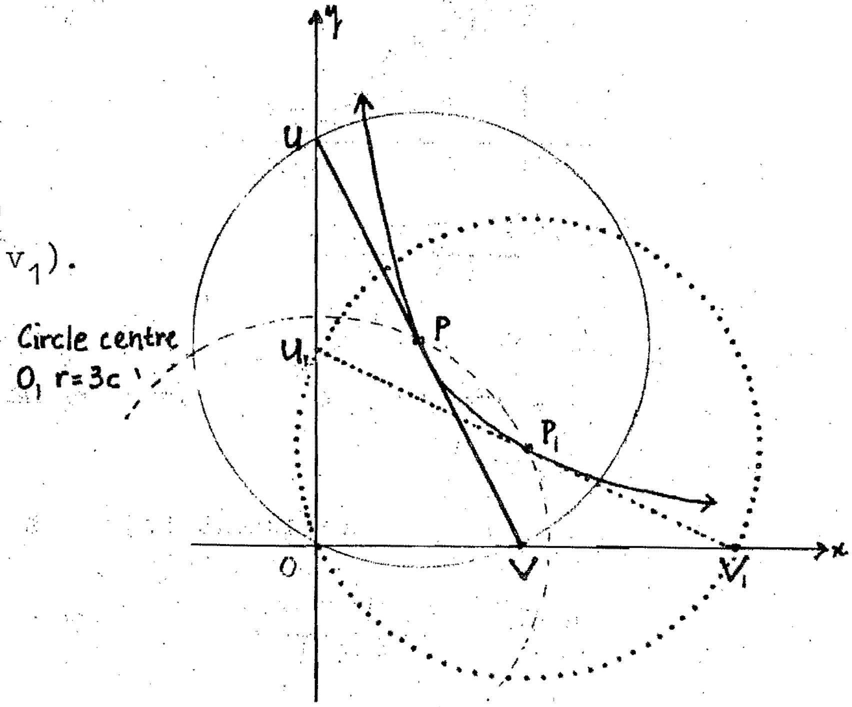
Midpt. of UV $\Rightarrow (ct, \frac{c}{t}) \Rightarrow P$

CONSTRUCTION

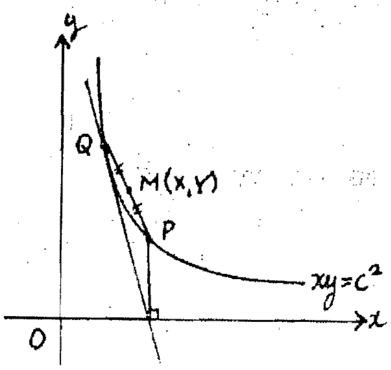
(1) Draw circle centre O, r = 3c to cut $xy = c^2$ at P, and/or at P_1.

(2) Draw circles centre P (or P_1) to cut axes at u,v (or at u_1, v_1).

(c) $\therefore$ Length of uv = 3c units (u_1, v_1 = 3c also).



Q5.



Let Q,P have coordinates $(cq, \frac{c}{q})$, $(cp, \frac{c}{p})$ respectively and M(X,Y) be the midpoint of PQ.

$\therefore X = \frac{c}{2}(p+q)$ and $Y = \frac{c}{2}(\frac{p+q}{pq})$

by using the midpoint formula.

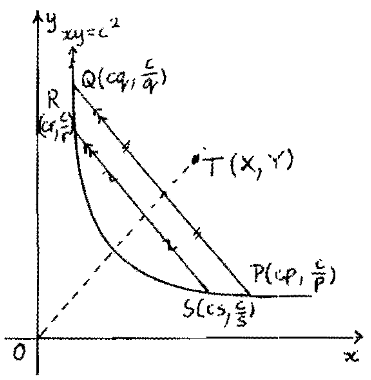
The tangent at Q $x + q^2y = 2cq$ cuts the x axis when y = 0 i.e. at $x = 2cq$, which is equivalent to the abscissa $x = cp$ of the point P.

$\therefore p = 2q$

On substituting this into X and Y we have $X = \frac{3cq}{2}$ and $Y = \frac{3c}{4q}$, which yields $XY = \frac{9c^2}{8}$ (on eliminating q.)

Hence the locus is a hyperbola having the same asymptotes as $xy = c^2$.

Q4.1.



Let PQ and SR to be any two elements of a set of parallel chords

Gradient of chord PQ:

$$m_1 = \frac{\frac{c}{p} - \frac{c}{q}}{cp-cq} = \frac{\frac{a - p}{pq}}{p - q} = -\frac{1}{pq}$$

Similarly; gradient of SR $= -\frac{1}{sr} - m_1$

Since SR is parallel to PQ $-\frac{1}{pq} = -\frac{1}{sr}$ ...(1)

Show; line through midpoint PQ and SR is a diameter.

Through $\left[\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right]$ and $\left[\frac{c(r+s)}{2}, \frac{c(r+s)}{2rs}\right]$ the gradient

$$m = \frac{\frac{c(r+s)}{2rs} - \frac{c(p+q)}{2pq}}{\frac{c(r+s)}{2} - \frac{c(p+q)}{2}} = \frac{\frac{r+s}{rs} - \frac{p+q}{pq}}{r + s - p - q} \quad \text{.............(2)}$$

(1) → (2) $\quad m = \frac{-(r+s)m_1 + (p+q)m_1}{r+s - p-q} = \frac{m_1(-r - s + p + q)}{r+s - p-q}$

$$m = -m_1 \quad (\text{i.e. } m = \frac{1}{pq})$$

Now, if the equation of the line through the midpoint of PQ $\left[\frac{c\,(p+q)}{2}, \frac{c(p+q)}{2pq}\right]$ with the gradient $m = \frac{1}{pq}$ is in the form of

$$y = mx + b$$

then

$$\frac{c(p+q)}{2pq} = \frac{1}{pq} \cdot \frac{c(p+q)}{2} + b$$

$$\therefore b = 0$$

So the equation of the line through midpoint PQ and SR in

$y = -mx$. i.e. passes through (0,0) and is ∴ a diameter.

The line that bisects parallel chords of the hyperbola $xy = c^2$ is a diameter.

(By Katherine Merrick Year 12)

Alternatively; using the above diagram
(None parametric approach)
Let the equation of the family of parallel chords be $y = mx$ + constant.
Let $y = mx + d$ be the equation of chord PQ (i.e. a member of the family of || chords) with T(X,Y) being its midpoint.

(cont. next p.)

$y = mx + d \cap xy = c^2$

$x(mx+d) = c^2$

$\therefore mx^2 + dx - c^2 = 0.$ If x_1 and x_2 are the roots of this equation then $x_1 + x_2 = -\frac{d}{m}$

$$\text{hence } X = \frac{x_1 + x_2}{2} = -\frac{d}{2m}$$

$$Y = m\left(\frac{-d}{2m}\right) + d$$

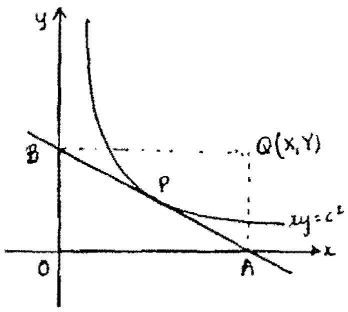
$$\text{ie. } Y = \frac{d}{2}$$

Now $\frac{X}{Y} = \frac{d/2}{-d/2m} \longleftrightarrow Y = -mX$ which is the equation of the locus of T.

This equation represents a straight line through O
$\therefore$ it is the diameter conjugate to the chord $y = mx + d$

Conclusion: as above.

Q4. 2.



$$\left.\begin{array}{l} x = ct \\ y = \frac{c}{t} \end{array}\right\} \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\frac{dx}{dt} = c$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{1}{t^2}$$

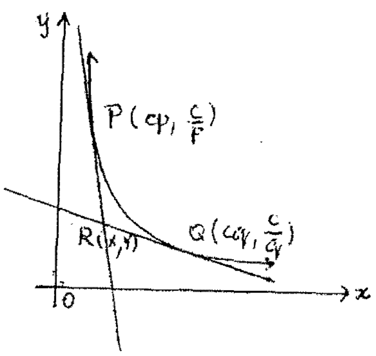
Tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

As $x = 0$ $y = \frac{2c}{t}$ at $y = 0$ $x = 2ct$

$$\therefore \text{Equ. of locus } Y = \frac{2c}{X/2c}$$

$XY = 4c^2$ is the equation of Q.

Q4. 3.



Let R (X,Y) be any point on the locus.
By solving the equations of the tangents
at P; $x + p^2y = 2cp$ (1)
at Q; $x + q^2y = 2cq$ (2) simultaneously
we obtain;

$$Y = \frac{2c}{p + q} \qquad (3)$$

Substitute (3) into (1) then

$$X = 2cp\left(1 - \frac{p}{p + q}\right)$$

$$X = \frac{2cpq}{p + q} = pqy \quad \text{(using (3))}$$

So $X = kY$ is the equation of the locus, which is a straight line through the origin. (i.e. a diameter)