

# *Integration Using Substitution*

Try to convert the integral into a “standard integral”

## ***STANDARD INTEGRALS***

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln(x + \sqrt{x^2 + a^2})$$

## **“NON” STANDARD INTEGRALS**

Not listed in the standard integrals, however will save time if you know

$\int \frac{du}{\sqrt{u}}$	$= 2\sqrt{u}$
$\int \frac{du}{u^2}$	$= -\frac{1}{u}$
$\int \ln x dx$	$= x \ln x - x$
$\int \tan x dx$	$= \log \sec x$
$\int$ complementary trig ratio	$= -\text{complement of the answer}$

e.g.

$$\int \cot x dx$$

$$= -\log \operatorname{cosec} x$$

$$= \log \sin x$$

cot is the complement of tan

so the answer is  
minus and the  
complement of sec is  
cosec

e.g. (i)  $\int x\sqrt{x^2+4}dx$   $= \frac{1}{2} \int 2x\sqrt{x^2+4}dx$   $u = x^2 + 4$

$\frac{1}{2} du$   $u$   $du = 2xdx$

$$= \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{2}{\frac{3}{2}} u^{\frac{3}{2}} + c$$

$$= \frac{1}{3} u^{\frac{3}{2}} + c$$

$$= \frac{1}{3} (x^2 + 4)^{\frac{3}{2}} + c$$

$$= \frac{1}{3} (x^2 + 4)\sqrt{x^2 + 4} + c$$


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$$\begin{aligned} (ii) \int \frac{x+1}{4x^2+8x-7} dx &= \frac{1}{8} \int \frac{du}{u} & u &= 4x^2+8x-7 \\ &= \frac{1}{8} \log u + c & du &= (8x+8)dx \\ &= \frac{1}{8} \log(4x^2+8x-7) + c \end{aligned}$$

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$$\begin{aligned} (iii) \int \frac{dx}{x(\log x)^3} &= \int \frac{du}{u^3} & u &= \log x \\ &= \int u^{-3} du & du &= \frac{dx}{x} \\ &= -\frac{1}{2} u^{-2} + c \\ &= -\frac{1}{2u^2} + c \\ &= -\frac{1}{2(\log x)^2} + c \end{aligned}$$

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$$\begin{aligned}
 (iv) \int \frac{x}{\sqrt{1-x}} dx &= \int \frac{1-u^2}{u} \cdot -2u du && x = 1-u^2 \Rightarrow u = \sqrt{1-x} \\
 &= 2 \int (u^2 - 1) du && dx = -2u du \\
 &= 2 \left( \frac{1}{3} u^3 - u \right) + c \\
 &= \frac{2}{3} (\sqrt{1-x})^3 - 2\sqrt{1-x} + c \\
 &= \frac{2}{3} (1-x)\sqrt{1-x} - 2\sqrt{1-x} + c
 \end{aligned}$$

$$\begin{aligned}
 (v) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x \cos x dx &= \int_{\frac{\sqrt{3}}{2}}^1 u^5 du && u = \sin x && x = \frac{\pi}{3}, u = \sin \frac{\pi}{3} \\
 &&& du = \cos x dx && u = \frac{\sqrt{3}}{2} \\
 &= \frac{1}{6} [u^6]_{\frac{\sqrt{3}}{2}}^1 && && x = \frac{\pi}{2}, u = \sin \frac{\pi}{2} \\
 &= \frac{1}{6} \left\{ 1^6 - \left( \frac{\sqrt{3}}{2} \right)^6 \right\} && && u = 1 \\
 &= \frac{37}{384}
 \end{aligned}$$

$$\begin{aligned} (vi) \int_3^4 \frac{x dx}{\sqrt{25-x^2}} &= -\frac{1}{2} \int_{16}^9 \frac{du}{\sqrt{u}} \\ &= \frac{1}{2} \int_9^{16} u^{-\frac{1}{2}} du \\ &= \left[ u^{\frac{1}{2}} \right]_9^{16} \\ &= \sqrt{16} - \sqrt{9} \\ &= \underline{1} \end{aligned}$$

$$u = 25 - x^2$$

$$du = -2x dx$$

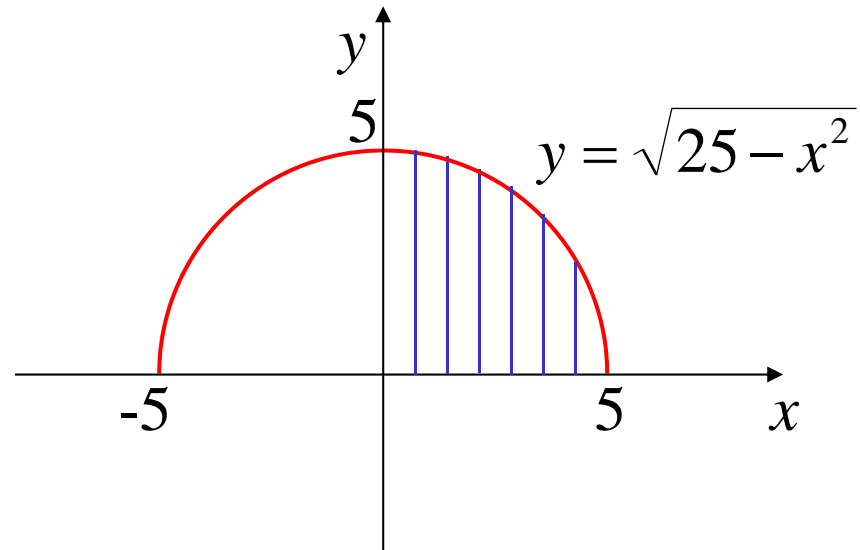
$$x = 3, u = 16$$

$$x = 4, u = 9$$

$$\begin{aligned}
\text{(vii)} \int_0^5 \sqrt{25 - x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{25 - 25 \sin^2 u} \cdot 5 \cos u du & x = 5 \sin u \Rightarrow u = \sin^{-1} \frac{x}{5} \\
&= \int_0^{\frac{\pi}{2}} \sqrt{25 \cos^2 u} \cdot 5 \cos u du & dx = 5 \cos u du \\
& & x = 0, u = \sin^{-1} 0 \\
& & u = 0 \\
& & x = 5, u = \sin^{-1} 1 \\
& & u = \frac{\pi}{2} \\
&= 25 \int_0^{\frac{\pi}{2}} \cos^2 u du \\
&= \frac{25}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2u) du \\
&= \frac{25}{2} \left[ u + \frac{1}{2} \sin 2u \right]_0^{\frac{\pi}{2}} \\
&= \frac{25}{2} \left\{ \frac{\pi}{2} + \frac{1}{2} \sin \pi - 0 \right\} \\
&= \frac{25\pi}{4}
\end{aligned}$$



$$\begin{aligned}
 (vii) \int_0^5 \sqrt{25 - x^2} dx \\
 &= \frac{1}{4} \pi (5)^2 \\
 &= \frac{25\pi}{4}
 \end{aligned}$$



**Exercise 6C; 1, 2ace, 3, 4ace, 5ace etc, 6, 7 & 8ac, 11, 12**

**Exercise 6D; 1, 2a, 3, 4b, 5ac, 6ace etc, 7ab(i,ii)**

**8ab (i,iii,vi), 9ab (ii,iv,vi), 11, 13**

*Patel Exercise 2A ace in all*

***NOTE: substitution is not given in Extension 2***