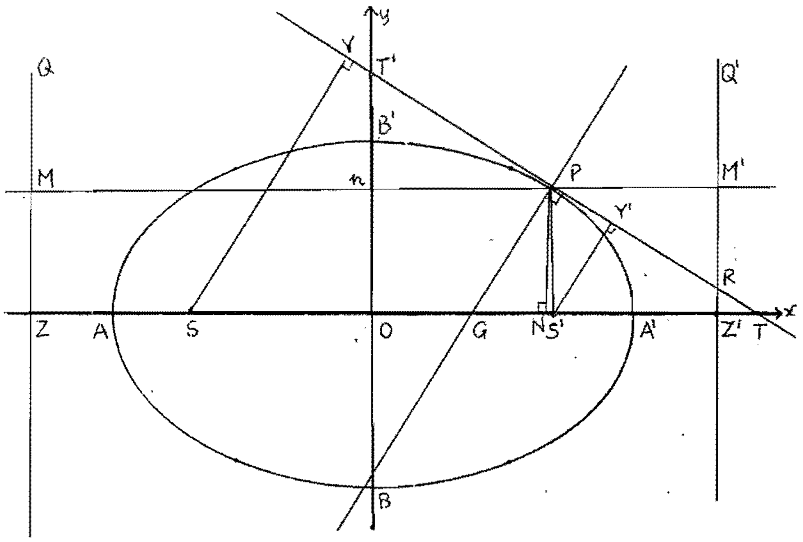
U N I T F O U R .



(Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. S, S^1 are the foci, ZQ and Z^1Q^1 the directrixes, PG, PT, PN are the normal, tangent and the ordinate at P.)

Q1. 1.

Tangent; $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$

$y = 0 \therefore x = \frac{a}{\cos \theta}$

$ON = a \cos \theta$

$OT = a/\cos \theta$

$ON \cdot OT = a \cos \theta \cdot \frac{a}{\cos \theta}$

$\therefore ON \cdot OT = a^2$

Q1. 2.

$x = 0 \therefore \frac{y \sin \theta}{b} = 1$

$On = b \sin \theta$

$OT^1 = \frac{b}{\sin \theta}$

$\therefore On \cdot OT^1 = b \sin \theta \cdot \frac{b}{\sin \theta}$

$= b^2$

Q1. 3.

Equ. of normal $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b$

$= a^2e^2$

$SG = SO + OG$

at G $y = 0 \therefore \frac{ax}{\cos \theta} = a^2 - b^2 \quad \therefore x = \frac{\cos \theta(a^2 - b^2)}{a}$

$\therefore SG = ae + ae^2 \cos \theta$

$= e(a + ae \cos \theta) \Longrightarrow ae(1 + e \cos \theta)$

(continued on next page)

Q1. 3. (cont'd)

$SP^2 = (-ae - a\cos\theta)^2 + (-b\sin\theta)^2$

$= a^2e^2 + 2a^2e\cos\theta + a^2\cos^2\theta + (b^2\sin^2\theta)$ $\qquad b^2 = a^2 - a^2e^2$

$= a^2e^2 + 2a^2e\cos\theta + a^2\cos\theta + (a^2 - a^2e^2)\sin^2\theta$

$= a^2(e^2 + 2e\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} - e^2\sin^2\theta)$

$= a^2(1 + 2e\cos\theta + e^2(1 - \sin^2\theta))$

$= a^2(1 + 2e\cos\theta + e^2\cos^2\theta)$

$= (a(1 + e\cos\theta)^2)$ $\quad \therefore SP = a(1 + e\cos\theta)$

$eSP = ae(1 + e\cos\theta) =$

$= SG$

Q1. 4. $S^1G = eS^1P$ (to prove)

$S^1G = OG - OS^1$

$= ae^2\cos\theta - ae$

$= ae(e\cos\theta - 1)$

$eS^1P^2 = e((a\cos\theta - ae)^2 + (b\sin\theta)^2)$ from 1.(3)

$= e(a(\cos\theta - 1))^2$

$eS^1P = ae(\cos\theta - 1)$

$= S^1G$

Q1. 5. $SG = e^2PM$ (to prove)

$SG = ae + ae^2\cos\theta$

$PM = \frac{a}{e} + a\cos\theta$

$e^2PM = ae + ae^2\cos\theta$

$= SG$

Q1. 6. Prove: $\angle GPS^1 = \angle PM^1S^1$

Gradient of $S^1M = \dfrac{b\sin\theta}{\frac{a}{e} - ae} = \dfrac{eb\sin\theta}{a(1-e^2)} = \dfrac{ae\sin\theta}{b} = \tan\angle PM^1S^1$ $\quad$ {Note: $1 - e^2 = \frac{b^2}{a^2}$

Gradient of $GP = \dfrac{b\sin\theta}{a\cos\theta - ae^2\cos\theta}$

$= \dfrac{a\sin\theta}{b\cos\theta}$

Gradient of $S^1P = \dfrac{b\sin\theta}{a\cos\theta - ae}$

$$\tan\angle GPS^1 = \frac{\dfrac{a\sin\theta}{b\cos\theta} + \dfrac{b\sin\theta}{ae - a\cos\theta}}{1 + \dfrac{ab\sin^2\theta}{-abe\cos\theta + ab\cos^2\theta}}$$

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Q1. 6. (cont'd)

$$= \frac{a^2 e \sin\theta + a^2 \sin\theta\cos\theta + b^2 \sin\theta\cos\theta}{-abe\cos\theta = ab\cos^2\theta + ab\sin^2\theta}$$

$$= \frac{a^2 e \sin\theta + \sin\theta\cos\theta\,(b^2 - a^2)}{-abe\cos\theta + ab(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{a^2 e \sin\theta - a^2 e^2 \sin\theta\cos\theta}{ab - abe\cos\theta}$$

$$= \frac{a^2 e \sin\theta(1 - e\cos\theta)}{ab(1 - e\cos\theta)}$$

$$= \frac{ae\sin\theta}{b}$$

$$= \tan \angle PM^1S^1$$

$$\therefore \angle GPS^1 = \angle PM^1S^1$$

Q1. 7. $\angle PS^1R = 90^\circ$ (to prove)

Gradient of $PS^1 = \dfrac{b\sin\theta}{a\cos\theta - ae} = m_1$

Gradient of $RS^1 = \dfrac{k - 0}{\frac{a}{e} - ae}$

$R(\frac{a}{e}, k)$

to find k;

$\dfrac{\cos\theta}{e} + \dfrac{y\sin\theta}{b} = 1$

$\therefore y = k = \dfrac{b(e-\cos\theta)}{e\sin\theta}$

$$= \frac{\frac{b(e-\cos\theta)}{c\sin\theta}}{\frac{a}{e} - ae}$$

$$= \frac{be(e - \cos\theta)}{ae\sin\theta(1 - e^2)}$$

$$= \frac{be(e-\cos\theta)}{ae\sin\theta\ (b)^2}(a^2)$$

$$= \frac{a(e-\cos\theta)}{b\sin\theta} = m_2$$

$$m_1 m_2 = \frac{b\sin\theta}{a(\cos - e)} \cdot \frac{a(e-\cos\theta)}{b\sin\theta}$$

$$= -1$$

$$\therefore \angle PS^1R = 90^\circ$$

Q1. 8. $SY \cdot SY^1 = b^2$ (to prove)

Tangent $YY^1 \equiv \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

$S(-ae,0)$, $S^1(ae,0)$

$$d_1 = \left|\frac{-aeb\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$d_2 = \left|\frac{aeb\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$d_1 d_2 = \left|\frac{a^2e^2b^2\cos^2\theta - a^2b^2}{b^2\cos^2\theta + a^2(1-\cos^2\theta)}\right|$$

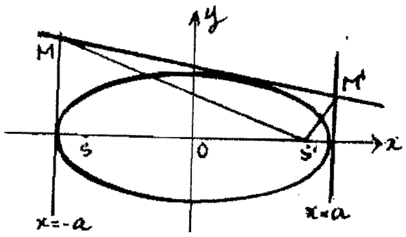
$$= \left| \frac{a^2b^2(e^2\cos^2\theta - 1)}{(b^2 - a^2)\cos^2\theta + a^2} \right|$$

$$= \left| \frac{a^2b^2(e^2\cos^2\theta - 1)}{-a^2e^2\cos^2\theta + a^2} \right|$$

$$= \left| \frac{a^2b^2(e^2\cos^2\theta - 1)}{a^2(1 - e^2\cos^2\theta - 1)} \right|$$

$$SY \cdot SY^1 = b^2$$

Q1. 9.



To find M^1 and M. $\underrightarrow{x = a}$ $\frac{x}{a}\cos\theta + \frac{y}{b}(\sin\theta) = 1$

$$y = \frac{b(1 - \cos\theta)}{\sin\theta}$$

$M^1\left(a, \frac{b(1 - \cos\theta)}{\sin\theta}\right.$ and $M\left(-a. \frac{b(1 + \cos\theta)}{\sin\theta}\right.$

$$m_{S^1M^1} = \frac{b(1 - \cos\theta)}{a\sin\theta(1-e)}$$

$$m_{S^1M} = \frac{-b(1 + \cos\theta)}{(1 + e)a\sin\theta}$$

$$m_{S^1M^1} \cdot m_{S^1M} = \frac{-b^2(1 - \cos^2\theta)}{a^2\sin^2\theta(1 - e^2)}$$

$$= -\frac{b^2}{a^2} \cdot \frac{1}{{}^2/{}_{b^2}}$$

$= -1$ (The result is identical when using S instead of S^1.)

Hence MM^1 subtends a right angle at either focus.

Q1. 10. $y = mx - c \cap \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{m^2x^2 + c^2 - 2mcx}{b^2} = 1$$

$$bx^2 + a^2m^2x^2 + a^2c^2 - 2a^2mc - a^2b^2 = 0$$

$$x^2(b^2 + a^2m^2) - 2a^2mcx + a^2(c^2 - b^2) = 0$$

If $y = mx - c$ is a tangent then

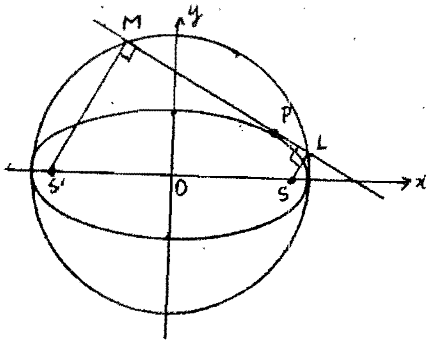
$$\Delta = b^2 - 4ac = 0 \quad \text{i.e.}$$

$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2b^2m^2 = 0$$

$$b^2c^2 = a^2b^2m^2 + b^4$$

$$\therefore c = \pm\sqrt{a^2m^2 + b^2}$$

Q1. 11.



Aim: to prove that the foot of the perpendicular from the foci lie on the auxiliary circle $x^2 + y^2 = a^2$.

(This question lends itself to a rather straightforward geometric solution, which is not part of the syllabus.) It is complicated to find the point of $\cap$ of $x^2 + y^2 = a^2$ with $\dfrac{x \cos \theta}{a} + \dfrac{y \sin \theta}{b} = 1$.

Instead; find the point of intersection of the tangent with the perpendiculars through S and S^1, then show that those points lie on the circle $x^2 + y^2 = a^2$.

Equation of tangent at $P(a \cos \theta, b \sin \theta)$;

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \qquad \text{——— (1)}$$

Gradient of a line $\perp$ to tangent $= \dfrac{a \sin \theta}{b \cos \theta}$

Equation of perpendicular through S;

$$y = \frac{a \sin \theta}{b \cos \theta}(x - ae)$$

$$y = \frac{ax \sin \theta - a^2e \sin \theta}{b \cos \theta} \qquad \text{——— (2)}$$

(2) $\longrightarrow$ (1) $\dfrac{x \cos \theta}{a} + \dfrac{(xa \sin \theta - a^2e \sin \theta)\sin \theta}{b^2 \cos \theta} = 1$

$$xb^2\cos^2\theta + xa^2\sin^2\theta - a^3e \sin^2\theta = ab^2\cos \theta$$

$$\therefore \quad x = \frac{ab^2\cos \theta + a^3e \sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

From (2) $\quad x = \dfrac{by \cos \theta + a^2}{a \sin \theta}$ ——— (3)

(3) $\rightarrow$ (1) $\dfrac{\cos \theta}{a} \cdot \dfrac{by \cos \theta + a^2e \sin \theta}{a \sin \theta} + \dfrac{y \sin \theta}{b} = 1$

$$b^2y \cos^2\theta + ya^2\sin^2\theta + a^2be \cos \theta \sin \theta - a^2b \sin \theta = 0$$

(cont'd)

$$y = \frac{a^2b\sin\theta - a^2be\cos\theta\sin\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

Now substitute $\dfrac{ab^2\cos\theta + a^3e\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2O}, \dfrac{a^2b\sin\theta - a^2be\cos\theta\sin\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$

into $x^2 + y^2 = a^2$

LHS $= x^2 + y^2$

$$\text{LHS} = \frac{a^2b^4\cos^2\theta + a^6e^2\sin^4\theta + 2a^4b^2e\cos\theta\sin^2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^2} +$$

$$\frac{a^4b^2\sin^2\theta + a^4b^2e^2\cos^2\theta\sin^2\theta - 2a^4b^2e\sin^2\theta\cos\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

substitute $a^2e^2 = a^2 - b^2$

$$= \frac{a^2\left[(a^4-a^2b^2)\sin^4\theta + b^4\cos^2\theta + (a^2b^2-b^4)\cos^2\theta\sin^2\theta + a^2b^2\sin^2\theta\right]}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

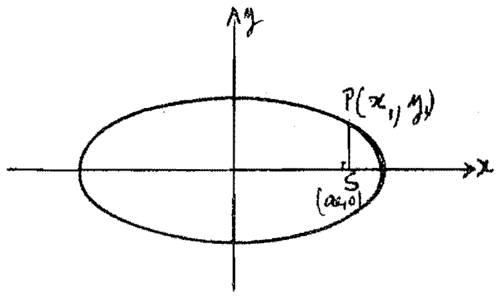
$$= \frac{a^2\left[a^4\sin^4\theta + (a^2b^2\sin^2\theta - a^2b^2\sin^4\theta) + a^2b^2\cos^2\theta\sin^2\theta + (b^4\cos^2\theta - b^4\cos^2\theta\sin^2\theta)\right]}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

$$= \frac{a^2(a^4\sin^4\theta + a^2b^2\cos^2\sin^2\theta + a^2b^2\cos^2\sin^2\theta + b^4\cos^4\theta}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

$= a^2$

$=$ RHS Hence M,L lie on $x^2 + y^2 = a^2$

Q1. 12.



(i) If PS is a semi latus rectum

$x_1 = ae$ and

$$\frac{a^2e^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\therefore y_1^2 = \left(\frac{a^2 - a^2e^2}{a^2}\right)b^2$$

$$= \frac{a^2(1-e^2)}{a^2}b^2$$

$$= \frac{b^4}{a^2}$$

$$y_1 = \frac{b^2}{a}$$

So the length of PS $= \dfrac{b^2}{a}$ units.

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Q1. 12. (cont'd)

OR (ii) If $P(a\cos\theta, b\sin\theta)$

$$PS^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$$

$$= a^2e^2 + a^2\cos^2\theta + b^2\sin^2\theta$$

Note $(b^2 = a^2 - a^2e^2)$

$$= a^2[e^2 + \cos^2\theta - 2e\cos\theta + (1-e^2)\sin^2\theta]$$

$$= a^2[e^2 + \cos^2\theta - 2e\cos\theta + (1-e^2)(1-\cos^2\theta)]$$

$$= a^2[e^2 + \cos^2\theta - 2e\cos\theta + 1 - \cos^2\theta - e^2 + e^2\cos^2\theta]$$

$$= a^2[1 - e\cos\theta]^2$$

but $ae = a\cos\theta$

$\therefore$ $e = \cos\theta$

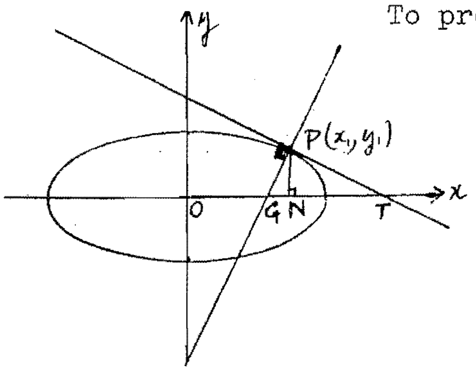
$$= a^2(1-e^2)^2$$

but $1-e^2 = \frac{b^2}{a^2}$

$$= a^2\frac{b^4}{a^4}$$

$$\therefore PS = \frac{b^2}{a}$$

Q1. 13.



To prove: $OT \cdot NG = b^2$

At T we have $\frac{xx_1}{a^2} = 1$

$$x = \frac{a^2}{x_1}$$

at G we have $\frac{xa^2}{x_1} = a^2 - b^2$

$$x = \frac{(a^2-b^2)}{a^2}x_1$$

$$\therefore OT = \frac{a^2}{x_1}$$

$$NG = ON - OG$$

$$= x_1 - \frac{x_1}{a^2}(a^2 - b^2)$$

$$= x_1 - \frac{x_1}{a^2}(a^2e^2)$$

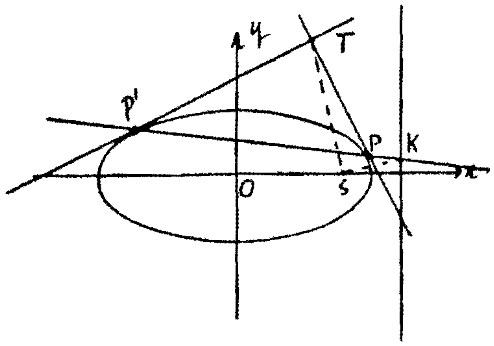
$$= x_1(1-e^2)$$

$$= x_1\frac{b^2}{a^2}$$

$$OT \cdot NG = \frac{a^2}{x_1} \cdot \frac{b^2x_1}{a^2}$$

$$= b^2$$

Q1. 14.



Prove: $\angle TSK = 90^\circ$

Note that PP^1 is a chord of contact with equation

$$\frac{xx_o}{a^2} + \frac{yy_o}{b^2} = 1 \qquad [\text{and } T(x_o, y_o)]$$

PP^1 intersects directrix $x = \frac{a}{e}$ when $\frac{x_o}{ae} + \frac{yy_o}{b^2} = 1$

$$\therefore\ y = \frac{b^2(ae - x_o)}{y_o ae}$$

Hence $K\left(\frac{a}{e}, \frac{b^2(ae - x_o)}{y_o ae}\right)$ (2)

Then the gradient of the line joining $T(x_o, y_o)$ $S(ae, 0)$ is

$$m_1 = \frac{y_o}{x_o - ae} \quad (1)$$

The gradient of the line joining $S(ae,0)$ $K\left(\frac{a}{e}, \frac{b^2(ae - x_o)}{y_o ae}\right)$ is

$$m_2 = \frac{\frac{b^2(ae - x_o)}{y_o ae} - 0}{\frac{a}{e} - ae} \quad (1)$$

$$m_2 = \frac{ae - x_o}{y_o} \quad (1)$$

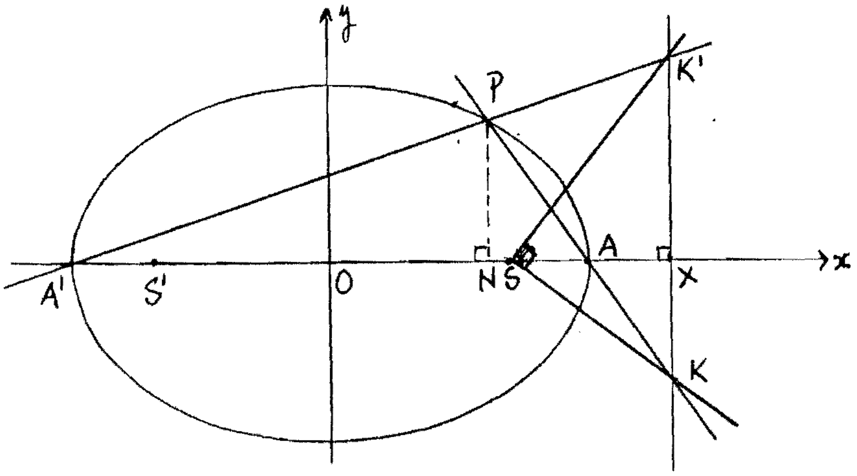
$$m_1 m_2 = \frac{y_o}{x_o - ae} \cdot \frac{ae - x_o}{y_o}$$

$$= -1 \quad (1)$$

Hence $\hat{TSK} = 90^\circ$

(Notice; as $P^1 \longrightarrow P$; $T \longrightarrow P$, $TS \longrightarrow PS$, so this property applies for a single tangent at P also. See example page 42.)

Q1. 15.



(i) (a) Let P be $(a\cos\theta, b\sin\theta)$

Gradient of $PA^1 = \frac{b\sin\theta}{a(\cos\theta + 1)}$

Equation of $PA^1 \Rightarrow y = \frac{b\sin\theta}{a(1+\cos\theta)}(x + a)$

At K^1 $x = \frac{a}{e}$, hence we have $y = \frac{b\sin\theta(1+e)}{e(1+\cos\theta)}$ $\therefore K^1\left(\frac{a}{e}, \frac{b\sin\theta(1+e)}{e(1+\cos\theta)}\right)$

Gradient of PA $= \frac{b\sin\theta}{a(\cos\theta-1)}$

Equation of PA; $y = \frac{b\sin\theta}{a(\cos\theta-1)}(x-a)$

At K, $x = \frac{a}{e}$ hence we have $y = \frac{(1-e)b\sin\theta}{e(\cos\theta-1)}$ $\therefore K\left(\frac{a}{e}, \frac{(1-e)b\sin\theta}{e(\cos\theta-1)}\right)$

m_1 = gradient of PK $= \frac{(1-e)b\sin\theta}{e(\cos\theta-1)} \Big/ \left(\frac{a}{e}-ae\right) = \frac{(1-e)b\sin\theta}{a(1-e^2)(\cos\theta-1)}$

m_2 = gradient of $P^1K = \frac{(1+e)b\sin\theta}{e(\cos\theta+1} \Big/ \left(\frac{a}{e}-ae\right) = \frac{(1+e)b\sin\theta}{a(1-e^2)(\cos\theta+1)}$

$$m_1m_2 = \frac{(1-e^2)b^2\sin^2\theta}{a^2(1-e^2)^2(\cos^2\theta-1)} = \frac{b^2}{a^2}\cdot\frac{\sin^2\theta}{(-\sin^2\theta)}\cdot\frac{1}{1-e^2} = -1$$

Hence $\angle KSK^1 = 90^\circ$

(b) if $P(x_1,y_1)$ instead of $(a\cos\theta, b\sin\theta)$

Gradient of $A^1P = \frac{y_1}{x_1+a}$

Equ. of A^1P $y = \frac{y_1}{x_1+a}(x+a)$

Cuts $x = \frac{a}{e}$ where $y = \frac{y_1a(1+e)}{e(x_1+a)}$

$\therefore K^1$ $\frac{a}{e}, \frac{ay_1(1+e)}{e(x_1+a)}$

m_1 = gradient of SK^1

$$= \frac{\frac{y_1a(1+e)}{e(x_1+a)}}{\frac{a}{e}-ae}$$

$$= \frac{y_1 a e(1+e)}{e(x_1+a)a(1-e^2)}$$

$$= \frac{y_1(1+e)}{(x_1+a)(1-e^2)}$$

Gradient of AP $= \frac{y_1}{x_1-a}$

Equ. of AP $y = \frac{y_1}{x_1-a}(x-a)$

Cuts $x = \frac{a}{e}$ when $y = \frac{ay_1(1-e)}{e(x_1-a)}$

$\therefore K\left(\frac{a}{e}, \frac{ay_1(1-e)}{e(x_1-a)}\right)$

m_2 = gradient of SK

$$= \frac{\frac{ay_1(1-e)}{e(x_1-a)}}{\frac{a}{e}-ae}$$

$$= \frac{y_1 a e(1-e)}{e(x_1-a)a(1-e^2)}$$

$$= \frac{y_1(1-e)}{(x_1-a)(1-e^2)}$$

$$m_1 \times m_2 = \frac{y_1(1+e)}{(x_1+a)(1-e^2)}\cdot\frac{y_1(1-e)}{(x_1-a)(1-e^2)}$$

$$= \frac{y_1^2(1-e^2)}{(x_1^2-a^2)(1-e^2)^2}$$

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Q1. 15. (i) (b) cont'd)

$$= \frac{y_1^2}{(x_1^2 - a^2)(1 - e^2)}$$

$$= \frac{b^2(a^2 - x_1^2)}{a^2(x_1^2 - a^2)} \cdot \frac{a^2}{b^2}$$

$$= -1 \quad \therefore \widehat{KSK} = 90^\circ$$

{Note $1 - e^2 = \frac{b^2}{a^2}$

$y_1^2 = \frac{b^2}{a^2}(a^2 - x_1^2)$}

(ii) $K^1X \cdot KX = XS^2$ (to prove)

(a) using similar triangles ($\triangle KXS \sim \triangle SXK^1$)

$$\frac{KX}{XS} = \frac{XS}{K^1X} \quad \therefore K^1X \cdot KX = XS^2$$

OR

(b) $K^1X = \left|\frac{b\sin\theta(1+e)}{e(1+\cos\theta)}\right|$, $KX = \left|\frac{(1-e)\,b\sin\theta}{e(\cos\theta - 1)}\right|$

$$SX = \left|\frac{a}{e} - ae\right| = \left|\frac{a}{e}(1 - e^2)\right| = \frac{a}{e} \cdot \frac{b^2}{a^2} = \frac{b^2}{ae}$$

$$K^1X \cdot KX = \left|\frac{b^2\sin^2\theta(1 - e^2)}{e^2(-\sin^2\theta)}\right|$$

$$= \frac{b^2(1 - e^2)}{e^2}$$

$$= \frac{b^4}{a^2e^2}$$

$$= SX^2$$

(iii) $PN : NA^1 = XK^1 : XA^1$ (to prove)

(a) Using similar triangles ($\triangle A^1PN \sim \triangle A^1K^1X$)

$$\frac{PN}{NA^1} = \frac{K^1X}{XA^1}$$

OR (b) $PN : NA^1 = XK^1 : XA^1 \iff PN \cdot XA^1 = XK^1 \cdot NA^1$

LHS $= PN \cdot XA^1$

$$= b\sin\theta\left(a + \frac{a}{e}\right)$$

$$= \frac{ab}{e}\sin\theta(e + 1)$$

RHS $= XK^1 \cdot NA^1$

$$= \frac{b\sin\theta(1+e)}{e(1+\cos\theta)} \cdot (a + a\cos\theta)$$

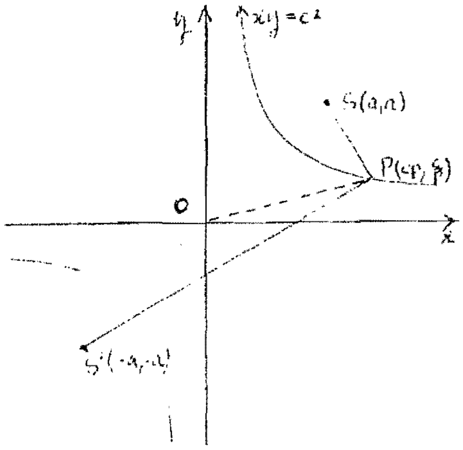
$$= \frac{ab\sin\theta(1+\cos\theta)(1+e)}{e(1+\cos\theta)}$$

$$= \frac{ab\sin\theta(1+e)}{e}$$

$$= \text{LHS}$$

$$\therefore PN : NA^1 = XK^1 : XA^1$$

Q2. 1.



Let P be $(cp, \frac{c}{p})$

$S(a,a)$ and $S^1(-a,-a)$ for $xy = c^2$

$$OP^2 = c^2p^2 + \frac{c^2}{p^2}$$

$$PS = \sqrt{(cp-a)^2 + (\frac{c}{p} - a)^2} = \sqrt{(cp-2c)^2 + (\frac{c}{p} - 2c)^2}$$

since $\frac{a}{2} = c$.

$$PS^1 = \sqrt{(cp + 2c)^2 + (\frac{c}{p} + 2c)^2}$$

$$PS^1 \cdot PS = \sqrt{c^2 \left[(p+2)^2 + (\tfrac{1}{p}+2)^2\right] \cdot c^2 \left[(p-2)^2 + (\tfrac{1}{p} - 2)^2\right]}$$

$$\therefore PS^1 \cdot PS = c^2\sqrt{(p^2+2\sqrt{2}p+1/p^2+2\sqrt{2}/p+4)(p^2-2\sqrt{2}p+1/p^2-2\sqrt{2}/p+4)}$$

$$= c^2\sqrt{1/p^2(p^4 2\sqrt{2}p^3+8p^2+2\sqrt{2}p+4)1/p^2(p^4-2\sqrt{2}p^3+8p^2-2\sqrt{2}p+4)}$$

$$= \frac{c^2}{p^2}\sqrt{p^8 + 2p^4 + 1} \quad \text{(on expansion)}$$

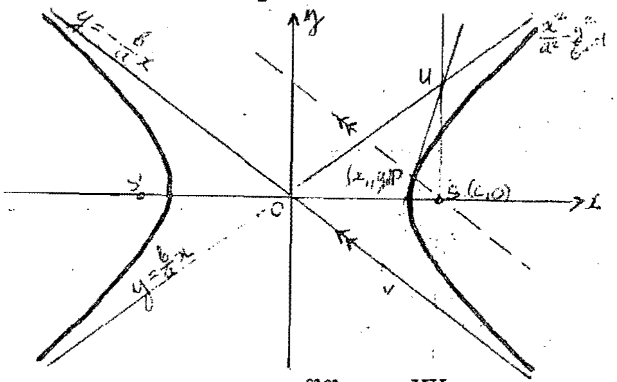
$$= \frac{c^2}{p^2}\sqrt{(p^4 + 1)^2}$$

$$= \frac{c^2}{p^2}(p^4 + 1)$$

$$= cp^2 + \frac{c^2}{p}$$

$= OP^2$ as required.

Q2. 2.



Let $(c,0)$ be the coordinates of S, (x_1, y_1) the coordinates of P and U is the point of concurrence of the tangent at P, the asymptote and of the perpendicular through S.

The tangent at P $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ intersects the asymptote

$y = \frac{b}{a}x$ at U_1 so we have

$$\frac{xx_1}{a^2} - \frac{bxy_1}{b^2a} = 1$$

$$x = \frac{a^2b}{bx_1 - ay_1} \qquad \left(\text{and } y = \frac{ab^2}{bx_1 - ay_1}\right)$$

Q2. 2. (cont'd)

but at U $x = c$ $\therefore \frac{a^2b}{bx_1 - ay_1} = c$ is the condition for concurrency.

Since $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ i.e., $bx_1^2 - ay_1^2 = a^2b^2$ the above equation

can be simplified to $\frac{a^2b}{bx_1 - ay_1} \cdot \frac{bx_1 + ay_1}{bx_1 + ay_1} = \frac{a^2b(bx_1 + ay_1)}{b^2x_1^2 - a^2y_1^2} = \frac{bx_1 + ay_1}{b}$

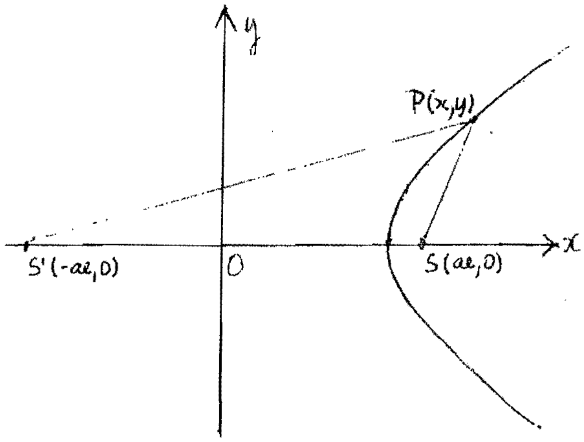
Hence we have $\frac{bx_1 + ay_1}{b} = c$ ——— (1)

The gradient of SP is $\frac{y_1}{x_1 - c} = m$ ——— (2)

(1) $\longrightarrow$ (2) $m = \frac{y_1}{x_1 - \frac{(bx_1 + ay_1)}{b}} = \frac{by_1}{bx_1 - bx_1 - ay_1}$ $\therefore m = -\frac{b}{a}$ which

is the same as the gradient of the other asymptote $\therefore$ PS// $y = -\frac{b}{a}x$

Q2. 3.



Prove that as $P(x,y)$ moves so that $S^1P - SP$ = constant, then the equation of the locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let S and S^1 be the points (ae,0) and (-ae,0) respectively, and the constant 2a.

Hence $S^1P - SP = 2a$

i.e., $\sqrt{(x + ae)^2 + y^2} - \sqrt{(x - ae)^2 + y^2} = 2a$

$$\sqrt{(x + ae)^2 + y^2} = 2a + \sqrt{(x - ae)^2 + y^2}$$

$$x^2 + 2aex + a^2e^2 + y^2 = 4a^2 + (x - ae)^2 + y^2 + 4a\sqrt{(x - ae)^2 + y^2}$$

$$x^2 + 2aex + a^2e^2 + y^2 = 4a^2 + x^2 - 2aex + a^2e^2 + y^2 + 4a\sqrt{(x - ae)^2 + y^2}$$

$$4aex - 4a^2 = 4a\sqrt{(x - ae)^2 + y^2}$$

$$(ex - a)^2 = (x - ae)^2 + y^2$$

$$e^2x^2 - 2aex + a^2 = x^2 - 2aex + a^2e^2 + y^2$$

$$a^2(1 - e^2) = x^2(1 - e^2) + y^2 \qquad \div \text{ by } a^2(1 - e^2)$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)}$$

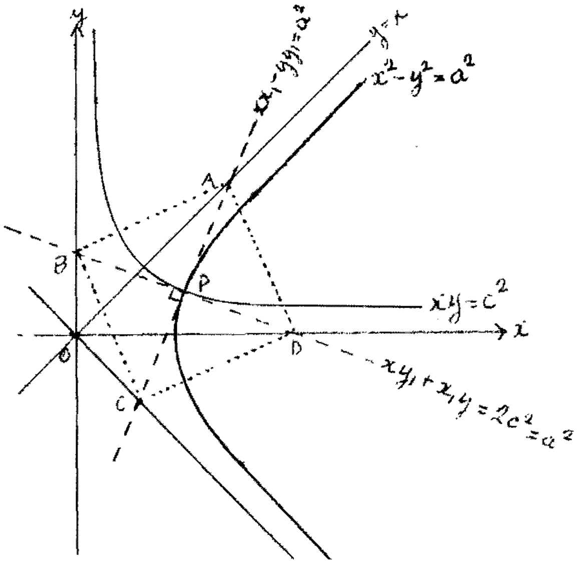
(continued on next page)

Q2. 3. (cont'd)

Let $b^2 = (1 - e^2)a^2$ but for the hyperbola $e > 1$ $\therefore 1 - e^2$ is negative and since b^2 must be positive, we must let $b^2 = -a^2(1 - e^2)$ or $b^2 = a^2(e^2 - 1)$, so the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{as required.}$$

Q2. 4.



For the hyperbola $x^2 - y^2 = a^2$ the equation of tangent at $P(x_1, y_1)$ is $xx_1 - yy_1 = a^2$, and the asymptotes are $y = \pm x$.

The tangent at $P(x_1, y_1)$ meets the asymptotes when

$$xx_1 \pm xy_1 = a^2$$

$$\therefore x = \frac{a^2}{x_1 - y_1} \text{ or } x = \frac{a^2}{x_1 + y_1}$$

$$\text{then } y = \frac{a^2}{x_1 - y_1} \text{ or } y = \frac{a^2}{x_1 + y_1}$$

Hence A is $\left(\frac{a^2}{x_1 - y_1}, \frac{a^2}{x_1 - y_1}\right)$ and B is $\left(\frac{a^2}{x_1 + y_1}, \frac{-a^2}{x_1 + y_1}\right)$

$$AB^2 = \left(\frac{a^2}{x_1 - y_1} - \frac{a^2}{x_1 + y_1}\right)^2 + \left(\frac{a^2}{x_1 - y_1} + \frac{a^2}{x_1 + y_1}\right)^2$$

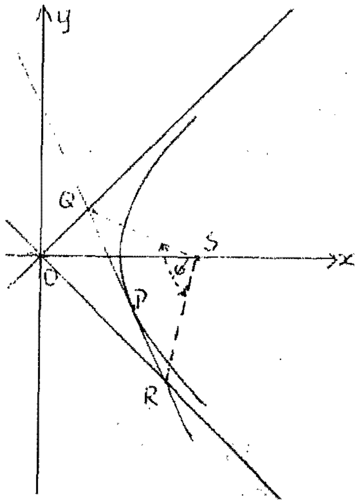
$$= \frac{4a^4(x_1^2 + y_1^2)}{(x_1^2 - y_1^2)} \quad \text{but } x_1^2 - y_1^2 = a^2 \quad \therefore AB^2 = 4(x_1^2 + y_1^2)$$

For the hyperbola $xy = c^2$ the equation of tangent at $P(x_1, y_1)$ is $xy_1 + yx_1 = 2c^2$ and it meets the asymptote $x = 0$ at $B(\frac{2c^2}{y_1}, 0)$ and the asymptote $y = 0$ at $D(0, \frac{2c^2}{x_1})$.

But $2c^2 = 2x_1y_1$ $\therefore$ $B(2x_1, 0)$ and $D(0, 2y_1)$ and $BD^2 = 4(x_1^2 + y_1^2)$.

Hence the diagonals of quadrilateral ABCD are equal. Gradient of AC is $\frac{x^1}{y_1}$ and of BD is $-\frac{y_1}{x_1}$ $\therefore$ AC $\perp$ BD and ABCD is a square.

Q2. 5.



Let $(c,0)$ and $(a\sec\theta,\ b\tan\theta)$ be the coordinates of S and P respectively.

The equation of the tangent at P is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ ——(1)
and the equations of the asymptotes are $y = \pm\frac{b}{a}x$ ——(2)
Then solving (1) and (2) we obtain $\frac{x\sec\theta}{a} \pm \frac{x\tan\theta}{a} = 1$

$$\therefore\ x = \frac{a}{\sec\theta - \tan\theta} \qquad \text{or } x = \frac{a}{\sec\theta + \tan\theta}$$

$$x = \frac{a\cos\theta}{1-\sin\theta} \qquad \text{or } x = \frac{a\cos\theta}{1+\sin\theta}$$

Since $y = \pm\frac{bx}{a}$

$$y = \frac{b\cos\theta}{1-\sin\theta} \qquad \text{or } y = \frac{-b\cos\theta}{1+\sin\theta}$$

Thus the coordinates of the points of intersection are;

$Q\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ $R\left(\frac{a\cos\theta}{1+\sin\theta}, \frac{b\cos\theta}{1+\sin\theta}\right)$

The gradient of QS is $\dfrac{\frac{b\cos\theta}{1-\sin\theta}}{\frac{a\cos\theta}{1-\sin\theta} - c} = \dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta} = m_1$

The gradient of RS is $\dfrac{-b\cos\theta}{a\cos\theta - c - c\sin\theta} = m_2$

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta} + \dfrac{b\cos\theta}{a\cos\theta - c - c\sin\theta}}{1 - \dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta}\cdot\dfrac{b\cos\theta}{a\cos\theta - c - c\sin\theta}}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{(a\cos\theta - c + c\sin\theta)(a\cos\theta - c - c\sin\theta) - b^2\cos^2\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{a^2\cos\theta - 2ac\cos\theta + c^2 - c^2\sin^2\theta - b^2\cos^2\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{a^2\cos^2\theta - b^2\cos^2\theta + c^2\cos^2\theta - 2ac\cos\theta + c^2 - c^2}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{(a^2 + c^2 - b^2)\cos^2\theta - 2ac\cos\theta}$$

(since $ae = c$
$a^2 + c^2 - b^2 = 2a^2$)

(cont'd)

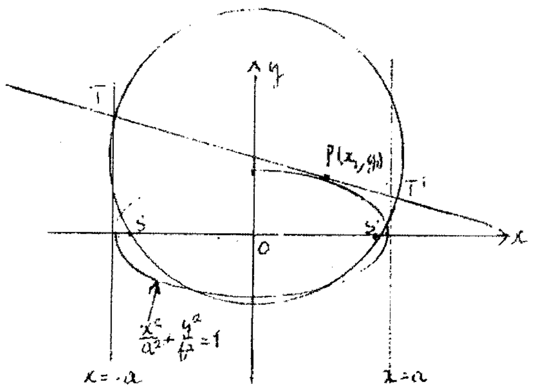
Q2. 5. (cont'd)

$$= \frac{2b\cos\theta(a\cos\theta - c)}{2a^2\cos^2\theta - 2ac\cos\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{2a\cos\theta(a\cos\theta - c)}$$

$\therefore \tan\phi = \frac{b}{a}$. Hence QR subtends a constant angle; $\tan^{-1}(\frac{b}{a})$ at S.

Q3. 1.



Let the coordinates of S and P be $(ae, 0)$ and (x_1, y_1) respectively.

The tangent at P $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

cuts $x = \pm a$ at $y = \frac{b^2}{y_1}(1 \pm \frac{x_1}{a})$

$\therefore$ $T\left[-a, \frac{b^2}{y_1}(1 + \frac{x_1}{a})\right]$ and

$T^1\left[a, \frac{b^2}{y_1}(1 - \frac{x_1}{a})\right]$

The equation of a circle on a given diameter $A(x_3, y_3)$, $B(x_4, y_4)$ is $(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$ (see page 21).

The equation of the circle with diameter TT^1 is

$$(x-a)(x+a) + \left[y - \frac{b^2}{y_1}(1 - \frac{x_1}{a})\right]\left[y - \frac{b^2}{y_1}(1 + \frac{x_1}{a})\right] = 0$$

The circle intersects the x axis when $y = 0$ i.e. when

$$x^2 - a^2 + \left[\frac{b^2}{y_1}(1 - \frac{x_1}{a})\right]\left[\frac{b^2}{y_1}(1 + \frac{x_1}{a})\right] = 0$$

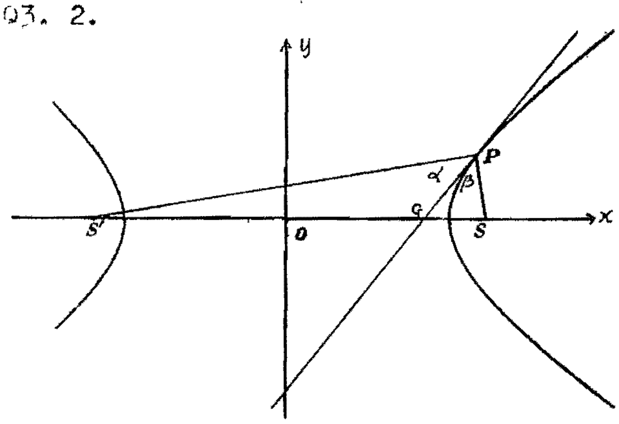
$$x^2 - a^2 + \frac{b^4}{y_1^2}(1 - \frac{x_1^2}{a^2}) = 0 \qquad \left(\text{but } \frac{y_1^2}{b^2} = 1 - \frac{x_1^2}{a^2}\right)$$

$$x^2 - a^2 + \frac{b^4}{y_1^2} \cdot \frac{y_1^2}{b^2} = 0 \qquad (b^2 = a^2(1 - e^2) = a^2 - a^2e^2)$$

$$x^2 = a^2 - b^2$$

$\therefore$ $x^2 = a^2e^2$ so $x = \pm ae$. i.e. the circle passes through S and S^1.

Q3. 2.



Let P be the point (x_1, y_1) S (ae,0) and S^1(-ae,0).

The tangent at P

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$ meets the transverse axis at G.

$$\therefore \frac{xx_1}{a^2} = 1 \quad \text{i.e. } x = \frac{a^2}{x_1}$$

hence $G(\frac{a^2}{x_1}, 0)$

$$GS = ae - \frac{a^2}{x_1} \text{ and } GS^1 = ae + \frac{a^2}{x_1}$$

$$PS^2 = (x_1 - ae)^2 + y_1^2 = x_1^2 - 2aex_1 + a^2e^2 + y_1^2$$

$$= x_1^2 - 2aex_1 + a^2e^2 + \frac{b^2}{a^2}(x_1^2 - a^2) \text{ --------(NOTE: } y_1^2 = \frac{b^2}{2}(x_1^2 - a^2)$$

and $b^2 = a\ (e^2 - 1)$)

$$= x_1^2 - 2aex_1 + a^2e^2 + (e^2 - 1)(x_1^2 - a^2)$$

$$= x_1^2e^2 - 2aex_1 + a^2$$

$$PS = \sqrt{(x_1e - a)^2} = x_1e - a \qquad \text{similarly } PS^1 = x_1e + a$$

$$\frac{PS}{SG} = \frac{x_1e - a}{ae - \frac{a^2}{x_1}} = \frac{x_1^2e - x_1a}{aex_1 - a^2} = \frac{x_1(x_1e - a)}{a(x_1e - a)} = \frac{x_1}{a}$$

$$\frac{PS^1}{GS^1} = \frac{x_1e + a}{ae + \frac{a^2}{x_1}} = \frac{x_1^2e + x_1a}{aex_1 + a^2} = \frac{x_1(x_1e + a)}{a(x_1e + a)} = \frac{x_1}{a}$$

$$\therefore \frac{PS}{SG} = \frac{PS^1}{GS^1}$$

Assume that PG makes $S^1\hat{P}G = \alpha$ and $S\hat{P}G = \beta$ i.e. divides $S\hat{P}S$ unequally.

Using the sine rule in △'s S^1PG and SPG we have

$$\frac{GS^1}{\sin\alpha} = \frac{S^1P}{\sin S^1\hat{G}P} \quad \text{and} \quad \frac{SG}{\sin\beta} \quad \frac{SP}{\sin S\hat{G}P}$$

$$\text{i.e., } \frac{GS^1}{S^1P} = \frac{\sin\alpha}{\sin(180^\circ - S\hat{G}P)} \text{ and } \frac{SG}{SP} = \frac{\sin\beta}{\sin S\hat{G}P}$$

Q3. 2. (cont.)

but $\frac{GS^1}{S^1P} = \frac{SG}{SP}$ $\therefore$ $\frac{\sin\alpha}{\sin(180^\circ - S\hat{G}P)} = \frac{\sin\beta}{\sin S\hat{G}P}$

Since $\sin(180^\circ - S\hat{G}P) = \sin S\hat{G}P$

$\therefore \sin\alpha = \sin\beta$

and $\alpha = \beta$

i.e. the tangent PG bisects the angle S^1PS.

OR

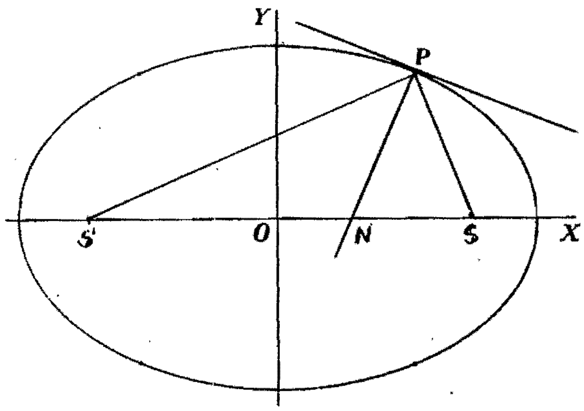
<u>Theorem</u>: The bisector of one angle of a triangle divides the opposite side in the ratio of the sides about that angle.

Hence it is sufficient to prove that

$$\frac{GS^1}{S^1P} = \frac{SG}{SP}$$

Conclusion: PG bisects the angle S^1PS.

Q3. 3.



PN is the normal at P.

The normal $\frac{xa}{\cos\theta} - \frac{yb}{\sin\theta} = a^2 - b^2 = a^2e^2$ meets the x axis at N.

$\therefore$ we have $x = \frac{a^2e^2\cos\theta}{a}$

So $N(e^2 a\cos\theta, 0)$. If S and S^1 are the points $(ae, 0)$, $(-ae, 0)$ respectively, P is $(a\cos\theta, b\sin\theta)$ as given.

Then the gradient of PN $= \frac{b\sin\theta}{a\cos\theta - ae^2\cos\theta} = \frac{b\sin\theta}{a\cos\theta(1-e^2)} = \frac{a^2 b\sin\theta}{a\cos\theta\cdot b^2} = \frac{a\sin\theta}{b\cos\theta}$

The gradient of PS $= \frac{b\sin\theta}{a\cos\theta - ae}$ and

the gradient of PS^1 $= \frac{b\sin\theta}{a\cos\theta + ae}$

(continued on next page)

Q3. 3.(cont'd)

$$\tan N\hat{P}S^1 = \left| \frac{\dfrac{b\sin\theta}{a\cos\theta + ae} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta + ae}\cdot\dfrac{a\sin\theta}{b\cos\theta}} \right| = \left| \frac{b^2\sin\theta\cos\theta - a^2\sin\theta\cos\theta - a^2e\sin\theta}{ab\cos^2\theta + ab\sin^2\theta + abe\cos\theta} \right|$$

$$= \left| \frac{(b^2-a^2)\sin\theta\cos\theta - a^2e\sin\theta}{ab(\cos^2\theta + \sin^2\theta) + abe\cos\theta} \right| = \left| \frac{-a^2e^2\sin\theta\cos\theta - a^2e\sin\theta}{ab(1 + e\cos\theta)} \right|$$

$$= \left| \frac{-a^2e\sin\theta(e\cos\theta + 1)}{ab(1 + e\cos\theta)} \right|$$

$\tan N\hat{P}S^1 = \dfrac{ae\sin\theta}{b}$. Similarly it can be shown that

$$\tan N\hat{P}S = \frac{\dfrac{b\sin\theta}{a\cos\theta - ae} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta - ae}\cdot\dfrac{a\sin\theta}{b\cos\theta}} = \frac{a^2e\sin\theta(1 - e\cos\theta)}{ab(1 - e\cos\theta)} = \frac{ae\sin\theta}{b}$$

$$= \tan N\hat{P}S^1$$

i.e. the normal at P bisects the angle SPS.