2c)

$$\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\cos^2 x$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)x \cos^2\left(\frac{\pi}{2}-x\right)$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)x \sin^2 x$$

$$2\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\cos^2 x\,dx = \int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\left(\sin^2 x+\cos^2 x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}x - x^2\right)dx$$

$$= \left[\frac{\pi}{4}x^2 - \frac{1}{3}x^3\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^3}{16} - \frac{\pi^3}{24}$$

$$= \underline{\underline{\frac{\pi^3}{48}}}$$

$$\therefore \int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\cos^2 x\,dx = \frac{\pi^3}{96}$$

2a) $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)} + \sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$$

$$\therefore\ 2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \Big[\,x\,\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$$

3b)
$$\int_0^{\pi} \frac{\sin x \, dx}{a + b\cos^2 x}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{a + b\cos^2 x}$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -2\int_1^0 \frac{du}{a + bu^2}$$

$$= \frac{2}{b}\int_0^1 \frac{du}{\frac{a}{b} + u^2}$$

$$= \frac{2}{b} \times \frac{\sqrt{b}}{\sqrt{a}} \left[\tan^{-1}\frac{\sqrt{b}\,u}{\sqrt{a}}\right]_0^1$$

$$= \frac{2}{\sqrt{ab}} \tan^{-1}\frac{\sqrt{b}}{\sqrt{a}}$$