Question 1 (12 Marks) **Marks**

(a) Find the value of $\int_0^{\pi} \tan\left(\frac{x}{4}\right) dx$, expressing your answer in the form $a \ln b$ where a and b are rational numbers. **3**

(b) A 240 metre tall tower stands on a large flat plain. From a point on the plain East of the tower James measures the angle of elevation of the top of the tower as $30°$. Bruce, who is South of the tower, measures the angle of elevation of the top of the tower as $45°$.

(i) Draw a neat sketch showing the above information. **1**

(ii) Show that James is $240\sqrt{3}$ metres from the base of the tower and also find the distance of Bruce from the base of the tower. **2**

(iii) Find the distance between James and Bruce. **2**

(c) Use the substitution $u^2 = x$ $(u > 0)$ to find the exact value of $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{x - x^2}}$. **4**

Question 2 START A NEW PAGE (12 Marks) **Marks**

(a) (i) Prove that the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ is given by $px - y - ap^2 = 0$. **2**

(ii) The tangent at P meets the directrix at the point T. Find the co-ordinates of T. **1**

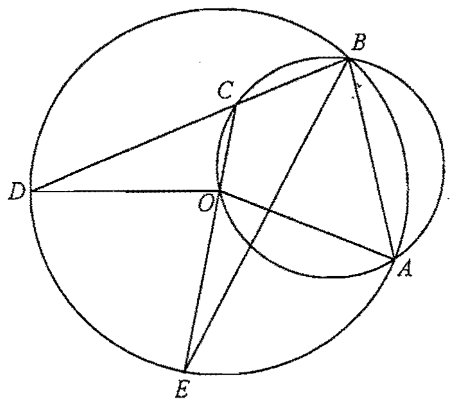
(iii) If F is the focus of the parabola prove that PF is perpendicular to FT. **3**

(b) (i) Sketch the curve $y = 1 + \sin x$ for $0 \le x \le 2\pi$. **2**

(ii) Find the exact volume of the solid formed when the area bounded by the curve $y = 1 + \sin x$ and the x-axis for $0 \le x \le 2\pi$ is rotated one revolution about the x-axis. **4**

Question 3 START A NEW PAGE (12 Marks) **Marks**

(a) A, B and D are three points on a circle with centre O. A smaller circle is drawn through the points O, A and B. The chord BD of the larger circle cuts the smaller circle at C and chord CO extended cuts the larger circle at E.



(i) Copy the diagram onto your examination paper and explain why $\angle CBA = \angle EOA$. **1**

(ii) Prove that BE bisects $\angle DBA$. **3**

(b) (i) The curve $y = x^4$ is rotated one revolution about the y-axis to form a container for storing water. Calculate the volume of water that can be stored if the container is filled to a depth of h cm. **2**

(ii) Water is poured into the above container at a rate of 60 ml/minute. Find the rate at which the depth is increasing when the depth is 16 cm. **2**

(c) The equation of motion of a particle moving along a horizontal straight line is given by the formula $x = 3\cos\left(\frac{1}{4}t\right) + \sin\left(\frac{1}{4}t\right)$, where x metres is the displacement of the particle at time t seconds.

(i) Explain whether the particle is initially moving to the right or left, and whether it is speeding up or slowing down. **2**

(ii) Find the time for the particle to first reach the origin. Give your answer correct to one decimal place. **2**

Question 4 **START A NEW PAGE** (12 Marks) **Marks**

(a) (i) Prove that $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ for $-1 \le x \le 1$. **2**

(ii) Find the acute angle between the curves $y = \sin^{-1} x$ and $y = \cos^{-1} x$ at the point where they intersect. Give your answer correct to the nearest degree. **3**

(b) Find the smallest positive solution, in radians, of the equation $\cos 3\theta = \sin 2\theta$. **3**

(c) (i) Write down the coefficient of x^k when the binomial product $(5+3x)^{20}$ is expanded in ascending powers of x. **1**

(ii) Which two adjacent terms in the above expansion have their coefficients in the ratio 2:3? **3**

Question 5 **START A NEW PAGE** (12 Marks) **Marks**

(a) (i) If $\theta = \tan^{-1} A + \tan^{-1} B$ show that $\tan\theta = \dfrac{A+B}{1-AB}$. **1**

(ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$. **3**

(b) Use Mathematical Induction to prove that for all positive integers $n \ge 1$, **4**

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}$$

(c) At training, a coach decides to organise a practise game between two teams using 5 players for each team. The coach has 12 players to choose from, including the Ruse twins James and Bruce.

(i) How many different practice games could be organised if there are no restrictions on who plays on each team? **2**

(ii) Find the probability that in a game chosen at random, the Ruse twins would not be playing against each other. **2**

Question 6 **START A NEW PAGE** (12 Marks) **Marks**

(a) $A_n = 1^2 + 5^2 + 9^2 + \ldots + (4n-3)^2$ and $B_n = 3^2 + 7^2 + 11^2 + \ldots$.

(i) Write down the n^{th} term of the sequence B_n. **1**

(ii) If $S_{2n} = A_n - B_n$, show that $S_{2n} = -8n^2$. **3**

(iii) Hence evaluate $101^2 - 103^2 + 105^2 - 107^2 + \ldots + 2009^2 - 2011^2$ **2**

(b) The number (N) of ants in an ant colony at time t weeks is given by the formula $N = 150\,000 - Be^{-kt}$, where B and k are positive constants. The initial size of the colony when discovered was 2 000 and 5 weeks later the size had increased to 50 000.

(i) Show that the instantaneous rate of increase in the size of the colony can be given by the equation $\dfrac{dN}{dt} = k(150\,000 - N)$. **1**

(ii) Find the exact values of B and k. **2**

(iii) Find the maximum size of the colony. **1**

(iv) Find the size of the colony 20 weeks after its discovery. Give your answer correct to the nearest 1000 ants. **2**

Question 7 **START A NEW PAGE** (12 Marks) **Marks**

(a) (i) Write down an expression for the expansion of $\cos(A+B)$ and hence prove that $\cos 2\theta = 2\cos^2\theta - 1$. **2**

(ii) ABC is a triangle with sides a, b, c and a perimeter of length p. **4**

Prove that $\cos\left(\dfrac{A}{2}\right) = \dfrac{1}{2}\sqrt{\dfrac{p(p-2a)}{bc}}$.

(b) An object is projected from the origin O with initial speed $U\ m/s$ at an angle of elevation of α. At the same instant another object is projected from a point A which is h units above the origin O. The second object is projected with initial speed $V\ m/s$ at an angle of elevation of β, where $\beta < \alpha$. Both objects move freely under gravity in the same plane.

(i) Given that the equations of motion for the object projected from the origin are: **2**

$$x = Ut\cos\alpha \quad \text{and} \quad y = Ut\sin\alpha - \frac{1}{2}gt^2,$$

write down the equations of motion for the object projected from the point A.

(ii) If the objects collide T seconds after they are projected, prove that $T = \dfrac{h\cos\beta}{U\sin(\alpha-\beta)}$. **4**

THIS IS THE END OF THE EXAMINATION PAPER

JRAHS M. EXT I SOLUTIONS 2010

3U TRIAL **MATHEMATICS Extension 1 : Question** 1 - 2010

Suggested Solutions	Marks	Marker's Comments

Solution:

(a)
$$\int_0^{\pi}\tan\left(\frac{x}{4}\right)dx=\int_0^{\pi}\frac{\sin\left(\frac{x}{4}\right)}{\cos\left(\frac{x}{4}\right)}dx$$

$$=-4\int_0^{\pi}\left\{\frac{-\frac{1}{4}\sin\left(\frac{x}{4}\right)}{\cos\left(\frac{x}{4}\right)}\right\}dx$$ 1 — 50% success rate

$$=-4\left[\ln\left(\cos\left(\frac{x}{4}\right)\right)\right]_0^{\pi}$$

$$=-4\left\{\ln\left(\cos\left(\frac{\pi}{4}\right)\right)-\ln(\cos(0))\right\}$$ 1

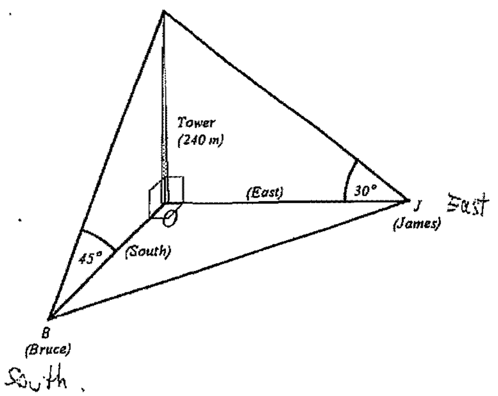
$$=-4\left\{\ln\left(\frac{1}{\sqrt{2}}\right)-\ln(1)\right\}$$

$$=-4\ln\left(\frac{1}{\sqrt{2}}\right)$$

$$=-4\ln\left(2^{-\frac{1}{2}}\right)$$ 1

$$=2\ln 2$$

(b)



1

:010

2U TRIAL **MATHEMATICS Extension 1 : Question** 1 2010

Suggested Solutions	Marks	Marker's Comments

(ii) **Solution:**

$$\frac{240}{OJ}=\tan 30^\circ$$

$$OJ=\frac{240}{\tan 30^\circ}$$

$$=\frac{240}{\left(\frac{1}{\sqrt{3}}\right)}$$

$$=240\sqrt{3}$$

Distance from base to James $=240\sqrt{3}$ metres 1

Distance from base to Bruce $=240$ metres (triangle is isosceles) 1

(iii) Find the distance between James and Bruce.

Solution:

$BJ^2=\left(240\sqrt{3}\right)^2+240^2$ (*Pythagoras' Theorem*) ½ — ½ Pythagoras

$=240^2(3+1)$ 1½ ½

$=240^2(4)$ — ½ if no attempt to clarify Bruce 2 James

$BJ=240\times 2$ 2

$=480$

Distance between James and Bruce = 480 m

(c)

Solution:

$u^2=x\quad(u>0)$

$2udu=dx$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\frac{dx}{\sqrt{x-x^2}}=\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{2udu}{\sqrt{u^2-u^4}}$$ 1 — limits

$$=\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{2udu}{u\sqrt{1-u^2}}$$ 1

$x=\frac{1}{4}\Rightarrow u^2=\frac{1}{4}$

$u=\frac{1}{2}\quad(u>0)$

$$=\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\frac{2du}{\sqrt{1-u^2}}$$

$$=2\left[\sin^{-1}u\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$ 1

$x=\frac{1}{2}\Rightarrow u^2=\frac{1}{2}$

$u=\frac{1}{\sqrt{2}}\quad(u>0)$

$$=2\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{2}\right)\right)$$

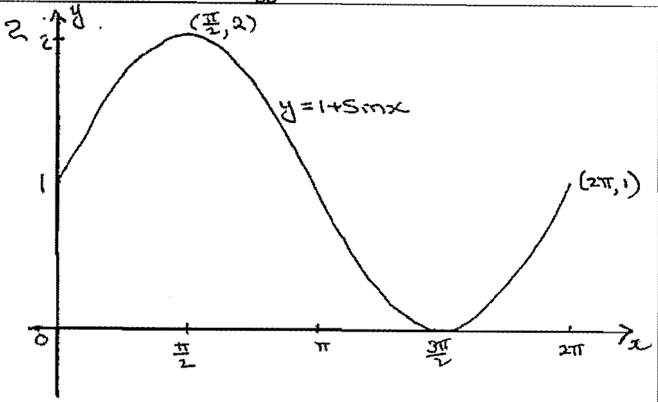
$$=2\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\frac{dx}{\sqrt{x-x^2}}==\frac{\pi}{6}$$ 1

TRIAL 2010 **MATHEMATICS Extension 1 : Question 2**

Suggested Solutions	Marks	Marker's Comments
Qu 2 (a)(i) $y=\frac{x^2}{4a}$ $\frac{dy}{dx}=\frac{x}{2a}$ when $x=2ap$ $\frac{dy}{dx}=p$ Eqn. of tangent is: $y-ap^2=p(x-2ap)$ $y-ap^2=px-2ap^2$ $\therefore px-y-ap^2=0$	 1 1	
(ii) Directrix has an eqn. $y=-a$ ....(1) Eqn. of tangent from (i) is: $px-y-ap^2=0$ (2) Sub (1) in (2) $px+a-ap^2=0$ $px=ap^2-a$ $x=\frac{a(p^2-1)}{p},\ p\neq 0$ $\therefore$ Co-ords. of T are $\left(\frac{a(p^2-1)}{p},-a\right)$	 1	No penalty for not mentioning $p\neq 0$ No half marks awarded
(iii) $F(0,a)$, $P(2ap,ap^2)$, $T\left(\frac{a(p^2-1)}{p},-a\right)$ $m(FP)=\frac{ap^2-a}{2ap-0}$ $=\frac{a(p^2-1)}{2ap}$ $\therefore m(FP)=\frac{(p^2-1)}{2p}$ $m(FT)=\frac{a--a}{0-\frac{a(p^2-1)}{p}}$ $=\frac{2ap}{-a(p^2-1)}$ $=\frac{-2p}{p^2-1}$ For perpendicular lines $m(FP)\times m(FT)=-1$ $m(FP)\times m(FT)=\frac{p^2-1}{2p}\times\frac{-2p}{p^2-1}$ $=-1$ $\therefore PF\perp FT$	 1 1 1	

MATHEMATICS Extension 1 : Question 2

Suggested Solutions	Marks	Marker's Comments
2 	2	½ mark deducted for any arrow heads 1 mark shape 1 mark for scale Needed to show $\frac{\pi}{2}$, $\frac{3\pi}{2}$ + 2π on x axis.
(b) $V=\pi\int_0^{2\pi}(1+\sin x)^2\,dx$ $=\pi\int_0^{2\pi}(1+2\sin x+\sin^2 x)\,dx$ $=\pi\int_0^{2\pi}\left(1+2\sin x+\frac{1-\cos 2x}{2}\right)dx$ $=\frac{\pi}{2}\int_0^{2\pi}(3+4\sin x-\cos 2x)\,dx$ $=\frac{\pi}{2}\left[3x-4\cos x-\frac{1}{2}\sin 2x\right]_0^{2\pi}$ $=\frac{\pi}{2}\left[\left(6\pi+3\cos 2\pi-\frac{1}{2}\sin 4\pi\right)-\left(0+3\cos 0-\frac{1}{2}\sin 0\right)\right]$ $=\frac{\pi}{6}\left[(6\pi+3)-(0+3)\right]$ $\therefore V=3\pi^2$ $\therefore$ Vol. $=3\pi^2 u^2$ If the area rotated was limited to between 0 and $\frac{3\pi}{2}$ $V=\pi\int_0^{\frac{3\pi}{2}}(1+\sin x)^2\,dx$ $\therefore V=\frac{\pi}{4}(9\pi+8)\ u^3$	 1 1 1 1	Marks were awarded if the area was taken between 0 and $\frac{3\pi}{2}$ OR between $\frac{3\pi}{2}$ and 2π.

MATHEMATICS Extension 1 : Question 3

Suggested Solutions	Marks	Marker's Comments
3)(i) $x = 3\cos(\frac{t}{4}) + \sin(\frac{t}{4})$		If they use the auxillary angle method then 1mk if done correctly
$\dot{x} = -\frac{3}{4}\sin(\frac{t}{4}) + \frac{1}{4}\sin(\frac{t}{4})$	} ½	velocity — ½ mk
$\ddot{x} = -\frac{3}{16}\cos(\frac{t}{4}) - \frac{1}{16}\cos(\frac{t}{4})$		acceleration — ½ mk
when $t=0$, $\dot{x} = \frac{1}{4}$ $\ddot{x} = -\frac{3}{16}$	½	
since $v>0$, particle is moving to the right	½	[auxillary angle $\sqrt{10}\cos(\frac{t}{4} - 0.32$
since $v>0$ and $a<0$, particle is slowing down	½	
(ii) particle at origin when $x=0$		
ie $0 = 3\cos(\frac{t}{4}) + \sin(\frac{t}{4})$	½	* using Auxillary angle method here only
$3\cos(\frac{t}{4}) = -\sin(\frac{t}{4})$		1mk - correct expression
$\tan(\frac{t}{4}) = -3$	½	1mk for solving correctly and getting $t=7.6s$
$\frac{t}{4} = k\pi + \tan^{-1}(-3)$ where k is an integer	} ½	
$t = 4k\pi + 4\tan^{-1}(-3)$		
when $k=0$, $t<0$		* If the auxillary angle method was used in part (i) 2mks for solving correctly
when $k=1$, $t = 4\pi + 4\tan^{-1}(-3) = 7.57$		
time taken is 7.6 seconds (1 dp).	½	* If they get $t=-4.996$ ⇒ 1mark only
		* If they got $t=433.7$ ⇒ 1mk only

MATHEMATICS Extension 1 : Question 2

Suggested Solutions	Marks	Marker's Comments
(a)(i) copy diagram neatly	½	lose the mark if drawn badly.
$\hat{CBA} = \hat{EOA}$ (exterior angle of cyclic quad OCBA equals the interior opposite angle)	½	no reason = no marks.
(ii) let $\hat{EBA} = x$		
$\hat{EOA} = 2x$ (angle at centre of circle is twice angle at circumference on the same arc EA)	½	
$\therefore \hat{CBA} = 2x$ (as $\hat{CBA} = \hat{EOA}$)	½	
$\hat{DBE} = \angle CBA - \angle EBA$ (subtraction of adjacent angles)	½	
$\therefore \hat{DBE} = 2x - x = x$	½	
$\therefore \hat{DBE} = \hat{EBA}$ (both x)	} 1	
$\therefore$ BE bisects $\angle DBA$		
(b)(i) $V = \pi\int_0^h x^2\,dy$		
$= \pi\int_0^h \sqrt{y}\,dy$	½	If they wrote the initial statement wrong or around the x-axis or 0 marks
$= \frac{2\pi}{3}\left[y^{3/2}\right]_0^h$	½	
$= \frac{2\pi}{3}\left[h^{3/2} - 0^{3/2}\right]$	½	
$\therefore V = \frac{2\pi}{3}h\sqrt{h}$		
$\therefore$ Volume is $\frac{2\pi}{3}h\sqrt{h}$ cm^3	½	
(ii) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{60}{\pi\sqrt{h}}$ → ½; $V = \frac{2\pi}{3}h^{3/2}$, $\frac{dV}{dh} = \frac{3}{2} \times \frac{2}{3} \times \pi h^{1/2} = \pi\sqrt{h}$	½	
when $h=16$, $\frac{dh}{dt} = \frac{60}{\pi \times 4} = \frac{15}{\pi}$	½	15π → 1½ mks
		15π and a sentence with units → 2mks
$\therefore$ Rate of water is increasing at $\frac{15}{\pi}$ cm/min.	½	

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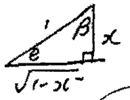
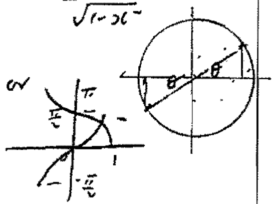
TRIAL 2010 Ext 1 **MATHEMATICS: Question** 4

Suggested Solutions	Marks	Marker's Comments
a i) 2m $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $-1<x<1$		many students did 
$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$ $-1<x<1$		
$\therefore \frac{d}{dx}\left[\sin^{-1}(x)+\cos^{-1}(x)\right] = 0$	1	or 
$\therefore \sin^{-1}(x)+\cos^{-1}(x) = \text{constant}$		
when $x=0$ $\sin^{-1}(0)+\cos^{-1}(0) = 0+\frac{\pi}{2}=\frac{\pi}{2}$	1	
Since $\sin^{-1}(x)+\cos^{-1}(x)$ is continuous for $-1<x<1$		only get 1 m
$\therefore \sin^{-1}(x)+\cos^{-1}(x) = \frac{\pi}{2}$, $-1\le x\le 1$		
ii) 3m $\sin^{-1}x = \cos^{-1}x$		many students did not prove and state only $x=\frac{1}{\sqrt{2}}$ get $\frac{1}{2}$m
$\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$		
$2\sin^{-1}x = \frac{\pi}{2}$		
$\sin^{-1}x = \frac{\pi}{4}$ $\therefore x = \frac{1}{\sqrt{2}}$	1	
$\frac{d}{dx}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$		
$\frac{d}{dx}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$	1	
$\tan\theta = \left\lvert\frac{m_1-m_2}{1+m_1\times m_2}\right\rvert = 2\sqrt{2}$	$\frac{1}{2}$	
$\theta = \underline{\underline{71^\circ}}$ (nearest degree)	$\frac{1}{2}$	forgot to round off lost $\frac{1}{2}$m
$\cos(2\theta+\theta) = 2\sin\theta\cos\theta$		
3m $\cos\theta\,(4\sin^2\theta + 2\sin\theta - 1) = 0$	1	many students forgot $\cos\theta=0$ lost $\frac{1}{2}$m
$\cos\theta = 0$ or $\sin\theta = \frac{-1\pm\sqrt{5}}{4}$	1	
$\theta = \frac{\pi}{2}$ or $0.314\ldots$ $\therefore \theta = 0.314$	$\frac{1}{2}+\frac{1}{2}$	Accept $\theta = \frac{\pi}{10}$

MATHEMATICS: Question 4

Suggested Solutions	Marks	Marker's Comments
b) $\cos 3\theta = \sin 2\theta$		
3m $\cos 3\theta = \cos\left(\frac{\pi}{2}-2\theta\right)$		
$3\theta = 2n\pi \pm \left(\frac{\pi}{2}-2\theta\right)$ $n\in Z$	1	forget $n\in Z$ $-\frac{1}{2}$m
$\theta = \frac{2}{5}n\pi + \frac{\pi}{10}$ or $2n\pi - \frac{\pi}{2}$	1	
$\theta = \frac{\pi}{10}$ is the smallest possible soln.	1	
or $\sin\left(\frac{\pi}{2}-3\theta\right) = \sin(2\theta)$		
$2\theta = n\pi + (-1)^n\left[\frac{\pi}{2}-3\theta\right]$	1	
n even $\theta = n\frac{\pi}{5}$ n odd $\theta = \frac{\pi}{2}-n\pi$	1	
$\theta = \frac{\pi}{10}$ smallest possible soln	1	
c(i) 1m coeff of $x^k = \binom{20}{k}3^k5^{20-k}$	1	
(ii) 3m $\frac{2}{3} = \binom{20}{k}3^k5^{20-k}\Big/\binom{20}{k+1}3^{k+1}5^{19-k}$	1	many students did $\frac{3}{2}$ and can't solve for k to be integer max 1 m.
$\frac{2}{3} = \frac{5}{3}\frac{(k+1)}{(20-k)}$	1	
$k=5$	$\frac{1}{2}$	
terms involving x^5 and x^6 or sixth and seventh term	$\frac{1}{2}$	many students lost $\frac{1}{2}$ if they write T_6 T_7 when they define $\frac{T_6}{T_7}=\frac{2}{3}$
OR $\dfrac{\binom{20}{k-1}3^{k-1}\cdot 5^{21-k}}{\binom{20}{k}3^k\cdot 5^{20-k}} = \frac{2}{3}$	1	
$\frac{k}{21-k}\cdot\frac{5}{3} = \frac{2}{3}$	1	$k=\frac{189}{19}$, or $k=\frac{120}{19}$ as a result of $\frac{3}{2}$ get 1 m.
$k=6$		
sixth and seventh term (or term involve x^5 and x^6)	1	

MATHEMATICS Extension 1 : Question 5

Suggested Solutions	Marks	Marker's Comments
(a) (i) If $\theta = \tan^{-1} A + \tan^{-1} B$ show that $\tan\theta = \frac{A+B}{1-AB}$.	1	
Solution: $\theta = \tan^{-1} A + \tan^{-1} B$ $\theta = \alpha + \beta$ where $\alpha = \tan^{-1} A$ and $\beta = \tan^{-1} B$ $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ $= \frac{A+B}{1-AB}$		Must show working
(ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$.	3	
Solution: $\tan\left(\frac{\pi}{4}\right) = \frac{3x+2x}{1-(3x)(2x)}$		1 substitution
$1 = \frac{5x}{1-6x^2}$ $1-6x^2 = 5x$ $6x^2+5x-1=0$ $(6x-1)(x+1)=0$		$\frac{1}{2}$ quadratic equation + factorisation
$x = \frac{1}{6}$ or -1		$\frac{1}{2}$ 2 solutions
but $\tan^{-1} 3x$ and $\tan^{-1} 2x$ are both acute (since their sum $< \frac{\pi}{2}$), therefore $x > 0$ $\therefore x = \frac{1}{6}$		1 rejecting $x=-1$
(b) Let $P(n)$ be the proposition that: $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$ for all positive integers $n \geq 1$,	4	
Test $P(1)$ when $n=1$, $LHS = \frac{1}{1}$, $RHS = \frac{2(1)}{1+1} = 1$ $\therefore LHS = RHS$ $\therefore P(1)$ is true		1 Test for $n=1$

MATHEMATICS Extension 1 : Question 5

Suggested Solutions	Marks	Marker's Comments
Assume $P(k)$ is true for $n=k$, $k \in Z^+$ i.e. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} = \frac{2k}{k+1}$		$\frac{1}{2}$ assumption including $k \in Z^+$
To prove true for $n = k+1$ i.e. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+k+(k+1)} = \frac{2(k+1)}{(k+1)+1}$ $= \frac{2k+2}{k+2}$		$\frac{1}{2}$ required to prove statement
Now $LHS = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+k+(k+1)}$ $= \frac{2k}{k+1} + \frac{1}{1+2+3+\ldots+k+(k+1)}$ (by assumption)		$\frac{1}{2}$ substitution of assumption including "by assumption"
$= \frac{2k}{k+1} + \frac{1}{\frac{1}{2}(k+1)(k+2)}$ $= \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)}$		$\frac{1}{2}$ showing AP sum
$= \frac{2k(k+2)+2}{(k+1)(k+2)}$ $= \frac{2(k^2+2k+1)}{(k+1)(k+2)}$ $= \frac{2(k+1)^2}{(k+1)(k+2)}$ $= \frac{2(k+1)}{(k+2)}$ $= \frac{2k+2}{k+2}$ $= RHS$ $\therefore P(k+1)$ is true $P(n)$ is true by the Principle of Mathematical Induction for $n = 1, 2, 3, \ldots$		1 completion

Q6 MARKING SCHEME.

2010 TRIAL ME1.

MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
(a) (i) $B_n = 3^2 + 7^2 + 11^2$ Now 3, 7, 11 is an AP as $d = 4$, $a = 3$ $\therefore T_n = 3 + (n-1)4 = 4n-1$ $\therefore$ n^{th} Term of B_n is $(4n-1)^2$	1	$\boxed{1}$
(ii) $S_{2n} = A_n - B_n$ $= \sum_{r=1}^{n}(4r-3)^2 - \sum_{r=1}^{n}(4r-1)^2 = \sum_{r=1}^{n}\left[(4r-3)^2 - (4r-1)^2\right]$ $= \sum (8r-4)(-2)$ $= -8\sum_{r=1}^{n}(2r-1) =$ $= -8[1+3+5+\dots+(2n-1)]$ A.Series $= -8 \times \frac{n}{2}[1 + 2n-1]$ $\therefore S_{2n} = -8n^2$ q.e.d.		$\frac{1}{2}$ For "$S_{2n} = -16n+8$" 1 For n^{th} term S_{2n} is $8-16n$ $\boxed{3}$
OR $S_{2n} =$ $= 1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \dots + (4n-3)^2 - (4n-1)^2$ n pairs $(\frac{1}{2})$ $= -2.4 + -2.12 + -2.20 + \dots + -2(8n-4)$ (1) $= -2[4 + 12 + 20 + \dots + 8n-4]$ $= -8[1+3+5+\dots+(2n-1)]$ $= -8 \times \frac{n}{2}[1 + 2n-1] = -8n^2$	$(1\frac{1}{2})$ (2)	1 For using A.series formula
(iii) $\overset{A_{26}}{101^2} - \overset{B_{26}}{103^2} + 105^2 - 107^2 + \dots + \overset{A_{503}}{2009^2} - \overset{B_{503}}{2011^2}$ $= A_{503} - B_{503} - (A_{25} - B_{25})$ $(\frac{1}{2})$ $= S_{2.503} - S_{2.25}$ $= -8 \times 503^2 - (-8 \times 25^2)$ $= -2\,019\,072$	$\frac{1}{2}$ $\frac{1}{2}$ 1	$4n-3 = 101$, $4n-1 = 103$ $\Rightarrow n = 26$ $4n-3 = 2009$, $4n-1 = 2011$ $\Rightarrow n = 503$ For: $n = 503$ For $n = 26-1 = 25$ For $-8 \times 503^2 + 8 \times (\dots)^2$ $\boxed{2}$
OR Sum $= -2 \times 204 + -2 \times 212 + \dots + -2 \times 4020$ $= -2[204 + 212 + 220 + \dots + 4020]$ $= -8[51 + 53 + 55 + \dots + 1005]$ Arith. Series as $d = 2$, $a = 51$, $n = 478$ as $51 + (n-1)2 = 1005$ $= -8 \times \frac{478}{2}[51 + 1005] = -2\,019\,072$		

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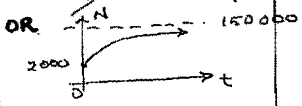
(b) (i) $N = 150\,000 - Be^{-kt}$ —(*) LHS: $\frac{dN}{dt} = +kBe^{-kt}$ RHS $= k(150\,000 - (150\,000 - Be^{-kt}))$ $= kBe^{-kt}$ $\therefore \frac{dN}{dt} = k(150\,000 - N)$ q.e.d. **or** $\frac{dN}{dt} = kBe^{-kt}$ but/as $Be^{-kt} = 150\,000 - N$ (*) transposed so $\frac{dN}{dt} = k(150\,000 - N)$		$\frac{1}{2}$ For EACH $\boxed{1}$
(ii) Data $t = 0$ $N = 2000$ $\therefore 2000 = 150\,000 - B$ $\therefore B = 148\,000$ so $N = 150\,000 - 148\,000e^{-kt}$ $t = 5$ $N = 50\,000$ $\therefore 50\,000 = 150\,000 - 148\,000e^{-5k}$ $-100\,000 = -148\,000e^{-5k}$ $e^{-5k} = \frac{100}{148} = \frac{25}{37}$ $-5k = \ln\frac{25}{37}$ $\therefore k = -\frac{1}{5}\ln\left(\frac{25}{37}\right)$		1 EACH $+\frac{1}{5}\ln\left(\frac{37}{25}\right)$ $\boxed{2}$
(iii) For possible max N, $\frac{dN}{dt} = 0$ i.e $k(150\,000 - N) = 0$ $\therefore N = 150\,000$ TEST $\frac{d^2N}{dt^2} = -k^2Be^{-kt} < 0$ as $k, B, e^{-kt} > 0$ $\therefore$ concave downwards $t \geq 0$ $\therefore$ max $N = 150\,000$ **OR** $N = 150\,000 - 148\,000e^{-kt}$ $t \to \infty$ $e^{-kt} \to 0^+$ $N \to 150\,000^-$ $\therefore$ max (limit) $N = 150\,000$ ants	1 $\frac{1}{2}$	$N' = kBe^{-kt} > 0$ $N'' = -k^2Be^{-kt} < 0$ $\boxed{1}$ **OR**, use Box method <table><tr><td>N</td><td>140 000</td><td>150 000</td><td>160 000</td></tr><tr><td>dN/dt</td><td>⊕</td><td>0</td><td>⊖</td></tr></table>**OR** 
(iv) $t = 20$ $N = ?$ $N = 150\,000 - 148\,000e^{-\frac{1}{5}\ln\frac{37}{25}\times 20}$ **OR** $e^{+\frac{1}{5}\ln\frac{25}{37}\times 20}$ $= 150\,000 - 148\,000e^{4\ln\frac{25}{37}}$ or $e^{-4\ln\frac{37}{25}}$ $= 150\,000 - 30\,847.1364\dots$ $= 119\,152.86\dots$ Calc display $\therefore$ No of ants 119 000 (nearest 1000)		$\boxed{2}$ 1 1 or Equiv.

Q7.

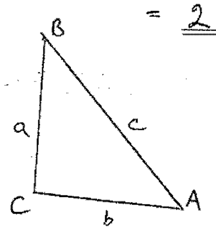
SUGGESTED SOLUTION	MARK	COMMENTS

a) i) $\cos(A+B) = \underline{\cos A\cos B - \sin A \sin B}$ — 1

Let $A = B = \theta$,

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= \underline{2\cos^2\theta - 1}\end{aligned}$$ — 1

ii)



$p = a+b+c$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ (Cosine Rule) — 1

$\therefore 2\cos^2\left(\frac{A}{2}\right) - 1 = \dfrac{b^2+c^2-a^2}{2bc}$ (Using i)

$\therefore 2\cos^2(A/2) = \dfrac{b^2+c^2-a^2}{2bc} + 1$

$\therefore 2\cos^2(A/2) = \dfrac{b^2+c^2-a^2+2bc}{2bc}$

$\cos^2(A/2) = \dfrac{(b+c)^2-a^2}{4bc}$ — 1

There were many untidy algebraic pathways used in this part.

$= \dfrac{(p-a)^2-a^2}{4bc}$ $(p=a+b+c)$

$= \dfrac{(p-a+a)(p-a-a)}{4bc}$ (Diff of squares)

$\cos^2 A/2 = \dfrac{p(p-2a)}{4bc}$

$\therefore \cos(A/2) = \pm \dfrac{1}{2}\sqrt{\dfrac{p(p-2a)}{bc}}$ — 1

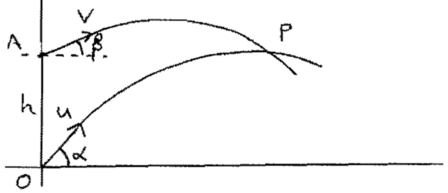
Too many people lost the last mark for not explaining the sign.

But A is angle of triangle.

$\therefore A < 180^\circ$, $\therefore A/2 < 90^\circ$ $\therefore \cos(A/2) > 0$

$\therefore \cos\left(\frac{A}{2}\right) = \dfrac{1}{2}\sqrt{\dfrac{p(p-2a)}{bc}}$ — 1

7 b)



i) For second particle

$\underline{x = Vt\cos\beta}$ — 1

$\underline{y = Vt\sin\beta - \frac{gt^2}{2} + h}$ — 1

Question said "write down". Many people wasted time by deriving these equations.

ii) At time $t=T$, the x and y values for each particle coincide.

i.e. $VT\cos\beta = UT\cos\alpha$ (x value)

$\therefore V = \dfrac{U\cos\alpha}{\cos\beta}$ (*) — 1

Rather untidily, many people carried 't' through the calculation when T is the correct value.

$VT\sin\beta - \frac{gT^2}{2} + h = UT\sin\alpha - \frac{gT^2}{2}$ (y value)

$\therefore h = UT\sin\alpha - VT\sin\beta$ — 1

A few people got lost in equations of paths.

Sub from * $h = UT\sin\alpha - \dfrac{UT\cos\alpha\sin\beta}{\cos\beta}$ — 1

Students must take care to distinguish U & V.

$\therefore h = \dfrac{UT\sin\alpha\cos\beta - UT\cos\alpha\sin\beta}{\cos\beta}$

$\therefore h = \dfrac{UT(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{\cos\beta}$

$\therefore h = \dfrac{UT\sin(\alpha-\beta)}{\cos\beta}$ — 1

$\therefore T = \dfrac{h\cos\beta}{U\sin(\alpha-\beta)}$ $(\alpha\neq\beta)$