# *Using Matrices to Solve Simultaneous Equations*

## 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

## Determinant of A

$$\det A = |A| = a_{11}a_{22} - a_{21}a_{12}$$

## Inverse of A

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

## Solving simultaneous equations

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} m \\ n \end{pmatrix}$$

e.g. $(i)$ $2x+3y=21\ldots(1)$

$5x+2y=3 \quad \ldots(2)$

$|A| = 2\times 2 - 5\times 3$

$$\begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-11}\begin{pmatrix} 2 & -3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 21 \\ 3 \end{pmatrix}$$

*swap*

*change signs*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-11}\begin{pmatrix} 2\times 21 - 3\times 3 \\ -5\times 21 + 2\times 3 \end{pmatrix}$$

*Multiply the row of the matrix with the column of the vector*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-11}\begin{pmatrix} 33 \\ -99 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

$\therefore x = -3, y = 9$

$(ii)\ 171x - 213y = 642 \ldots (1)$

$114x - 326y = 244 \quad \ldots (2)$

$$\begin{pmatrix} 171 & -213 \\ 114 & -326 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 642 \\ 244 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-31464} \begin{pmatrix} -326 & 213 \\ -114 & 171 \end{pmatrix} \begin{pmatrix} 642 \\ 244 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-31464} \begin{pmatrix} -157320 \\ -31464 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\therefore x = 5, y = 1$$

$n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$|A| = \sum_{k=1}^{n} (-1)^{k-1} a_{1k} |M_{1k}|$$

where $M_{1k}$ is the matrix with column 1 and row $k$ removed

e.g. $(i)$ $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix}$

$= 3(6) - 2(2) + (1)$

$= 15$

$$(ii)\ \begin{vmatrix}1 & 3 & 5 & 2\\ 0 & -1 & 3 & 4\\ 2 & 1 & 9 & 6\\ 3 & 2 & 4 & 8\end{vmatrix} = \begin{vmatrix}-1 & 3 & 4\\ 1 & 9 & 6\\ 2 & 4 & 8\end{vmatrix} - 3\begin{vmatrix}0 & 3 & 4\\ 2 & 9 & 6\\ 3 & 4 & 8\end{vmatrix} + 5\begin{vmatrix}0 & -1 & 4\\ 2 & 1 & 6\\ 3 & 2 & 8\end{vmatrix} - 2\begin{vmatrix}0 & -1 & 3\\ 2 & 1 & 9\\ 3 & 2 & 4\end{vmatrix}$$

$$= \left(-\begin{vmatrix}9 & 6\\ 4 & 8\end{vmatrix} - 3\begin{vmatrix}1 & 6\\ 2 & 8\end{vmatrix} + 4\begin{vmatrix}1 & 9\\ 2 & 4\end{vmatrix}\right) - 3\left(0\begin{vmatrix}9 & 6\\ 4 & 8\end{vmatrix} - 3\begin{vmatrix}2 & 6\\ 3 & 8\end{vmatrix} + 4\begin{vmatrix}2 & 9\\ 3 & 4\end{vmatrix}\right)$$

$$+5\left(0\begin{vmatrix}1 & 6\\ 2 & 8\end{vmatrix} + \begin{vmatrix}2 & 6\\ 3 & 8\end{vmatrix} + 4\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\right) - 2\left(0\begin{vmatrix}1 & 9\\ 2 & 4\end{vmatrix} + \begin{vmatrix}2 & 9\\ 3 & 4\end{vmatrix} + 3\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\right)$$

$$= \left(-4 - 3(-4) + 4(-11)\right) - 3\left(0 - 3(-2) + 4(-19)\right) + 5\left(0 + (-2) + 4(1)\right)$$

$$-2\left(0 + (-19) + 3(1)\right)$$

$$= \underline{160}$$

*or* convert to a triangular matrix (*create a triangle of 0's)*

e.g. (*i*)

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 0 & \frac{10}{3} & \frac{5}{3} \\ 0 & 1 & 2 \end{vmatrix} \quad \leftarrow 3\times row2 - row1$$

$$= \begin{vmatrix} 3 & 2 & 1 \\ 0 & \frac{10}{3} & \frac{5}{3} \\ 0 & 0 & \frac{3}{2} \end{vmatrix} \quad \leftarrow \frac{10}{3}\times row3 - row2$$

$$= 3\times\frac{10}{3}\times\frac{3}{2} \quad \leftarrow \text{multiply the diagonal}$$

$$\underline{=15}$$

$$(ii)\ \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 2 & 1 & 9 & 6 \\ 3 & 2 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & -5 & -1 & 2 \\ 0 & -7 & -11 & 2 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow row3-2\times row1 \\ \leftarrow row4-3\times row1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -16 & -18 \\ 0 & 0 & -32 & -26 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow row3-5\times row2 \\ \leftarrow row4-7\times row2 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -16 & -18 \\ 0 & 0 & 0 & 10 \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow row4-2\times row3 \end{matrix}$$

$$= \underline{160}$$

## Adjoint matrix

$$\operatorname{adj}A=\begin{pmatrix}|M_{11}| & \cdots & (-1)^{n-1}|M_{1n}| \\ \vdots & \ddots & \vdots \\ (-1)^{n(n-1)}|M_{n1}| & \cdots & (-1)^{nn-1}|M_{nn}|\end{pmatrix}^t$$

i.e signs alternate $\begin{pmatrix}+ & - & + \\ - & + & - \\ + & - & +\end{pmatrix}$

$t$ means transpose rows and columns

$$A^{-1}=\frac{\operatorname{adj}A}{|A|}$$

e.g

$$x+2y-z=-5 \quad \ldots(1)$$

$$2x-3y+4z=28 \quad \ldots(2)$$

$$4x+5y-3z=-10 \ldots(3)$$

$$A=\begin{pmatrix}1 & 2 & -1\\ 2 & -3 & 4\\ 4 & 5 & -3\end{pmatrix}$$

$$|A|=\begin{vmatrix}1 & 2 & -1\\ 2 & -3 & 4\\ 4 & 5 & -3\end{vmatrix}=\begin{vmatrix}1 & 2 & -1\\ 0 & -7 & 6\\ 0 & -3 & 1\end{vmatrix}=\begin{vmatrix}1 & 2 & -1\\ 0 & -7 & 6\\ 0 & 0 & -\frac{11}{7}\end{vmatrix}=11$$

$$\operatorname{adj}A=\begin{pmatrix}\begin{vmatrix}-3 & 4\\ 5 & -3\end{vmatrix} & -\begin{vmatrix}2 & 4\\ 4 & -3\end{vmatrix} & \begin{vmatrix}2 & -3\\ 4 & 5\end{vmatrix}\\ -\begin{vmatrix}2 & -1\\ 5 & -3\end{vmatrix} & \begin{vmatrix}1 & -1\\ 4 & -3\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}\\ \begin{vmatrix}2 & -1\\ -3 & 4\end{vmatrix} & -\begin{vmatrix}1 & -1\\ 2 & 4\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & -3\end{vmatrix}\end{pmatrix}^t=\begin{pmatrix}-11 & 1 & 5\\ 22 & 1 & -6\\ 22 & 3 & -7\end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 4 & 5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 28 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -11 & 1 & 5 \\ 22 & 1 & -6 \\ 22 & 3 & -7 \end{pmatrix}\begin{pmatrix} -5 \\ 28 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

$$\therefore x = 3, y = -2, z = 4$$

**Exercise 1H; 1, 2, 6**