2010

TRIAL
HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 1

General Instructions:

- Reading Time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen.
- Board - approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

Total marks – 84

Attempt Questions 1 –7

All questions are of equal value.

Question 1 (12 marks). Start on a SEPARATE page.

Marks

(a) The line $y = mx$ makes an angle of $45°$ with the line $y = 2x - 3$. Find the possible values of m. 2

(b) Find the coordinates of the point $P(x, y)$ which divides the interval joining $A(-4, -6)$ and $B(6, -1)$ externally in the ratio 3:2. 2

(c) Solve for x: $\dfrac{2x+1}{x-1} \geq 3$ 2

(d) Differentiate $y = x\tan^{-1}\dfrac{x}{2}$ 3

(e) Use the substitution $u = \sqrt{x}$ to evaluate $\displaystyle\int_1^4 \frac{dx}{x + \sqrt{x}}$ 3

Question 2 (12 marks). Start on a SEPARATE page.

(a) Find the coefficient of x^9 in the expansion of $\left(x^2+\dfrac{2}{x}\right)^{12}$ 2

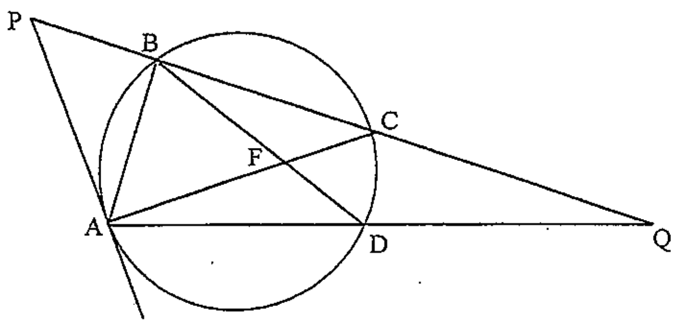
(b) Evaluate: $\lim\limits_{x\to 0} \dfrac{\sin 6x}{7x}$ 2

(c) If $f(x)=4\cos^{-1}\dfrac{x}{3}$, find

(i) the domain and range of $f(x)$. 2

(ii) Sketch the curve. 2

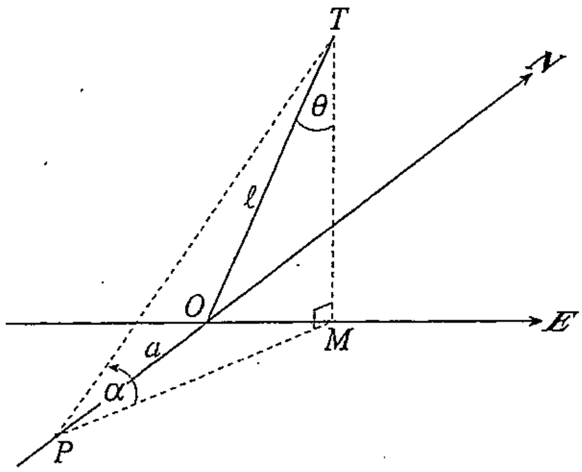
(d)



In the above figure, AP is a tangent to the circle at A. $PBCQ$ and ADQ are straight lines. Prove that $\angle PAB=\dfrac{1}{2}(\angle CFD+\angle CQD)$ 4

Question 3 (12 marks) Start on a SEPARATE page.

(a) A pole, OT, of length l metres stands on horizontal ground. The pole leans towards the east, making an angle of θ with the vertical. From P, a metres south of O, the elevation of T is α.



(i) Copy the diagram above onto your booklet.Find expressions, in terms of l and θ for OM and MT. 2

(ii) Prove that $PM = l\cos\theta\cot\alpha$. 1

(iii) Prove that $l^2 = \dfrac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$ 3

(iv) Find the length of the pole, to the nearest metre, if $a = 25$, $\theta = 20°$ and $\alpha = 24°$. 1

(b) A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated? 2

(c) In an election 40% of the voters favoured Party A. If an interviewer selected 5 voters at random, what is the probability that

(i) exactly three of them favoured Party A.

(ii) A majority of those selected favoured Party A

(iii) At most two favoured Party A. 3

Question 4 (12 marks). Start on a SEPARATE page.

(a) Prove the following by the Principle of mathematical induction.

$$\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots \quad + \log\left(\frac{n}{n-1}\right) = \log n \quad \text{for all}$$

integers $n \geq 2$. 3

(b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 16y$. The equation of the normal at P is given by $x + py = 4p^3 + 8p$.

(i) Find the point of intersection R of the normals at P and Q, the end points of focal chord PQ. 2

(ii) Find the locus of R. 2

(c) For the function $y = \dfrac{2x^2 - 2}{x^2 - 9}$

(i) Write down the equations of horizontal and vertical asymptotes. 2

(ii) Sketch the curve showing intercepts with axes and asymptotes. 3

Question 5 (12 marks)

(a) By expanding both sides of the identity $(1+x)^5(1+x)^5=(1+x)^{10}$, show that $\sum_{k=0}^{5}({}^5C_k)^2={}^{10}C_5$ 3

(b) (i) Write the expansion of $(1+x)^n$. 1

(ii) By integrating, show that

$${}^nC_0+\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2+\ldots\ldots\qquad+\frac{1}{n+1}{}^nC_n=\frac{2^{n+1}-1}{n+1}$$ 3

(c) The rate at which an object warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation

$$\frac{dT}{dt}=k(T-A),$$

Where t is the time in minutes, T and A are measured in degrees centigrade, and k is aconstant.

(i) Show that $T=A+Ce^{kt}$, where C is a constant is a solution of the differential equation. 2

(ii) An object warms from $10°C$ to $15°C$ in 20 minutes. The air temperature surrounding the object is $25°C$. Determine the temperature of the object after a further 30 minutes have passed. Give your answer to the nearest degree. 2

(iii) Using the equation for T, given in part (i), explain the behaviour of T as t increases to large values. 1

Question 6 (12 marks). Start on a SEPARATE page.

(a) (i) Given $f(x) = x\sin^{-1}x + \sqrt{1-x^2}$. Find $f'(x)$. 3

(ii) Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1}x \; dx$ 2

(b) (i) show that there exists a root of the equation $\tan x - x = 0$ between $x = 4$ and $x = 4.5$. 1

(ii) By halving the interval twice find an approximate value of the root Correct to 1 decimal place. 2

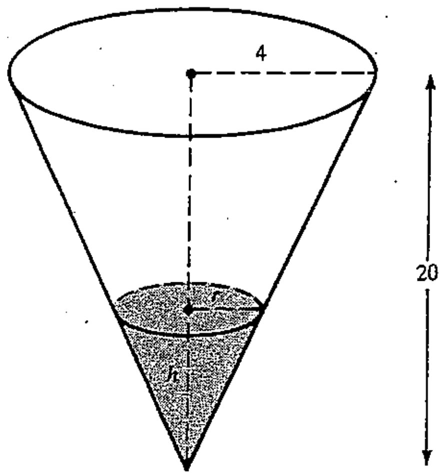
(c) Assume tides at a harbour rise and fall in SHM. At low tide the harbour is 12 m deep, and at high tide 17 m deep. Low tide is at 9-00 am and high tide at 3.00 pm.Assuming a ship needs 14 m to go safely,

(i) at what time can the ship go into the harbour. 3

(ii) if the ship take 30 minutes to go out, before what time must it depart the harbour. 1

Question 7 (12 marks). Start on a SEPARATE page.

(a)



A small funnel in the shape of a cone is being emptied of fluid at the rate of $12\,cm^3/s$. The height of the funnel is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone? 3

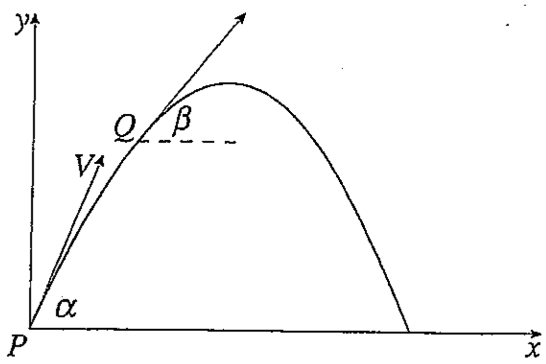
(b) Given that $x^3+x^2-10=0$ has a root between 1 and 2. By taking 2 as the initial value find an approximation to the root using Newton's method, correct to one decimal place. 2

(c) A particle is projected from a point P on horizontal ground, with initial speed $V\,m/s$ at an angle of elevation α to the horizontal. It's equations of motion are $x=0$ and $y=-g$. The horizontal and vertical component of velocity and displacement of the particle at any time t are given by

$$\frac{dx}{dt}=V\cos\alpha \quad \text{and} \quad \frac{dy}{dt}=V\sin\alpha-gt$$

$$x=Vt\cos\alpha \quad \text{and} \quad y=Vt\sin\alpha-\frac{1}{2}gt^2 \text{ (do not prove these)}$$

(i) Determine the time of flight of the particle. 2



(ii) The particle reaches a point Q, as shown, where the direction of the flight makes an angle β with the horizontal. Find an expression for $\tan\beta$. 1

(iii) Hence show that the time taken to travel from P to Q is $\dfrac{V\sin(\alpha-\beta)}{g\cos\beta}$ seconds. 2

(iv) Consider the case where $\beta=\dfrac{\alpha}{2}$. If the time taken to travel from P to Q is one third of the total time of flight, find the value of α. 2

End of paper

Trial HSC Extension 1, 2010 – Solutions

Question 1 (12 marks)

(a) $m_1 = m \qquad m_2 = 2$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$1 = \left|\frac{2-m}{1+2m}\right| \qquad 2$$

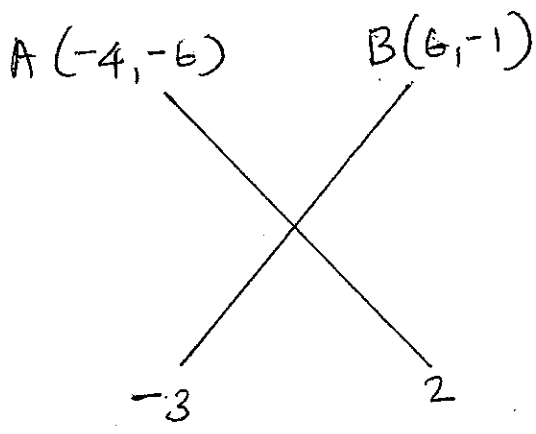
$$\frac{2-m}{1+2m} = 1 \quad \text{OR} \quad \frac{2-m}{1+2m} = -1$$

$$2 - m = 1 + 2m \qquad 2 - m = -1 - 2m$$

$$3m = 1 \qquad -m = 3$$

$$\underline{\underline{m = \frac{1}{3}}} \qquad \underline{\underline{m = -3}}$$

(b)



$$x = \frac{(-3 \times 6) + (2 \times -4)}{-3+2} = 26$$

$$y = \frac{(-3 \times -1) + (2 \times -6)}{-3+2} = 9$$

$$\underline{\underline{P(26, 9)}} \qquad 2$$

(c) $\dfrac{2x+1}{x-1} \geq 3$

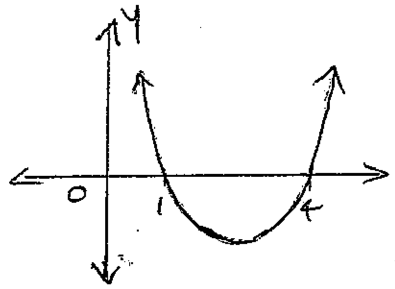
$$(x-1)(2x+1) \geq 3(x-1)^2, \ x \neq 1$$

$$3(x-1)^2 - (x-1)(2x+1) \leq 0$$

$$(x-1)\left[3(x-1) - (2x+1)\right] \leq 0$$

$$(x-1)(3x - 3 - 2x - 1) \leq 0$$

$$(x-1)(x-4) \leq 0$$



2

$$\underline{\underline{1 < x \leq 4}}$$

(d) $y = x \tan^{-1}\dfrac{x}{2}$

$$y' = x \times \frac{1}{1 + \frac{x^2}{4}} \times \frac{1}{2} + \tan^{-1}\frac{x}{2} \times 1$$

$$= x \times \frac{1}{\frac{4+x^2}{4}} \times \frac{1}{2} + \tan^{-1}\frac{x}{2}$$

$$= x \times \frac{4}{4+x^2} \times \frac{1}{2} + \tan^{-1}\frac{x}{2}$$

$$= \underline{\underline{\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}}} \qquad 3$$

(c) $\int_1^4 \frac{dx}{x+\sqrt{x}}, \quad u=\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$; $\frac{dx}{du} = 2\sqrt{x}$

$dx = 2\sqrt{x}\,du$
$= 2u\,du$

when $x=1$, $u=\sqrt{1}=1$

when $x=4$, $u=\sqrt{4}=2$

$\int_1^2 \frac{2u\,du}{u^2+u} = \int_1^2 \frac{2u\,du}{u(u+1)}$

$= \int_1^2 \frac{2\,du}{u+1} = 2\int_1^2 \frac{du}{u+1}$

$= 2\left[\log(u+1)\right]_1^2$ 3

$= 2\left[\log 3 - \log 2\right]$

$= 2\log\frac{3}{2}$

Question 2 (12 marks)

(a) $T_{r+1} = {}^{12}C_r (x^2)^{12-r}\left(\frac{2}{x}\right)^r$

$= {}^{12}C_r\, x^{24-2r}\, \frac{2^r}{x^r}$

$= {}^{12}C_r\, x^{24-3r}\, 2^r$

$24-3r = 9$

$3r = 15$

$r = 5$

$T_6 = {}^{12}C_5\, x^{24-15}\, 2^5$ 2

$= {}^{12}C_5 \times 2^5 x^9$

Coefficient of $x^9 = {}^{12}C_5 \times 2^5$

(b) $\lim_{x\to 0} \frac{\sin 6x}{7x}$

$= \lim_{x\to 0} \frac{\sin 6x}{6x} \times \frac{6x}{7x}$ 2

$= \frac{6}{7}$ $\left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right)$

(c) $f(x) = 4\cos^{-1}\frac{x}{3}$

(i) $D: -1 \le \frac{x}{3} \le 1$

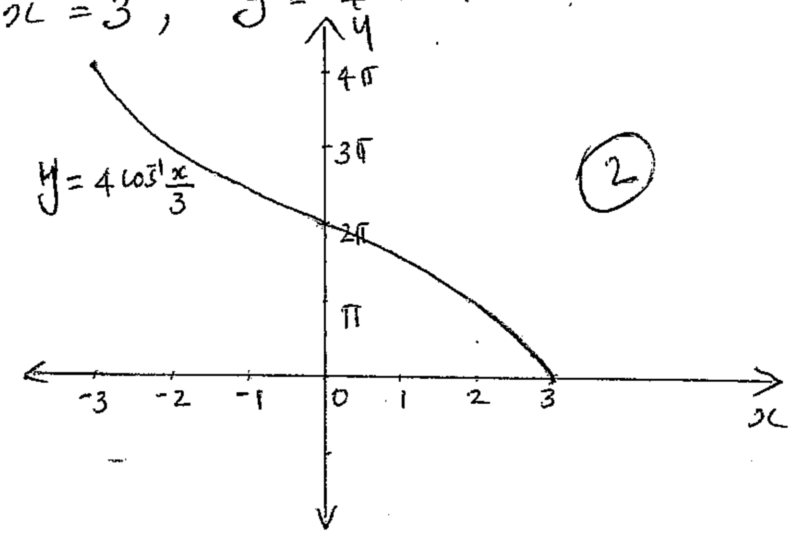
$-3 \le x \le 3$ 2

$R: 0 \le y \le 4\pi$

$x=-3$, $y = 4\cos^{-1}(-1) = 4\pi$

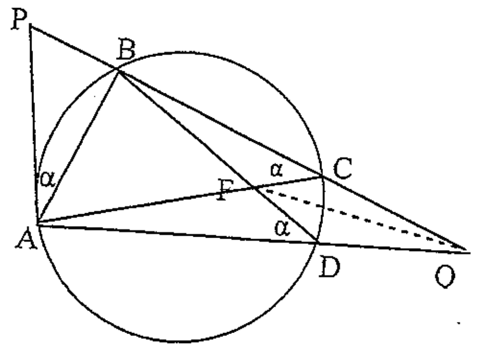
$x=0$, $y = 4\cos^{-1}0 = 4\times\frac{\pi}{2} = 2\pi$

$x=3$, $y = 4\cos^{-1}1 = 0$



2

(d)



$\angle PAB = \angle ACB$ (angle between tangent and chord is equal to the angle in the alternate segment)

$\angle ACB = \angle ADB$ (angles at the circumference standing on the same chord)

$\angle BCA = \angle CQF + \angle CFQ$ (exterior angle of ΔFQC)

$\angle ADF = \angle DFQ + \angle DQF$ (exterior angle of ΔFQD)

$\angle BCA + \angle ADF$

$= \angle CQF + \angle CFQ + \angle DFQ + \angle DQF$

$= \angle CQF + \angle DQF + \angle CFQ + \angle DFQ$

$= \angle CQD + \angle CFD$

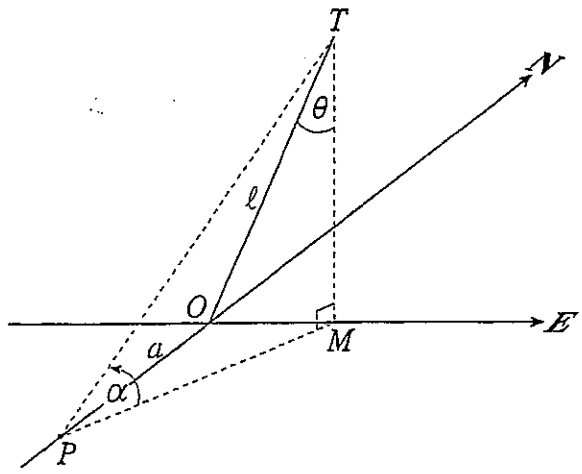
But $\angle BCA + \angle ADF = 2\alpha$

$2\alpha = \angle CQD + \angle CFD$ 4

$\alpha = \frac{1}{2}(\angle CQD + \angle CFD)$

Question 3 (12 marks)

(a)



(i) $\sin\theta = \frac{OM}{l}$

$OM = l\sin\theta$ 2

$\cos\theta = \frac{MT}{l}$; $MT = l\cos\theta$

(ii) $\cot\alpha = \frac{PM}{MT}$; $PM = MT\cot\alpha$ 1
$= l\cos\theta\cot\alpha$

(iii) $PM^2 - OM^2 = a^2$

$l^2\cos^2\theta\cot^2\alpha - l^2\sin^2\theta = a^2$

$l^2(\cos^2\theta\cot^2\alpha - \sin^2\theta) = a^2$

$l^2 = \frac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$ 3

(iv) $l^2 = \frac{25^2}{\cos^2 20^\circ \cot^2 24^\circ - \sin^2 20^\circ}$

$l = 12$ 1

(b) Total numbers of ways in which 16 people can be seated $= 8P_4 \times 8P_2 \times 10!$ (2)

(c) (i) $P(x=3) = {}^5C_3 (0.4)^3 (0.6)^2$

$= 0.2304$

(ii) $P(x=3) + P(x=4) + P(x=5)$

$= 0.2304 + {}^5C_4 (0.4)^4 (0.6)^1$

$+ {}^5C_5 (0.4)^5 (0.6)^0$

$= 0.31744$ (3)

(iii) $P(x=0) + P(x=1) + P(x=2)$

$= 1 - 0.31744$

$= \underline{\underline{0.68256}}$

Question 4 (12 marks)

(a) when $n = 2$,

$LHS = \log \frac{2}{2-1} = \log 2$

$RHS = \log 2$

$LHS = RHS$

$\therefore$ the result is true for $n=2$

Assume the result is true for $n = k$

ie $\log 2 + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{k}{k-1}\right)$

$= \log k$ —— (1)

To prove that the result is true for $n = k+1$

ie $\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots$

$\cdots + \log\left(\frac{k}{k-1}\right) + \log\left(\frac{k+1}{k}\right) = \log k+1$

Now

$\log 2 + \log\frac{3}{2} + \cdots + \log\left(\frac{k}{k-1}\right) + \log\left(\frac{k+1}{k}\right)$

$= \log k + \log\left(\frac{k+1}{k}\right)$

by assumption (1)

$= \log\left(k \times \frac{k+1}{k}\right)$

$= \log(k+1)$ (3)

$\therefore$ the result is true for $n = k+1$

Hence by the principle of mathematical induction, the result is true for $n \geq 2$

(b) (i) Normal at P

$x + py = 4p^3 + 8p$ —— (1)

Normal at Q

$x + qy = 4q^3 + 8q$ —— (2)

① − ② gives

$$y(p-q) = 4(p^3-q^3)+8(p-q)$$

$$y(p-q) = 4(p-q)(p^2+pq+q^2)+8(p-q)$$

$$y = 4(p^2+pq+q^2)+8$$
$$= 4(p^2-1+q^2)+8$$
$$= 4p^2-4+4q^2+8$$
$$= 4p^2+4q^2+4$$

Substitute in ①

$$x + p(4p^2+4q^2+4) = 4p^3+8p$$

$$x+4p^3+4pq^2+4p = 4p^3+8p$$

$$x + 4pq^2 = 4p$$

$$x + 4pq \times q = 4p$$

$$x - 4q = 4p$$

$$x = 4p+4q$$
$$= 4(p+q)$$

$$x = 4(p+q) \text{ — ③}$$

$$y = 4p^2+4q^2+4 \text{ — ④}$$

From ③ $p+q = \frac{x}{4}$

From ④ $\frac{y}{4} = p^2+q^2+1$

$$p^2+q^2 = \frac{y}{4}-1$$

$$p^2+q^2 = (p+q)^2 - 2pq$$
$$= (p+q)^2 - 2\times -1$$
$$= (p+q)^2+2$$

$$\frac{y}{4} - 1 = \left(\frac{x}{4}\right)^2+2$$

$$\frac{y}{4} - 1 = \frac{x^2}{16}+2$$

$$4y-16 = x^2+32$$ (3)

$$x^2 = 4y - 48$$

$$\underline{\underline{x^2 = 4(y-12)}}$$

(c) $y = \frac{2x^2-2}{x^2-9} = \frac{2(x+1)(x-1)}{(x+3)(x-3)}$

Vertical asymptotes

$x = -3$ and $x = 3$

$$\lim_{x\to\infty} \frac{2x^2-2}{x^2-9}$$ (3)

$$= \lim_{x\to\infty} \frac{2-\frac{2}{x^2}}{1-\frac{9}{x^2}}$$

$$= 2 \left(\text{since } \lim_{x\to\infty} \frac{1}{x} = 0\right)$$

Horizontal asymptote is

$$y = 2$$

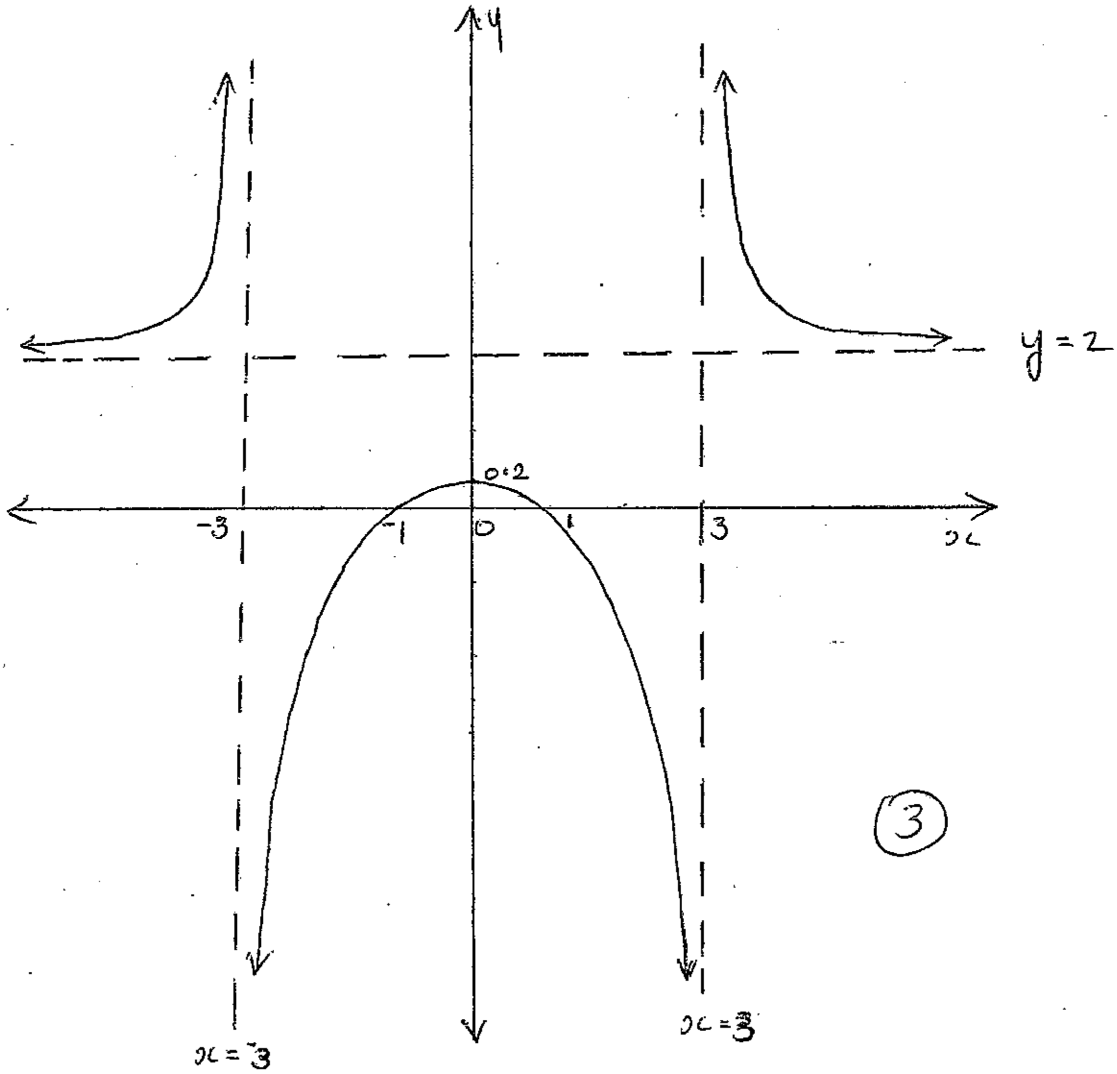
x intercepts

$y=0 \Longrightarrow 2(x+1)(x-1)=0$

$x=-1$ or $x=1$

y intercept

$x=0 \Longrightarrow y = \dfrac{-2}{-9} = \dfrac{2}{9} = 0{\cdot}2$



3

Question 5 (12 marks)

(a) $(1+x)^5\,(1+x)^5 = (1+x)^{10}$

$\left[{}^5C_0 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + {}^5C_4 x^4 + {}^5C_5 x^5\right]\left[{}^5C_0 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + {}^5C_4 x^4 + {}^5C_5 x^5\right] = {}^{10}C_0 + {}^{10}C_1 x + {}^{10}C_2 x^2 + {}^{10}C_3 x^3 + {}^{10}C_4 x^4$

$+ {}^{10}C_5 x^5 + \cdots + {}^{10}C_{10} x^{10}$

Equating coefficients of x^5 on both sides,

$$^5C_0 \times {}^5C_5 + {}^5C_1 \times {}^5C_4 + {}^5C_2 \times {}^5C_3 + {}^5C_3 \times {}^5C_2 + {}^5C_4 \times {}^5C_1$$

$$+ {}^5C_5 \times {}^5C_0 = {}^{10}C_5$$

$$^5C_0 \times {}^5C_0 + {}^5C_1 \times {}^5C_1 + {}^5C_2 \times {}^5C_2 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_4$$

$$+ {}^5C_5 \times {}^5C_5 = {}^{10}C_5 \quad (\text{since } {}^nC_r = {}^nC_{n-r})$$

$$({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2 = {}^{10}C_5$$

ie $$\sum_{k=0}^{5} ({}^5C_k)^2 = {}^{10}C_5 \qquad 3$$

(b) (i) $$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n \qquad 1$$

(ii) $$\int_0^1 (1+x)^n \, dx = \int_0^1 ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n) \, dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[{}^nC_0 x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + \cdots + {}^nC_n \frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \cdots + \frac{{}^nC_n}{n+1}$$

3

$$\underline{\underline{{}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \cdots + \frac{{}^nC_n}{n+1} = \frac{2^{n+1}-1}{n+1}}}$$

(c) (i) $T = A + Ce^{kt}$

$$\frac{dT}{dt} = Ce^{kt} \times k$$

$$= k \times Ce^{kt} \quad 2$$

$$= k(T-A) \quad (\text{since } Ce^{kt} = T-A)$$

$\therefore T = A + Ce^{kt}$ is a solution of the equation

$$\frac{dT}{dt} = k(T-A)$$

(ii) $T = 25 + Ce^{kt}$

when $t=0$, $T = 10°C$

$$10 = 25 + C$$

$$C = 10 - 25 = -15$$

$$\therefore T = 25 - 15e^{kt}$$

when $t = 20$, $T = 15^{\circ}$

$$15 = 25 - 15e^{20k}$$

$$15e^{20k} = 10$$

$$e^{20k} = \frac{10}{15}$$

$$20k = \log\left(\frac{10}{15}\right)$$

$$k = \frac{1}{20}\log\left(\frac{10}{15}\right)$$

when $t = 50$, $T = ?$

$$T = 25 - 15e^{50k}$$

$$= 25 - 15e^{50 \times \frac{1}{20}\log\left(\frac{10}{15}\right)} \quad 2$$

$= 20°C$ (to the nearest degree)

(iii) $k = \frac{1}{20}\log\left(\frac{10}{15}\right) = -0.02$

$$T = 25 - 15e^{-0.02t}$$

Since $k < 0$, as $t \to \infty$ $e^{kt} \to 0$

$\therefore$ as t increases indefinitely the object's temperature approaches air temperature. 1

Question 6 (12 marks)

(a) (i) $f(x) = x\sin^{-1}x + \sqrt{1-x^2}$

$$f'(x) = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 1 + \frac{1}{2\sqrt{1-x^2}} \times -2x$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

$$\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) = \sin^{-1}x \quad 3$$

$$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}}\right) - (0 + \sqrt{1})$$

$$= \frac{1}{2} \times \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$$

$$= 0.128 \quad 2$$

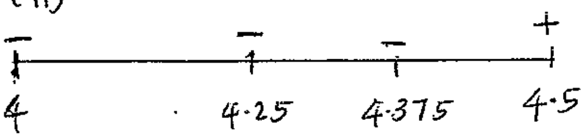
(b) (I) $\tan x - x = 0$

$\tan 4 - 4 = -2.8$

$\tan 4.5 - 4.5 = 0.14$

Since $f(4) < 0$ and $f(4.5) > 0$ there is a root of $f(x) = 0$ between $x = 4$ and $x = 4.5$ ①

(II)



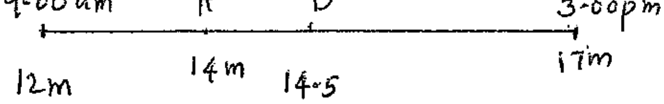
$f(4.25) = \tan 4.25 - 4.25$
$= -2.24$

$f(4.375) = \tan 4.375 - 4.375$
$= -1.52$

Approximate value of the root ②

$= \dfrac{4.375 + 4.5}{2} = 4.4$

(C) (i)



$T = 2 \times (3.00\text{pm} - 9.00\text{am})$
$= 2 \times 6h$
$= 12h$

$T = \dfrac{2\pi}{n}$ $\quad n = \dfrac{2\pi}{T} = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$

Let 9.00 am denote $t = 0$

$x = a\cos(nt + \pi)$
$= 2.5\cos\left(\frac{\pi}{6}t + \pi\right)$

when the harbour is 14 m deep

$x = -0.5$

$-0.5 = 2.5\cos\left(\frac{\pi}{6}t + \pi\right)$

$\cos\left(\frac{\pi}{6}t + \pi\right) = \frac{-1}{5}$

$\frac{\pi}{6}t + \pi = \pi - \cos^{-1}\left(\frac{1}{5}\right), \pi + \cos^{-1}\left(\frac{1}{5}\right)$

$\frac{\pi t}{6} = -1.3694, 1.369$

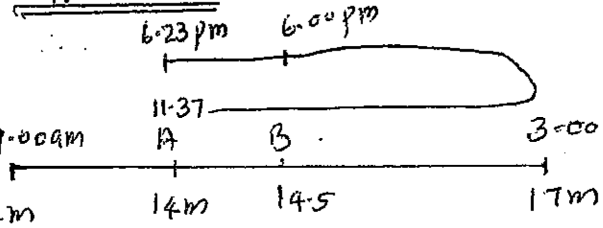
$\frac{\pi t}{6} = 1.3694$ (t can't be negative)

$t = 1.3694 \times \frac{6}{\pi}$
$= 2.6154$
$= 2$ hr 37 minutes. ③

Time taken from A to B
$= 3$ hr $- 2$ hr 37 min
$= 23$ min

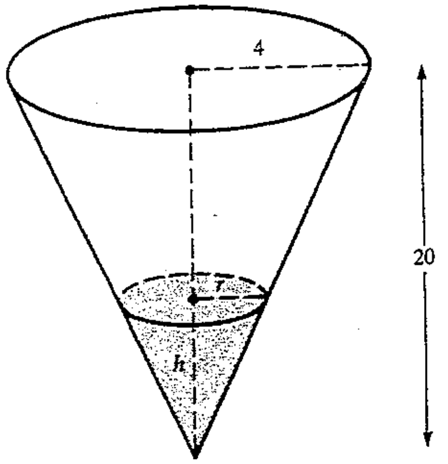
The ship can go into the harbour at 11.37 am

(ii)



The ship must depart before 5.53 pm. ①

Question 7 (12 marks)



$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$

By similar triangles

$\frac{r}{4} = \frac{h}{20}$

$r = \frac{4h}{20} = \frac{h}{5}$

$V = \frac{1}{3}\pi \times \frac{h^2}{25} \times h$

$= \frac{\pi h^3}{75}$ (3)

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\frac{dV}{dh} = \frac{\pi \times 3h^2}{75} = \frac{\pi h^2}{25}$

$12 = \frac{\pi h^2}{25} \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{12 \times 25}{\pi h^2}$

when $h = 5$,

$\frac{dh}{dt} = \frac{12 \times 25}{\pi \times 25} = 3 \cdot 82$

The fluid level is dropping at the rate of $3 \cdot 82$ cm/s.

(b) $f(x) = x^3 + x^2 - 10$

$f'(x) = 3x^2 + 2x$

$a = 2$

$a_1 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \left[\frac{8+4-10}{3\times 4+4}\right] = 2 - \frac{1}{8} = 1 \cdot 875$

$= 1 \cdot 9$

$a_2 = 1 \cdot 875 - \frac{f(1 \cdot 875)}{f'(1 \cdot 875)}$

$= 1 \cdot 875 - \left[\frac{(1 \cdot 875)^3 + (1 \cdot 875)^2 - 10}{3 \times (1 \cdot 875)^2 + 2 \times 1 \cdot 875}\right]$

$= 1 \cdot 867 = 1 \cdot 9$ (2)

$a_1 = a_2 = 1 \cdot 9$

$\therefore$ the root of $f(x) = 0$ correct to one decimal place is $1 \cdot 9$

(c) (i) When the particle strikes the ground $y = 0$

$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(V\sin\alpha - \frac{1}{2}gt) = 0$

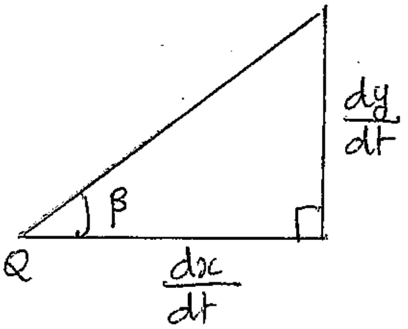
$t = 0$ or $V\sin\alpha = \frac{1}{2}gt$

$t = 0$ or $2V\sin\alpha = gt$

$t = 0$ or $t = \dfrac{2V\sin\alpha}{g}$

Now $t=0$ refers to the instant of projection (2) and hence $t = \dfrac{2V\sin\alpha}{g}$ is the required time, the time of flight.

(ii)



(1)

$$\tan\beta = \frac{dy/dt}{dx/dt}$$

$$= \underline{\underline{\frac{V\sin\alpha - gt}{V\cos\alpha}}}$$

(iii) $$\frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$$

$$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - gt\cos\beta$$

$$gt\cos\beta = V\sin\alpha\cos\beta - V\sin\beta\cos\alpha$$

$$gt\cos\beta = V(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$gt\cos\beta = V\sin(\alpha - \beta)$$ (2)

$$\underline{\underline{t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}}}$$

(iv) when $\beta = \frac{\alpha}{2}$ we have

$$t = \frac{V\sin(\alpha - \frac{\alpha}{2})}{g\cos\frac{\alpha}{2}} = \frac{V\sin\frac{\alpha}{2}}{g\cos\frac{\alpha}{2}}$$

$$= \frac{V}{g}\tan\frac{\alpha}{2}$$

Given that $\dfrac{V}{g}\tan\dfrac{\alpha}{2} = \dfrac{1}{3}\,\dfrac{2V\sin\alpha}{g}$

$$\tan\frac{\alpha}{2} = \frac{1}{3}\times 2\sin\alpha$$

$$3\tan\frac{\alpha}{2} = 2\sin\alpha$$

$$= 2\times\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$3 = \frac{4}{1+\tan^2\frac{\alpha}{2}}$$

$$4 = 3 + 3\tan^2\frac{\alpha}{2}$$ (2)

$$3\tan^2\frac{\alpha}{2} = 1$$

$$\tan^2\frac{\alpha}{2} = \frac{1}{3}$$

$$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}} \quad \left(\frac{\alpha}{2} \text{ is acute}\right)$$

$$\frac{\alpha}{2} = \frac{\pi}{6}$$

$$\underline{\underline{\alpha = \frac{\pi}{3}}}$$