HORNSBY GIRLS HIGH SCHOOL

2010
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question in a new booklet

Total marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value

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Total Marks
Attempt Questions 1–7
All Questions are of equal value

Begin each question in a new booklet, writing your student number and question number in the boxes indicated. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The interval AB, where A is $(-3, 4)$ and B is $(1, -2)$, is divided **externally** in the ratio 1: 3 by the point $P\ (x, y)$. Find the values of x and y. **2**

(b) Differentiate $\cos^{-1}\left(x^3\right)$ with respect to x. **2**

(c) Prove that, if $x^4 - x^3 + kx - 4$ has a factor of $(x+1)$, then it also has a factor of $(x-2)$. **2**

(d) Solve $\dfrac{x+4}{x-2} \geq 3$ for x. **3**

(e) Use the substitution $u = 2x+1$ to evaluate $\displaystyle\int_0^1 \frac{4x}{2x+1}\,dx$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve (giving your answer to nearest degree): **3**

$3\cos x + 5\sin x = 5$ for $0° \le x \le 360°$

(b) The curves $y = x^2$ and $y = 2x$ meet at $x = 2$. Find the angle between these curves at this point of intersection. **2**

(c) Find the primitive function of $2\sin^2 x$. **2**

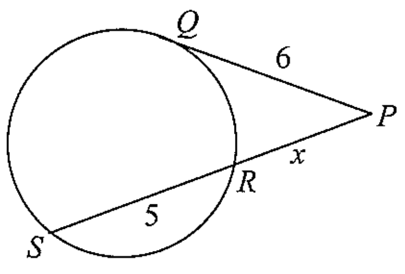
(d) In a large school, 5% of the students have blond hair. A group of 10 students is randomly chosen.

What is the probability that:

(i) exactly one student has blond hair? **1**

(ii) at least 2 students have blond hair? **2**

(e)



PQ is a tangent to a circle QRS, while PRS is a secant intersecting the circle at R and S, as shown in the diagram. **2**

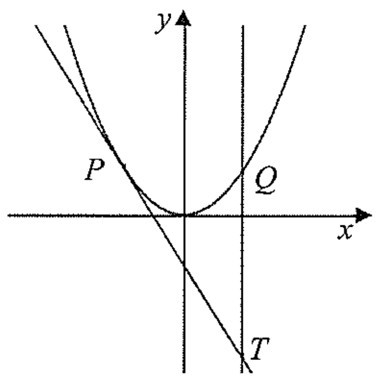
Given that $PQ = 6$, $RS = 5$ and $PR = x$, find the value of x.

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the volume of the solid of revolution formed when the region bounded by the curve $y = \frac{1}{\sqrt{4+x^2}}$, the x-axis, the y-axis and the equation $x = \frac{\pi}{2}$ is rotated about the x-axis. **3**

(b) Find the term independent of x in the expansion of $\left(3x^2 + \frac{2}{x}\right)^6$. **3**

(c)



NOT TO SCALE

Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The tangent at P and the line through Q parallel to the y axis intersect at point T.

The equation of the tangent at P is $y = px - ap^2$. **(Do NOT prove this)**

(i) Find the coordinates of T. **1**

(ii) Write down the coordinates of M, the midpoint of PT. **1**

(iii) Determine the locus of M when $pq = -1$. **1**

(d) A sphere is being heated so that its surface area is expanding at a constant rate of $0.025\ \text{cm}^2$ per second. **3**

Find the rate of change of the volume when the radius is 5 cm.

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x) = \dfrac{2x}{1-x^2}$.

(i) Show that the function is increasing for all values of x in its domain. **2**

(ii) Sketch the graph of $y = f(x)$ showing the intercepts on the axes and any asymptotes. **3**

(iii) Hence, or otherwise, find the values of k such that $\dfrac{2x}{1-x^2} = k$ has two solutions. **1**

(b) Evaluate $\lim\limits_{x \to 0} \dfrac{1-\cos^2 2x}{x^2}$. **2**

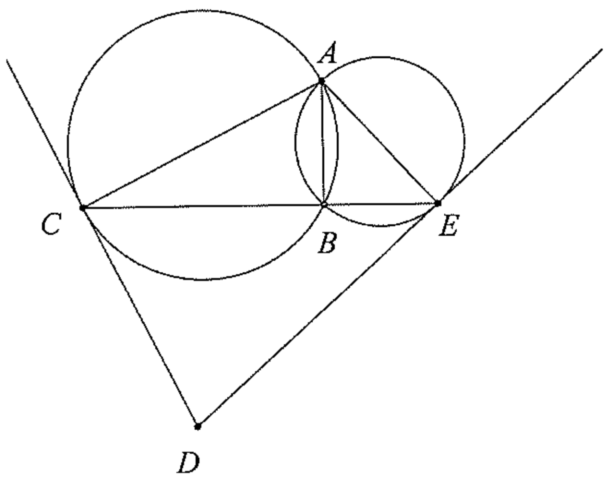
(c) Peter and his brother James are having a card night for themselves and 6 other friends. If they are to be seated at a round table, what is the probability that Peter and James do NOT sit next to each other? **2**

(d) Find the maximum value of $2x(1-x)$ and hence determine the range of $y = \sin^{-1}\left[2x(1-x)\right]$ for $0 \le x \le 1$. **2**

Question 5 (12 marks) Use a SEPERATE writing booklet. **Marks**

(a) Use mathematical induction to prove that $4^n > 2n+1$, where n is a positive integer. **3**

(b)



NOT TO SCALE

In the diagram above, two circles intersect at A and B. **3**

Points C and E lie on the circles and C, B and E are collinear.

Tangents at C and E meet at D.

Show that quadrilateral AEDC is concyclic.

(c) The population, P, of a mining town after t years satisfies the equation

$$\frac{dP}{dt} = k(P-1000).$$

The population was initially 10 000, and after five years it had decreased to 8 000.

(i) Show that $P = 1000 + Ae^{kt}$ is a solution of the equation. **1**

(ii) Find the value of A. **1**

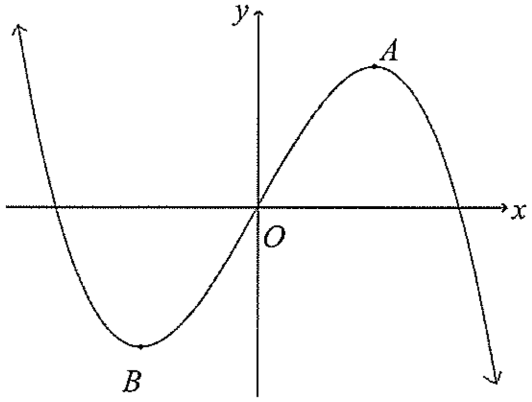
(iii) Find the value of k. (Give your answer correct to 3 significant figures) **1**

(iv) Find the number of years taken for the population to reach 5000. **2**

(v) Sketch the graph of the population against time. **1**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graph of $y = f(x)$, where $f(x) = 3x - x^3$, is shown in the diagram below.



NOT TO SCALE

(i) Find the coordinates of the turning points A and B. **2**

(ii) Find the largest domain containing the origin for which $f(x)$ has an inverse function. **1**

(iii) By considering the graph of $y = f(x)$, find the domain of $f^{-1}(x)$. **1**

(iv) By considering the gradient of $y = f(x)$, or otherwise, find the gradient of the inverse function $y = f^{-1}(x)$ at $x = 0$. **2**

(b) A particle moves in a straight line, so that its acceleration x cm from the origin is given by $\frac{d^2x}{dt^2} = 15 - 25x$. Initially the particle is at rest 1.6 cm to the right of the origin.

(i) Show that the speed is given by $s = \sqrt{30x - 25x^2 + 16}$. **2**

(ii) Given that the motion is simple harmonic, find the interval in which the particle moves. **2**

(iii) Find the maximum speed, and the displacement where this occurs. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Two of the roots of the equation $x^3 - ax^2 + b = 0$ are reciprocals.

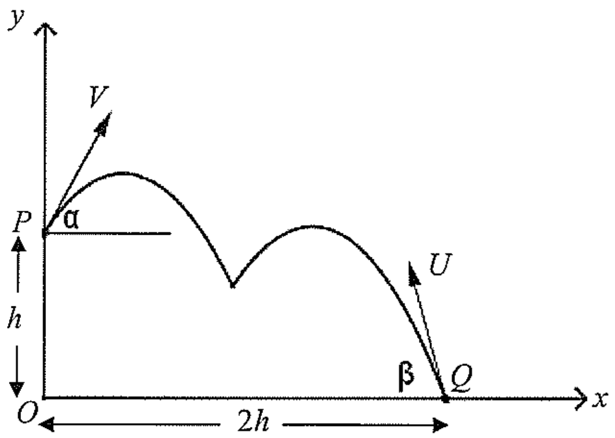
(i) Show that the third root is equal to $-b$. **2**

(ii) Show that $a = \frac{1}{b} - b$. **2**

(iii) Show that the 2 roots, which are reciprocals, will be real if $\frac{-1}{2} \le b \le \frac{1}{2}$. **2**

Question 7 continues on page 10

(b)



O and Q are two points $2h$ metres apart on horizontal ground. P is a point h metres directly above O. Particle A is projected from P towards Q with speed V m/s at an angle α above the horizontal.

At the same time particle B is projected from Q towards P with speed U m/s at an angle β above the horizontal. The two particles collide T seconds after projection.

For particle A the equations of motion are: $\ddot{x}_p = 0$ and $\ddot{y}_p = -g$.

It is known that its horizontal distance x_P from O, is given by:

$x_P = Vt\cos\alpha$, where t is time in seconds. **(Do NOT prove this)**

(i) Use calculus to show that at time t seconds, its vertical distance y_P from O is given by: **2**

$$y_P = Vt\sin\alpha - \tfrac{1}{2}gt^2 + h$$

(ii) For particle B, write down expressions for its horizontal distance x_Q from Q and its vertical distance y_Q from Q at time t seconds. **2**

(iii) By considering the point of collision, find an expression for $\dfrac{V}{U}$ in terms of α and β. **2**

End of paper

Question 1.

(a) $A(\overset{x_1}{-3}, \overset{y_1}{4})$ $B(\overset{x_2}{1}, \overset{y_2}{-2})$ externally $\overset{m}{1} : \overset{n}{3}$

$$X = \frac{1 \times -1 - 3 \times -3}{1-3} = \frac{1+9}{-2} = -5$$

$$Y = \frac{1 \times -2 - 3 \times 4}{1-3} = \frac{-14}{-2} = 7$$

$(-5, 7)$ //

(b) $$\frac{d}{dx}\cos^{-1}(x^3) = \frac{-1}{\sqrt{1-x^6}} \times 3x^2 = \frac{-3x^2}{\sqrt{1-x^6}} \quad //$$

(c) $(x+1)$ is a factor if $P(-1) = 0$

$$P(-1) = (-1)^4 - (-1)^3 + k(-1) - 4$$
$$= 1 - {}^-1 - k - 4$$
$$= -2 - k$$
$$= 0 \text{ when } k = -2 \quad //$$

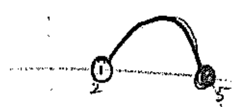
$\therefore$ $P(x) = x^4 - x^3 - 2x - 4$

if $P(x-2)$ is a factor $P(2) = 0$

$$P(2) = 2^4 - 2^3 - 2 \times 2 - 4$$
$$= 16 - 8 - 4 - 4$$
$$= 0 \quad //$$

(d) $$\frac{x+4}{x-2} \geqslant 3$$

$$(x+4)(x-2) \geqslant 3(x-2)^2$$
$$(x-2)[(x+4) - 3(x-2)] \geqslant 0$$
$$(x-2)[10 - 2x] \geqslant 0$$
$$2(x-2)(5-x) \geqslant 0$$
$$2 < x \leqslant 5 \quad //$$



(e) $$\int_0^1 \frac{4x}{2x+1}\,dx$$

$$= \int_1^3 \frac{2u-2}{u} \cdot \frac{du}{2}$$
$$= \int_1^3 \frac{u-1}{u} \cdot du$$
$$= \int_1^3 \left(1 - \frac{1}{u}\right) du$$
$$= \left[u - \ln u\right]_1^3$$
$$= (3 - \ln 3) - (1 - 0)$$
$$= 2 - \ln 3 \quad //$$

$u = 2x+1 \Rightarrow 2x = u - 1$, $4x = 2u - 2$

$\frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

$x = 1$ $u = 3$

$x = 0$ $u = 1$

Q2. a) $3\cos x + 5\sin x = 5 \quad 0° \le x \le 360°$

$R = \sqrt{9+25} = \sqrt{34} \qquad \tan\alpha = 5/3 \quad$ ie $\alpha = 59°2'$

ie $\sqrt{34}\cos(x-\alpha) = 5$

$\cos(x-\alpha) = \frac{5}{\sqrt{34}}$

$x - 59°2' = 30°58', \ 329°2'$

$\therefore x = 90°, 388°$

since $0° \le x \le 360°, \quad x = 90°, \ 388-360 = 28°$

ie $x = 90°, 28°$.

b) $y = x^2 \qquad y = 2x$

$\frac{dy}{dx} = 2x \qquad m = 2$

at $x = 2$, $m_1 = 4 \qquad \therefore m_2 = 2$

$\tan\theta = \left|\frac{4-2}{1+4\times2}\right|$

$= \frac{2}{9}$

$\therefore \theta = 12°32'$

c) $\int 2\sin^2 x = \int 1 - \cos 2x \, dx$

$= x - \frac{1}{2}\sin 2x + C$

d) i) $P(B) = 0.05 \qquad P(\tilde{B}) = 0.95$

$P(1B) = {}^{10}C_1 (0.05)(0.95)^9$

$= 0.31512$

ii) $P(\ge 2B) = 1 - [P(0B) + P(1B)]$

$= 1 - [0.95^{10} + 0.31512]$

$= 0.0861$

e) $x(x+5) = 6^2 \qquad PS.PR = QP^2$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0$

$x = -9, 4$

$\therefore x = 4 \quad (x \ge 0)$

Question 3

(a) $V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$

$= \pi \int_0^{\pi/2} \frac{1}{4+x^2} \, dx$

$= \frac{\pi}{2} \left[\tan^{-1} \frac{x}{2} \right]_0^{\pi/2}$

$= \frac{\pi}{2} \left[\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0 \right]$

$= \frac{\pi}{2} \tan^{-1} \frac{\pi}{4} \; u^3$ (exact) (3)

(1.05 2dp)

$\frac{dV}{dt} = ?$ $\frac{dS}{dt} = 0.025$ at $r = 5$

(d) $\frac{dV}{dt} = \frac{dS}{dt} \times \frac{dV}{dS}$

$= 0.025 \times \frac{r}{2}$

$= 0.025 \times \frac{5}{2}$

$= 0.0625 \text{ cm}^3/\text{s}$ //

Need $\frac{dV}{dS}$

$\frac{dV}{dS} = \frac{dV}{dr} \times \frac{dr}{dS}$

$= 4\pi r^2 \times \frac{1}{8\pi r}$

$= \frac{r}{2}$

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

3

(b) $\left(3x^2 + \frac{2}{x}\right)^6$ has general term $\binom{6}{r}(3x^2)^{6-r}\left(\frac{2}{x}\right)^r$

$= \binom{6}{r} 3^{6-r} x^{12-2r} \cdot 2^r \cdot x^{-r}$

$= \binom{6}{r} 3^{6-r} \cdot 2^r \cdot x^{12-3r}$

term independent of x (power of $x = 0$)

$\therefore 12 - 3r = 0$

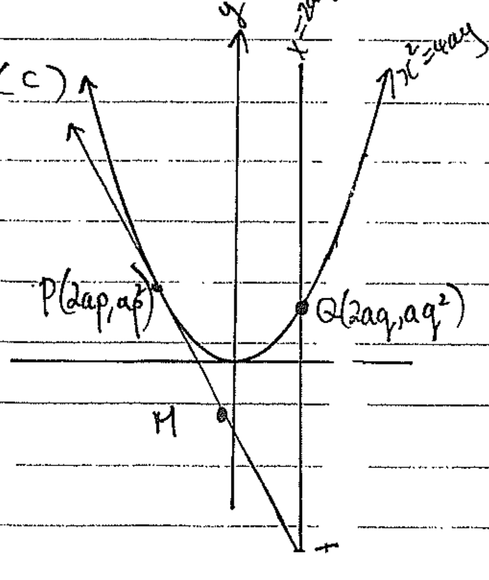
$3r = 12$

$r = 4$

$\therefore \binom{6}{4} 3^{6-4} \cdot 2^4 \cdot x^0 = 15 \times 9 \times 16$

$= 2160$ // (3)

(c)



(i) Finding T: $x = 2aq$ satisfies $y = px - ap^2$

$\therefore y = 2apq - ap^2$

$T(2aq, 2apq - ap^2)$ //

(ii) $M_{PT} = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + (2apq - ap^2)}{2}\right)$

$= \left(a(p+q), apq\right)$ //

(iii) given $pq = -1$ then $y = apq$

$= -a$

$\therefore$ M lies on the directrix of $x^2 = 4a$[y]

Question 4 Solutions.

$f(x) = \frac{2x}{1-x^2}$ $\quad x \neq \pm 1$

$f'(x) = \frac{2(1-x^2)+4x^2}{(1-x^2)^2}$

$= \frac{2x^2+2}{(1-x^2)^2}$

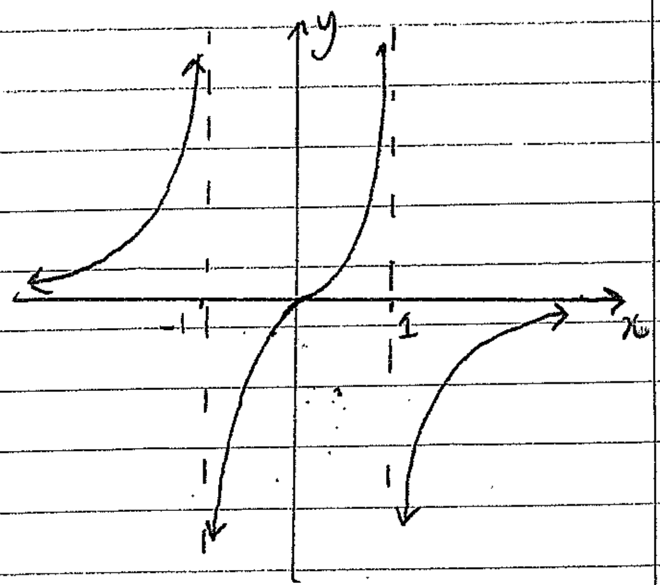
$2x^2+2 > 0$ for all x

$(1-x^2)^2 > 0$ for all x, $x \neq \pm 1$

$\therefore$ $f'(x) > 0$ for all x in domain

$f(x)$ is increasing.

ii)



ii) From graph, all k, $k \neq 0$.

b) $\lim_{x \to 0} \frac{1-\cos^2 2x}{x^2} = \lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

$= 4 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2$

$= 4$

(c) Arrangements not together $= 1 \times 5 \times 6!$

Total arrangements $= (8-1)!$

$= 7!$

$\therefore P(\text{not together}) = \frac{1 \times 5 \times 6!}{7!}$

$= \frac{3600}{5040}$

$= \frac{5}{7}$

(d) For $0 \leq x \leq 1$,

$y = 2x(1-x) = 2x - 2x^2$

$\frac{dy}{dx} = 2 - 4x$

Let $\frac{dy}{dx} = 0$

$x = \frac{1}{2}$

When $x = \frac{1}{2}$, max value is $\frac{1}{2}$

$\therefore$ Range of $f(x) = \sin^{-1}(2x(1-x))$ is $0 \leq f(x) \leq \frac{\pi}{6}$.

QUESTION 5

a) Prove true for $n=1$

L.H.S. $= 4$

R.H.S $= 2+1$

$= 3$

$\therefore$ L.H.S. > R.H.S.

$\therefore$ True for $n=1$

Assume true for $n=k$

i.e. $4^k > 2k+1$

Prove true for $n=k+1$

i.e. $4^{k+1} > 2(k+1)+1$

$4^{k+1} > 2k+3$

now $4^k > 2k+1$

$4 \cdot 4^k > 4(2k+1)$

$4^{k+1} > 8k+4$

$4^{k+1} > 2k+3+6k+1$

$\therefore 4^{k+1} > 2k+3$

$\therefore$ True for $n=k+1$

Since true for $n=1$ etc.

$A = 9000$

iii) when $t=5$ $P=8000$

$8000 = 1000 + 9000e^{5k}$

$e^{5k} = 7/9$

$5k = \ln(7/9)$

$k = -0.0503$

iv) when $P=5000$

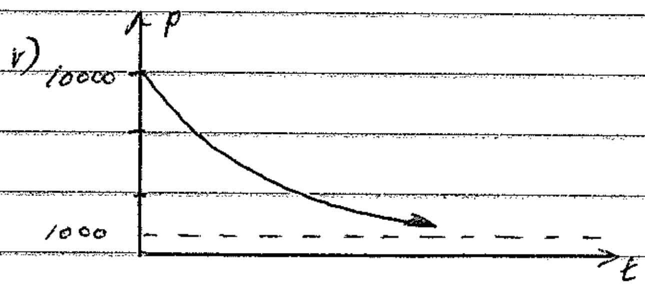
$5000 = 1000 + 9000e^{kt}$

$e^{kt} = 4/9$

$kt = \ln(4/9)$

$t = 16.12$ years

v)



b) $\angle DCE = \angle CAB = x$ ($\angle$ in alternate segment)

$\angle DEC = \angle EAB = y$ ($\angle$ in alternate segment)

$\therefore \angle CAE = x+y$

$\angle CDE + x + y = 180$ (angle sum $\triangle CED$)

$\therefore \angle CDE + \angle CAE = 180$

$\therefore$ AEDC cyclic quadrilateral (opposite $\angle$'s supplementary)

c) i) $P = 1000 + Ae^{kt} \Rightarrow Ae^{kt} = P - 1000$ —— (1)

$\frac{dP}{dt} = kAe^{kt}$

$= k(P-1000)$ by substituting (1)

ii) when $t=0$, $P = 10\,000$

$10\,000 = 1000 + Ae^0$

Question 6

$f(x) = 3x - x^3$

$f'(x) = 3 - 3x^2$

Let $f'(x) = 0$

$0 = 3 - 3x^2$

$x^2 = 1$

$x = \pm 1$

when $x = 1$, $f(1) = 3 - 1$
$= 2$

$x = -1$, $f(-1) = -3 - (-1)^3$
$= -2$

$\therefore A = (1, 2)$; $B \in (-1, -2)$

ii) Largest domain containing $(0,0)$: $-1 \leq x \leq 1$

iii) Range of $f(x)$ for restricted domain is $-2 \leq y \leq 2$

$\therefore$ Domain of $f^{-1}(x)$ is
$-2 \leq x \leq 2$

iv) $f'(x) = 3 - 3x^2$
$f'(0) = 3$
$\therefore \frac{dx}{dy} = \frac{1}{3}$

Gradient of inverse function at $x=0$ is $\frac{1}{3}$.

(b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 15 - 2x$

$\frac{1}{2}v^2 = \int (15 - 2x)\,dx$

(b) $\frac{1}{2}v^2 = 15x - \frac{25x^2}{2} + c$

When $t = 0$, $x = 1.6$, $v = 0$

$\therefore 0 = 15(1.6) - \frac{25}{2}(1.6)^2 + c.$

$c = 8$

~~$v^2 = 30x - 50$~~

$v^2 = 30x - 25x^2 + 18$

$s = \sqrt{30x - 25x^2 + 18}$

(ii) Let $s = 0$

$0 = 30x - 25x^2 + 18$

$x = \frac{30 \pm \sqrt{900 + 4(25)(16)}}{50}$

$= \frac{30 \pm 50}{50}$

$x = -\frac{2}{5}, \frac{8}{5}$

Particle moves on interval
$-\frac{2}{5} \leq x \leq \frac{8}{5}$.

(iii) Max speed at $x = \frac{-\frac{2}{5} + \frac{8}{5}}{2}$

$= \frac{3}{5}$

$s = \sqrt{30\left(\frac{3}{5}\right) - 25\left(\frac{3}{5}\right)^2 + 16}$

$= \sqrt{25}$

$\therefore$ Max speed is $5\,cms^{-1}$ when ~~x=~~ $\frac{3}{5}$ cm to right of origin.

Q7 a) $x^3 - ax^2 + b = 0$

i) let roots be $\alpha, \frac{1}{\alpha}, \beta$

Product of roots: $\alpha . \frac{1}{\alpha} . \beta = \frac{-d}{a}$

$\therefore \beta = -b$

ii) $-b$ is a root

$\therefore P(-b) = (-b)^3 - a(-b)^2 + b = 0$

$-b^3 - ab^2 + b = 0$

$ab^2 = -b^3 + b$

$a = -b + \frac{1}{b}$

iii) $x^3 - \left(b - \frac{1}{b}\right)x^2 + b = 0$

Sum of roots 2 @ a time: $\alpha . \frac{1}{\alpha} + \alpha.(-b) + \frac{(-b)}{\alpha} = \frac{c}{a} = 0$

$\therefore 1 - \alpha b - \frac{b}{\alpha} = 0$

$\alpha - \alpha^2 b - b = 0$

ie $b\alpha^2 - \alpha + b = 0$

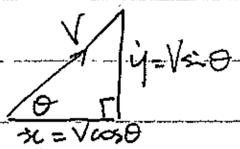
real roots $\Delta \geq 0$

$1 - 4b^2 \geq 0$

$(1 - 2b)(1 + 2b) \geq 0$

ie $-\frac{1}{2} \leq b \leq \frac{1}{2}$

Q7b) i) $\ddot{y}_p = -g$



$\dot{y}_p = \int -g \, dt$

$= -gt + C_1$

$t = 0$

$\dot{y} = V\sin\theta$

$\dot{y}_p = -gt + V\sin\alpha$

$y = \int -gt + V\sin\alpha \, dt$

$y = -\frac{gt^2}{2} + Vt\sin\alpha + C_2$

$t = 0$

$y = h$

$\therefore y = -\frac{gt^2}{2} + Vt\sin\alpha + h$

ii) $x_Q = 2h - Ut\cos\beta$

$y_Q = -\frac{gt^2}{2} + Ut\sin\beta$

iii) $2h - Ut\cos\beta = Vt\cos\alpha$

$Vt\cos\alpha + Ut\cos\beta = 2h$

$t = \frac{2h}{V\cos\alpha + U\cos\beta}$

$-\frac{gt^2}{2} + Vt\sin\alpha + h = -\frac{gt^2}{2} + Ut\sin\beta$

$h = Ut\sin\beta - Vt\sin\alpha$

$t = \frac{h}{U\sin\beta - V\sin\alpha}$

$\frac{2h}{V\cos\alpha + U\cos\beta} = \frac{h}{U\sin\beta - V\sin\alpha}$

$2U\sin\beta - 2V\sin\alpha = V\cos\alpha + U\cos\beta$

$V\cos\alpha + 2V\sin\alpha = 2U\sin\beta - U\cos\beta$

$V(\cos\alpha + 2\sin\alpha) = U(2\sin\beta - \cos\beta)$

$\frac{V}{U} = \frac{2\sin\beta - \cos\beta}{\cos\alpha + 2\sin\alpha}$