



**SYDNEY BOYS HIGH
SCHOOL**
MOORE PARK, SURRY HILLS

2009

**TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Start each **NEW** question in a separate answer booklet.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All answers must be given in exact simplified form unless otherwise stated.
- All necessary working should be shown in every question.

Total Marks – 84

- Attempt questions 1-7.

Examiner: *D.McQuillan*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 1 (12 marks)	Marks
(a) Solve $x(3-2x) > 0$.	2
(b) Find $\frac{d}{dx}(e^{-x} \cos^{-1} x)$	2
(c) The remainder when $x^3 + ax^2 - 3x + 5$ is divided by $(x+2)$ is 11. Find the value of a .	2
(d) Find the general solution of $2 \cos x + \sqrt{3} = 0$.	2
(e) Solve $\frac{x^2 - 9}{x} \geq 0$.	2
(f) Find $\int_0^2 (4+x^2)^{-1} dx$.	2

End of Question 1

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 2 (12 marks)

Marks

- (a) Use the substitution $x = \ln u$ to find $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$. **3**
- (b) Use one application of Newton's method to find an approximation to the root of the equation $\cos x = x$ near $x = 0.5$. Give your answer correct to two decimal places. **3**
- (c) The curves $y = e^{2x}$ and $y = 1 + 4x - x^2$ intersect at the point $(0, 1)$. Find the angle between the two curves at this point of intersection. **3**
- (d) **3**
- (i) In how many ways can a committee of 2 Englishmen, 2 Frenchmen and 1 American be chosen from 6 Englishmen, 7 Frenchmen and 3 Americans.
- (ii) In how many of these ways do a particular Englishman and a particular Frenchman belong to the committee?

End of Question 2

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 3 (12 marks)

Marks

- (a) Evaluate $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$. **1**
- (b) **3**
- (i) Expand $\cos(\alpha + \beta)$.
- (ii) Show that $\cos 2\alpha = 1 - 2\sin^2 \alpha$.
- (iii) Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$.
- (c) If $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ and $\beta = \cos^{-1}\left(\frac{4}{5}\right)$, calculate the exact value of $\tan(\alpha - \beta)$. **2**
- (d) A and B are points $(-1, 7)$ and $(5, -2)$; P divides AB internally in the ratio $k : 1$. **3**
- (i) Write down the coordinates of P in terms of k .
- (ii) If P lies on the line $5x - 4y = 1$, find the ratio of AP:PB.
- (e) Use mathematical induction to prove that **3**
- $$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1,$$
- where n is a positive integer.

End of Question 3

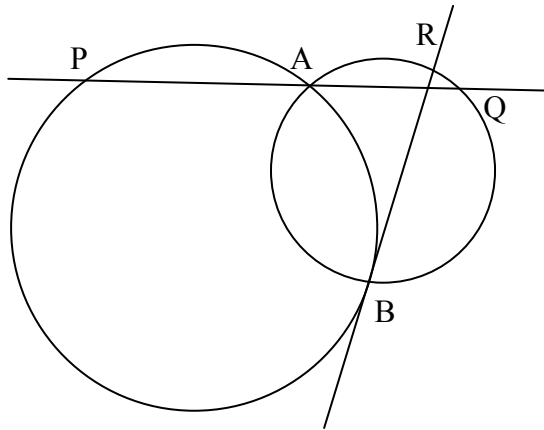
START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 4 (12 marks)

Marks

(a) If $\frac{dy}{dx} = 1 + y$ and when $x = 0$, $y = 2$ find y as a function of x . **3**

(b) Two circles cut at A and B. A line through A meets one circle at P and the other at Q. BR is a tangent to circle ABP and R lies on circle ABQ. Prove that $PB \parallel QR$. **3**



(c) The area bounded by the curve $y = \sin^{-1} x$ the y axis and $y = \frac{\pi}{2}$ is rotated about the y axis. Find the volume of the solid generated. **3**

(d) A particle moves in a straight line from a position of rest at a fixed origin O. Its velocity is v when displacement from O is x . If its acceleration is $\frac{1}{(x+3)^2}$, find v in terms of x . **3**

End of Question 4

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 5 (12 marks)

Marks

- (a) The speed $v \text{ ms}^{-1}$ of a particle moving along the x axis is given by $v^2 = 24 - 6x - 3x^2$, where $x \text{ m}$ is the distance of the particle from the origin. **4**
- (i) Show that the particle is executing Simple Harmonic Motion.
- (ii) Find the amplitude and the period of motion.
- (b) Five Jovians and four Martians are sitting around discussing galactic peace. **5**
- (i) In how many ways can they be arranged around the table?
- (ii) If Marvin the Martian will not sit next to any of the Jovians, how many arrangements are possible?
- (iii) If all the Jovians sit together and all the Martians sit together and Marvin will still not sit next to a Jovian, how many arrangements are possible?
- (c) If one root of $x^3 + px^2 + qx + r = 0$ equals the sum of the two other roots, prove that $p^3 + 8r = 4pq$. **3**

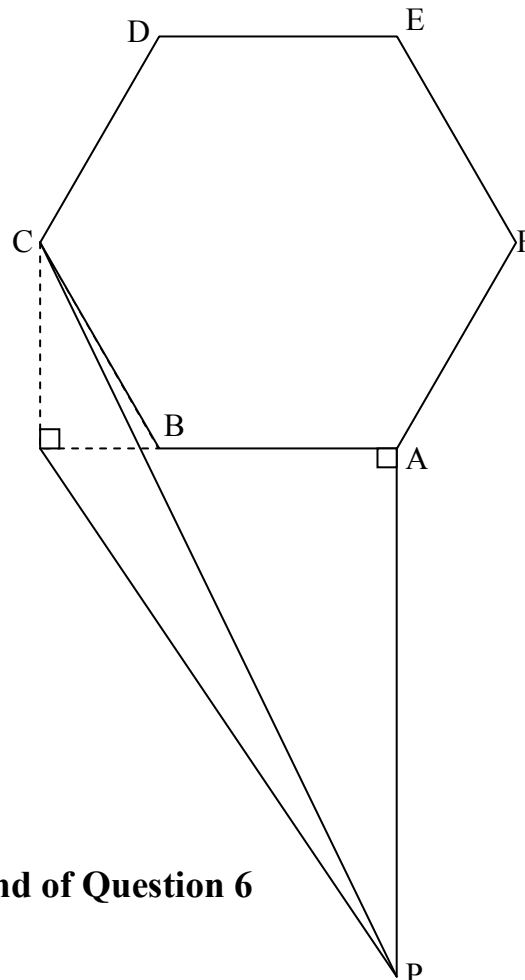
End of Question 5

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 6 (12 marks)

Marks

- (a) $f(x) = \cos x - \sqrt{3} \sin x$, where $0 \leq x \leq 2\pi$. **3**
- (i) Write $f(x)$ in the form $R \cos(x + \alpha)$ where $R > 0$ and α is in the first quadrant.
- (ii) Hence solve $f(x) = 1$.
- (b) Wheat falls from an auger onto a conical pile at the rate of $20 \text{ cm}^3 \text{ s}^{-1}$. The radius of the base of the pile is always equal to half its height. **5**
- (i) Show that $V = \frac{1}{12} \pi h^3$ and hence find $\frac{dh}{dt}$.
- (ii) Find the rate at which the pile is rising when it is 8 cm deep, in terms of π .
- (iii) Find the time taken for the pile to reach a height of 8 cm.
- (c) In a horizontal triangle APB, $AP = 2AB$, and the angle A is a right angle. On AB stands a vertical and regular hexagon ABCDEF. Prove that PC is inclined to the horizontal at an angle whose tangent is $\frac{\sqrt{3}}{5}$. **4**



End of Question 6

START EACH NEW QUESTION ON A NEW ANSWER BOOKLET

Question 7 (12 marks)

Marks

(a) Use mathematical induction to prove that $\cos(\pi n) = (-1)^n$, where n is a positive integer.

2

(b)

3

(i) Find the largest possible domain of positive values for which $f(x) = x^2 - 5x + 13$ has an inverse.

(ii) Find the equation of the inverse function, $f^{-1}(x)$.

(c) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

7

(i) Find the equation of the chord PQ and hence or otherwise show that $pq = -\frac{b}{a}$.

(ii) Prove that $p^2 + q^2 = 4m^2 + \frac{2b}{a}$.

(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and that N, the point of intersection of the normals at P and Q, has coordinates

$$\left[-apq(p+q), a(2+p^2+pq+q^2) \right],$$

express these coordinates in terms of a , m and b .

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line.

Verify that this line is a normal to the parabola.

End of Question 7

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

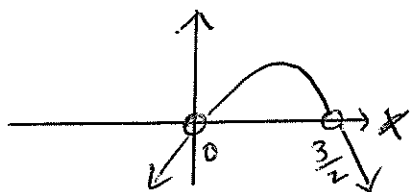
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1.

(a) $x(3-2x) > 0$



$\therefore \boxed{0 < x < \frac{3}{2}}$

(b)

$$\frac{d}{dx} [e^{-x} \cdot \cos^{-1} x]$$

$$= e^{-x} \cdot \frac{-1}{\sqrt{1-x^2}} + \cos^{-1} x \cdot -e^{-x}$$

$$= \frac{-e^{-x}}{\sqrt{1-x^2}} - e^{-x} \cos^{-1} x$$

(c) let $P(x) = x^3 + ax^2 - 3x + 5$

then $P(-2) = 11$ (Rem. Th)

$$\Rightarrow (-8) + 4a + 6 + 5 = 11$$

$$4a = 8$$

$$\boxed{a = 2}$$

(d) $2\cos x + \sqrt{3} = 0$

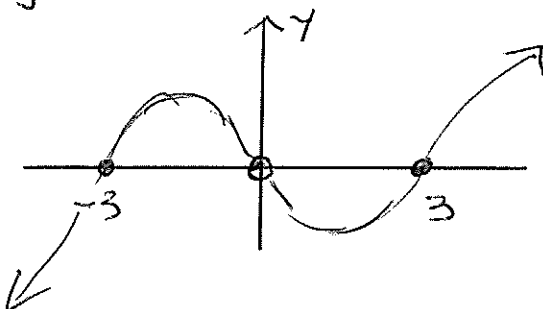
$$\Rightarrow \cos x = -\frac{\sqrt{3}}{2}$$

i. $\cos x = \cos \frac{5\pi}{6}$

$$\therefore \boxed{x = 2n\pi \pm \frac{5\pi}{6}}$$

(e) $\frac{x^2-9}{x} \geq 0 \quad [x \neq 0]$

$\times \text{ by } x^2 \Rightarrow x(x^2-9) \geq 0$



$$\boxed{-3 \leq x < 0 \text{ or } x \geq 3}$$

(f) $\int_0^2 \frac{dx}{4+x^2}$

$$= \frac{1}{2} \left[\tan^{-1} \frac{x}{2} \right]_0^2$$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi}{8}$$

QUESTION 2

$$(a) \quad x = \ln u \quad \frac{du}{dx} = e^x \\ u = e^x$$

$$\int \frac{u}{\sqrt{1-u^2}} \frac{du}{u} \quad dx = \frac{du}{e^x} \\ = \frac{du}{u}$$

$$= \sin^{-1} u + c \\ = \sin^{-1} e^x + c$$

$$(b) \quad \cos x - x = 0$$

$$f(x_1) = \cos 0.5 - 0.5 = 0.378$$

$$f'(x_1) = -\sin 0.5 - 1 = -1.479$$

$$x_2 = 0.5 - \frac{0.378}{-1.479}$$

$$= 0.7556$$

$$= 0.76 \text{ 2d.p.}$$

$$(c) \quad m_1 = 2e^{2x} \quad m_2 = 4 - 2x$$

$$x=0 \quad m_1 = 2, \quad m_2 = 4$$

$$\tan \theta = \left| \frac{2-4}{1+2 \cdot 4} \right|$$

$$= \frac{2}{9}$$

$$\theta = 12^\circ 32'$$

$$(d) \quad (i) \quad {}^6C_2 \times {}^7C_2 \times 3 = 945$$

$$(ii) \quad {}^5C_1 \times {}^6C_1 \times 3 \\ = 90$$

3 unit Trial ASC 2009

$$\begin{aligned}
 (3) \quad (a) \quad & \cos(\sin^{-1}(-\frac{1}{2})) \\
 &= \cos(-\frac{\pi}{6}) \\
 &= \cos \frac{\pi}{6} \quad \text{even fn} \\
 &= \frac{\sqrt{3}}{2} \quad (1)
 \end{aligned}$$

$$(b) \quad (i) \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (1)$$

$$\begin{aligned}
 (ii) \quad \text{let } \beta = \alpha. \quad & \cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\
 & \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \\
 \text{using } & \sin^2 \alpha + \cos^2 \alpha = 1.
 \end{aligned}$$

$$\begin{aligned}
 \cos 2\alpha &= 1 - \sin^2 \alpha - \sin^2 \alpha \\
 &= 1 - 2\sin^2 \alpha \quad (1)
 \end{aligned}$$

$$(iii) \quad \lim_{\alpha \rightarrow 0} \frac{1 - \cos 2\alpha}{\alpha^2} = \lim_{\alpha \rightarrow 0} \frac{2\sin^2 \alpha}{\alpha^2} = 2 \lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} \times \frac{\sin \alpha}{\alpha}$$

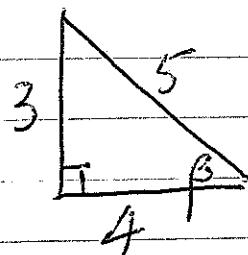
$$\begin{aligned}
 \text{since } \cos 2\alpha &= 1 - 2\sin^2 \alpha & &= 2 \times 1 \times 1 \\
 2\sin^2 \alpha &= 1 - \cos 2\alpha & &= 2. \quad (1)
 \end{aligned}$$

$$(c) \quad \alpha = \tan^{-1}\left(\frac{5}{12}\right) \quad \beta = \cos^{-1}\left(\frac{4}{5}\right)$$

$$\tan \alpha = \frac{5}{12}$$

$$\cos \beta = \frac{4}{5}$$

$$\text{So } \tan \beta = \frac{3}{4}$$



$$\begin{aligned}
 \text{So } \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\frac{5}{12} - \frac{3}{4}}{1 + \frac{5}{12} \times \frac{3}{4}} = \frac{-\frac{1}{3}}{\frac{15}{16} + \frac{9}{16}} \\
 &= \frac{-\frac{1}{3}}{\frac{24}{16}} = -\frac{1}{3} \quad (2)
 \end{aligned}$$

check $(-1, 7)$ $(5, -2)$ $17:16$
m n-

$$\frac{17 \times 5 + 16 \times -1}{17+16}, \quad \frac{17 \times -2 + 16 \times 7}{17+16}$$

$$= \frac{69}{33} = 2\frac{1}{11} \checkmark$$

$$\frac{78}{33} = 2\frac{4}{11} \checkmark$$

(*) $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1$
for n a positive integer

step 1 let $n=1$, LHS = $1 \times 1! = 1$
RHS = $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

So $n=1$ is true.

step 2 Assuming it is true for $n=k$,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1$$

we must prove that for $n=k+1$,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1) \times (k+1)! \\ = (k+2)! - 1$$

$$\text{LHS } (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! [1 + k + 1] - 1$$

$$= (k+1)! [k+2] - 1$$

$$= (k+2)! - 1.$$

$$= \text{RHS}.$$

step 3 Hence the statement is true for $n = k+1$
By the principle of math induction it is true for all $n \geq 1$.

(3)

$$4) a) \frac{dy}{dx} = 1+y$$

$$\frac{dx}{dy} = \frac{1}{1+y}$$

$$x = \ln(1+y) + C$$

$$\text{when } x=0, y=2$$

$$0 = \ln(3) + C$$

$$C = -\ln 3$$

$$x = \ln(1+y) - \ln 3$$

$$x = \ln\left(\frac{1+y}{3}\right)$$

$$\frac{1+y}{3} = e^x$$

$$1+y = 3e^x$$

$$y = 3e^x - 1$$

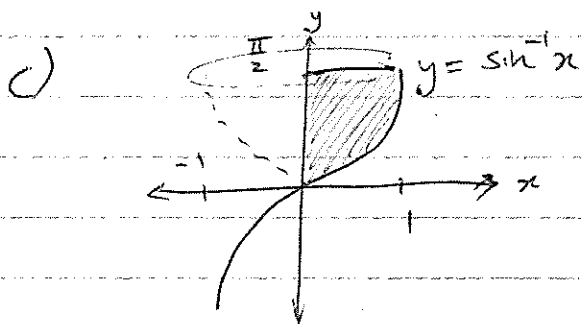
$$b) \text{ let } \hat{RQA} = x$$

$$\hat{ABR} = x \quad (\text{angles in same segment})$$

$$\hat{BPA} = x \quad (\text{alternate segment theorem})$$

since alternate angles equal ($\hat{RQP} = \hat{QPB}$)

$PB \parallel QR$



$$x = \sin y$$

$$V = \pi \int_a^b x^2 dy$$

$$V = \pi \int_0^{\frac{\pi}{2}} \sin^2 y dy$$

$$V = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2y) dy$$

$$V = \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}$$

$$V = \frac{\pi}{2} \left[\frac{\pi}{2} - \frac{1}{2} \sin 2\left(\frac{\pi}{2}\right) - \left(0 - \frac{1}{2} \sin 2(0)\right) \right]$$

$$V = \frac{\pi^2}{4} \text{ units}^3$$

$$d) \quad a = \frac{1}{(x+3)^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = (x+3)^{-2}$$

$$\frac{1}{2}v^2 = \frac{(x+3)^{-1}}{-1 \times 1} + C$$

$$\frac{1}{2}v^2 = -\frac{1}{x+3} + C$$

when $x=0, v=0$

$$0 = -\frac{1}{3} + C$$

$$C = \frac{1}{3}$$

$$\frac{1}{2}v^2 = \frac{1}{3} - \frac{1}{x+3}$$

$$v^2 = 2 \left(\frac{1}{3} - \frac{1}{x+3} \right)$$

$$v = \pm \sqrt{2 \left(\frac{1}{3} - \frac{1}{x+3} \right)}$$

but acceleration is always positive, & since it starts from rest,

$$v = \sqrt{2 \left(\frac{1}{3} - \frac{1}{x+3} \right)} \quad \text{OR} \quad \sqrt{\frac{2x}{3(x+3)}}$$

Question 5

(a) $\frac{d}{dx} (\frac{1}{2}v^2) = \ddot{x} = -3 - 3x$

(i) $= -3(x+1)$

Let $X = x+1$, so $\ddot{X} = \ddot{x}$

$\therefore \ddot{X} = -3X$ [2]

Hence, Simple Harmonic Motion.

(ii) From above, $n^2 = 3$

$v^2 = 3(8 - 2x - x^2)$

$= 3(8 - (x^2 + 2x + 1) + 1)$

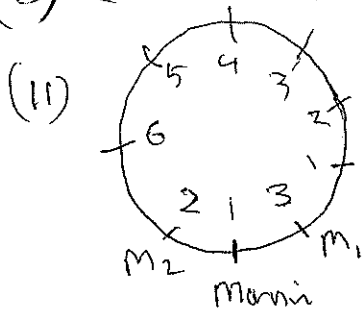
$= 3(9 - (x+1)^2)$

$\therefore v^2 = 3(9 - X^2)$

$\therefore a^2 = 9 \quad T = \frac{2\pi}{\sqrt{3}}$ [2]

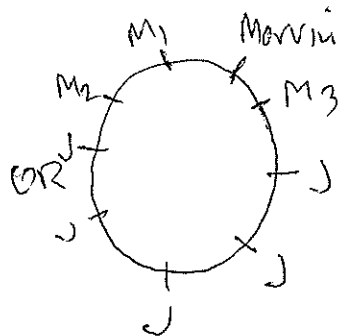
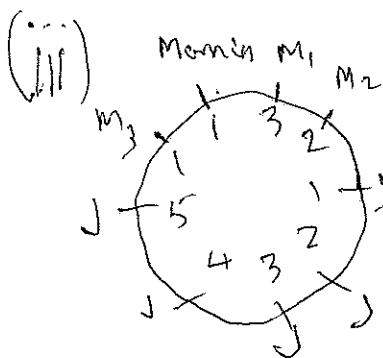
$a = 3$

(b) (i) $8! = 40320$ ways [1]



$2 \times 3 \times 6! = 4320$ ways

[2]



$\therefore 2 \times (3 \times 2 \times 5!) = 1440$ ways

[2]

(c) $x^3 + px^2 + qx + r = 0$

Let roots be $\alpha, \beta, \alpha + \beta$

Now $-p = 2(\alpha + \beta)$

$q = \alpha\beta + (\alpha^2 + \alpha\beta) + (\alpha\beta + \beta^2)$

$= 3\alpha\beta + \alpha^2 + \beta^2$

$-r = \alpha\beta(\alpha + \beta)$

$= \alpha^2\beta + \alpha\beta^2$ [1]

RTP: $p^3 + 8r = 4pq$

LHS $= -8(\alpha + \beta)^3 + 8(\alpha^2\beta + \alpha\beta^2)$

$= (8(\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3) +$

$8(\alpha^2\beta + \alpha\beta^2))$

$= (8\alpha^3 + 32\alpha^2\beta + 32\alpha\beta^2 + 8\beta^3)$

RHS $= -8(\alpha + \beta)(3\alpha\beta + \alpha^2 + \beta^2)$

$= -8(3\alpha^2\beta + \alpha^3 + \alpha\beta^2 + 3\alpha\beta^2 + \alpha^2\beta + \beta^3)$

$= -(8\alpha^3 + 32\alpha^2\beta + 32\alpha\beta^2 + 8\beta^3)$

$=$ LHS as required.

[2]

Alternatively

$\alpha + \beta = -\frac{p}{2}$

But $\alpha + \beta$ is a root.

$\therefore P(-\frac{p}{2}) = 0$

$(-\frac{p}{2})^3 + p(-\frac{p}{2})^2 + q(\frac{p}{2}) + r = 0$

$-\frac{p^3}{8} + \frac{p^3}{4} + (-\frac{pq}{2}) + r = 0$

$\frac{p^3}{8} + (-\frac{pq}{2}) + r = 0$

$\therefore p^3 + 8r = 4pq$ [3]

Question (6)

(a)

$$\cos x - \sqrt{3} \sin x = R \cos(x + \alpha)$$

(i)

$R = \sqrt{1+3} = 2$
$\tan \alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3}$

(ii) $2 \cos(x + \frac{\pi}{3}) = 1$

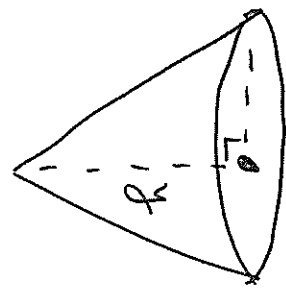
$\therefore \cos(x + \frac{\pi}{3}) = \frac{1}{2}$

$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore x = 0, \frac{4\pi}{3}, \frac{2\pi}{3}$

(3)

(b)



(i)

$$h = \frac{r}{2}$$

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \cdot \frac{r^2}{4} \cdot r$$

$$\therefore V = \frac{\pi r^3}{12}$$

$$\frac{dr}{dt} = \frac{dV}{dV} \times \frac{dV}{dt}$$

$$= \left(\frac{dV}{dr} \right) \times \frac{dV}{dt}$$

$$= \frac{4}{\pi r^2} \times 20$$

$$= \frac{80}{\pi r^2} \text{ cm/s}$$

(ii) $\frac{80}{\pi \times 64} = \frac{5}{4\pi}$ cm/s

(iii) $\frac{dr}{dt} = \frac{80}{\pi r^2}$

$\therefore \frac{dt}{dr} = \left(\frac{\pi}{80} \right) r^2$

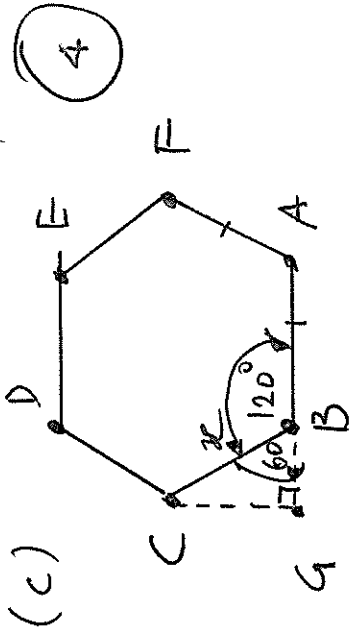
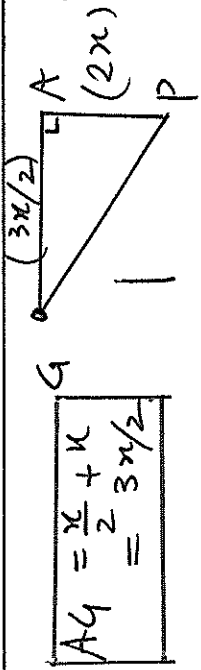
$$t = \int_0^8 \frac{\pi}{80} r^2 dr$$

$$= \left[\frac{\pi r^3}{240} \right]_0^8$$

$$= \frac{32\pi}{15} \text{ sec}$$

$$\approx 6.7 \text{ sec}$$

(5)



$AB = x = CB$

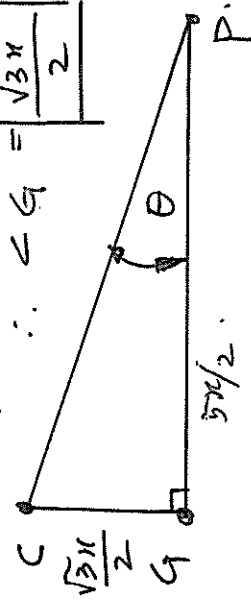
Express $\triangle CBG$ in terms of x

$\angle CBA = 120^\circ$ (regular hex)

$\Rightarrow \angle CBG = 60^\circ$

$$\frac{BG}{x} = \cos 60^\circ = \frac{x}{2}$$

$$\frac{CG}{x} = \sin 60^\circ = \frac{\sqrt{3}x}{2}$$



$$GP = \sqrt{4x^2 + 9x^2}$$

$$= \sqrt{\frac{25}{4}x^2} = \frac{5x}{2}$$

$$\tan \theta = \frac{CG}{GP} = \frac{\frac{\sqrt{3}x}{2}}{\frac{5x}{2}} = \frac{\sqrt{3}}{5}$$

(4)

2009 Mathematics Extension 1 Trial HSC: Question 7 solutions

7. (a) Use mathematical induction to prove that $\cos(\pi n) = (-1)^n$, where n is a positive integer. 2

Solution: Test for $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= \cos \pi, & \text{R.H.S.} &= (-1)^1, \\ &= -1. & &= -1. \end{aligned}$$

\therefore True when $n = 1$.

Now assume true when $n = k$, some particular integer,

i.e. $\cos(\pi k) = (-1)^k$.

Then test for $n = k + 1$, *i.e.* $\cos(\pi(k + 1)) = (-1)^{k+1}$.

$$\begin{aligned} \text{L.H.S.} &= \cos(\pi(k + 1)), \\ &= \cos(\pi k + \pi), \\ &= \cos \pi k \cos \pi - \sin \pi k \sin \pi, \\ &= (-1)^k \cdot (-1) - 0, \text{ using the assumption,} \\ &= (-1)^{k+1}, \\ &= \text{R.H.S.} \end{aligned}$$

\therefore True for all $n \geq 1$ by the principle of mathematical induction.

- (b) (i) Find the largest possible domain of positive values for which $f(x) = x^2 - 5x + 13$ has an inverse. 3

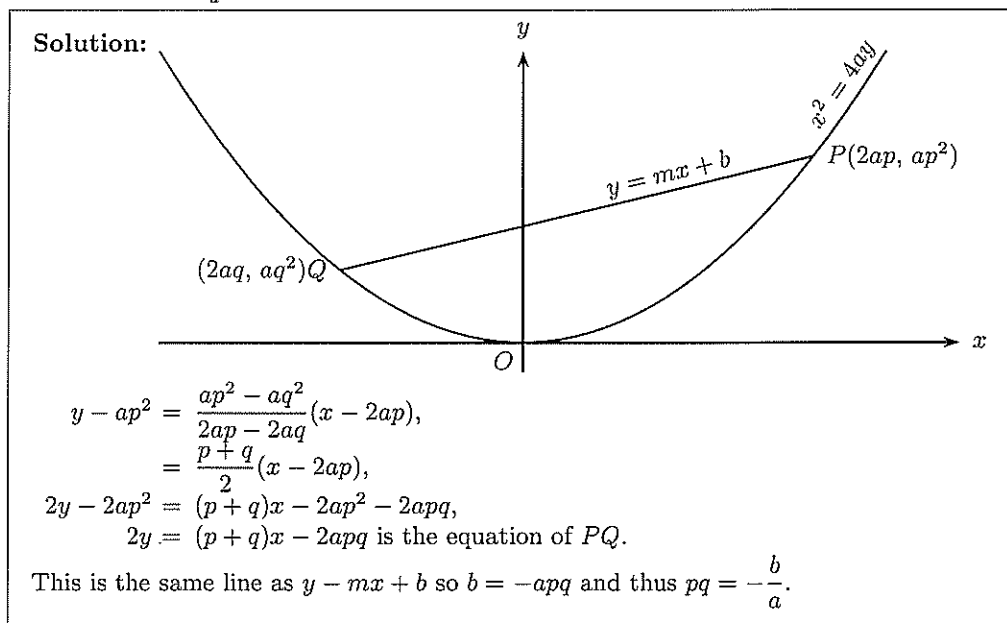
Solution: $f'(x) = 2x - 5$,
 $2x - 5 = 0$ when $x = 5/2$.
 \therefore Function is one-one if $x > 5/2$.

- (ii) Find the equation of the inverse function, $f^{-1}(x)$.

Solution: Put $x = y^2 - 5y + 13$,
 $= y^2 - 5y + \frac{25}{4} + 13 - \frac{25}{4}$,
 $x - \frac{27}{4} = (y - 5/2)^2$,
 $y - 5/2 = \frac{\pm\sqrt{4x - 27}}{2}$,
 $y = \frac{5 \pm \sqrt{4x - 27}}{2}$,
i.e. $f^{-1}(x) = \frac{5 + \sqrt{4x - 27}}{2}$, taking the positive root as $f^{-1}(x) > 5/2$.

(c) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$.

(i) Find the equation of the chord PQ and hence or otherwise show that $pq = -\frac{b}{a}$.



(ii) Prove that $p^2 + q^2 = 4m^2 + \frac{2b}{a}$.

Solution: $m = \frac{p+q}{2}$,

$$\therefore \text{R.H.S.} = 4\left(\frac{p+q}{2}\right)^2 + 2(-pq),$$

$$= p^2 + 2pq + q^2 - 2pq,$$

$$= p^2 + q^2,$$

$$= \text{L.H.S.}$$

(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and that N , the point of intersection of the normals at P and Q , has coördinates

$$[-apq(p+q), a(2 + p^2 + pq + q^2)],$$

express these coördinates in terms of a , m and b .

Solution: Now $-apq = b$, $p+q = 2m$, $p^2 + q^2 = 4m^2 + 2b/a$.

$$\therefore x_N = 2bm, \quad y_N = a(2 + 4m^2 + 2b/a - b/a),$$

$$= a(2 + 4m^2 + b/a).$$

$$\therefore N : [2bm, 2a + 4am^2 + b]$$

- (iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line.

Verify that this line is a normal to the parabola.

Solution: Method 1—

$$b = \frac{x}{2m},$$

$y = \frac{x}{2m} + 2a + 4am^2$ which is the locus of N
and a straight line with a slope of $1/2m$.

Rewriting, $x - 2my = -4am - 8am^3$,
then let $p = -2m$ so that $x + py = 2ap + ap^3$
which is in the form of a normal to the parabola $x^2 = 4ay$.

Solution: Method 2—

$$b = \frac{x}{2m},$$

$y = \frac{x}{2m} + 2a + 4am^2$ which is the locus of N
and a straight line with a slope of $\frac{1}{2m}$.

Where this locus of N meets the parabola $x^2 = 4ay$,

$$x^2 = 4a \left(\frac{x}{2m} + 2a + 4am^2 \right),$$

$$mx^2 - 2ax - 8a^2m + 16a^2m^3 = 0.$$

$$x = \frac{2a \pm \sqrt{4a^2 + 4a^2(8m^2 + 16m^4)}}{m},$$

$$= \frac{a \pm a\sqrt{1 + 8m^2 + 16m^4}}{m},$$

$$= \frac{a}{m} (1 \pm (1 + 4m^2)),$$

$$= \frac{a}{m} (2 + 4m^2) \text{ or } \frac{a}{m} (-4m^2).$$

In the limiting case when $x = -4am$, $p = q$ and $-4am = 2ap$,

$$\therefore p = -2m.$$

So the slope of the normal at this point is $\frac{1}{2m}$ which is the slope of the locus of N .