Sydney Grammar School

2011 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Wednesday 10th August 2011

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 126 boys

Examiner

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QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $\dfrac{(n+1)!}{n!}$. **1**

(b) Find $\displaystyle\int \frac{1}{9+x^2}\,dx$. **1**

(c) When the polynomial $P(x) = x^3 + 3x^2 + ax - 10$ is divided by $x - 2$, the remainder is 24. Find a. **2**

(d) Differentiate $y = \sin^{-1}(x^3)$. **2**

(e) Suppose that α, β and γ are the roots of the equation $x^3 - 3x^2 - 4x + 12 = 0$.

(i) Write down the value of $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Hence find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **1**

(f) (i) Without the use of calculus, sketch the polynomial $y = x(x+1)(x-4)$ showing all the intercepts with the axes. **2**

(ii) Hence, or otherwise, solve the inequation $\dfrac{x(x+1)}{x-4} \geq 0$. **2**

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\sin^{-1}(\sin \frac{2\pi}{3})$. **1**

(b) Find $\lim_{x \to \infty} \dfrac{3 - x}{2x + 3}$. **1**

(c) The point A is $(2, -4)$ and the point B is $(5, 2)$. The point P divides the interval AB externally in the ratio $4:1$. Find the coordinates of P. **2**

(d) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = \pi$. **2**

(e) A ball is projected vertically upwards from the ground. After t seconds, the height of the ball is given by $h = 45t - 5t^2$ metres.

(i) At what time does the ball returns to the ground? **1**

(ii) When is the ball instantaneously at rest? **1**

(iii) What is the greatest height attained by the ball? **1**

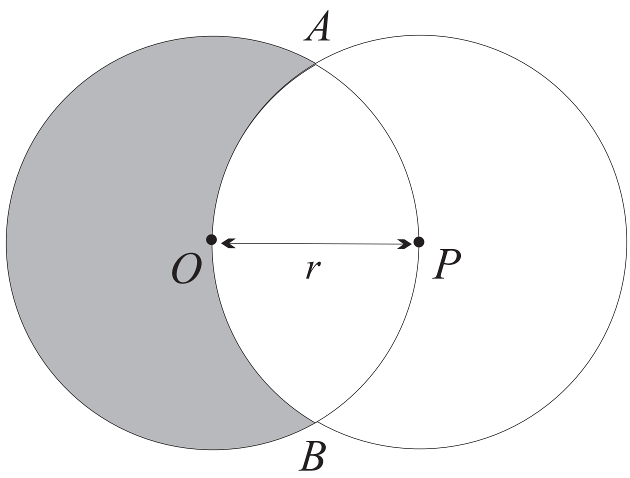
(f) (i) Sketch the graph of the function $y = \left|x^2 - 4\right|$. **2**

(ii) At what points is $f(x) = \left|x^2 - 4\right|$ not differentiable? **1**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) State the domain and range of $f(x) = 2\cos^{-1}\frac{x}{4}$. **2**

(b)



In the diagram above, two circles of equal radius r units are drawn such that their centres O and P are r units apart. The two circles intersect at A and B.

(i) Show that the quadrilateral $AOBP$ is a rhombus. **1**

(ii) Show that $\angle AOB = 120°$. **1**

(iii) Find the area of the shaded region in terms of r. **2**

(c) The function $f(x) = x\log x + x - 1{\cdot}1$ has a zero near $x = 1$. Take $x = 1$ as a first approximation and use Newton's method once to obtain a closer approximation to this zero. **3**

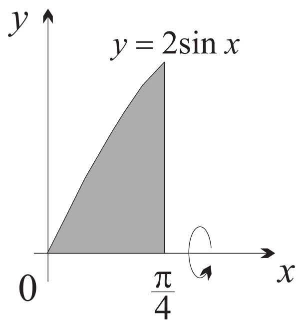
(d) Find the term independent of x in the expansion of $\left(4x^3 - \dfrac{1}{x}\right)^{12}$. **3**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Given that α is an acute angle and $\cos\alpha = \frac{3}{4}$, find the exact value of $\tan\frac{\alpha}{2}$. **2**

(b) Using the substitution $u = 4x + 1$, evaluate $\displaystyle\int_0^1 \frac{4x}{(4x+1)^2}\,dx$. **3**

(c)



The diagram above shows the region bounded by the curve $y = 2\sin x$, the x-axis and the line $x = \frac{\pi}{4}$. Find the exact volume of the solid generated when the shaded region is rotated about the x-axis. **3**

(d) A particle is moving in a straight line according to the equation

$$x = \sqrt{3}\cos 3t - \sin 3t\,,$$

where x metres is its displacement from the origin after t seconds.

(i) Show that the particle is moving in simple harmonic motion. **2**

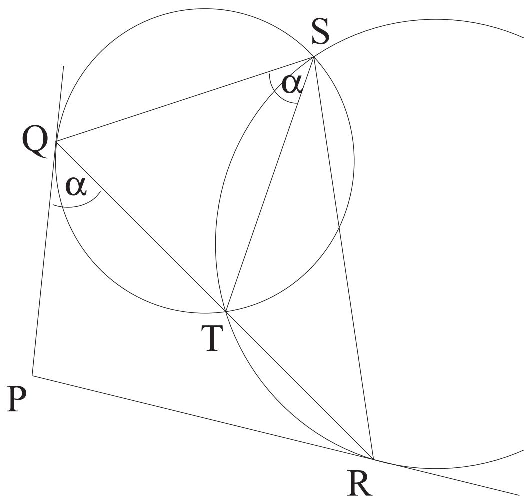
(ii) Find the time at which the particle first passes through the origin. **2**

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, **4**

$$\frac{1}{3} \times \frac{1}{1} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{7} \times \frac{1}{5} + \cdots + \frac{1}{(2n+1)} \times \frac{1}{(2n-1)} = \frac{n}{2n+1}.$$

(b)

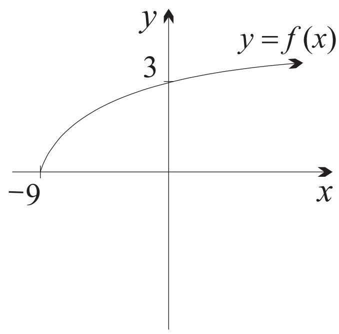


In the diagram above PQ and PR are tangents to the circles SQT and STR respectively, and the points Q, T and R are collinear.

(i) Given that $\angle QST = \alpha$, state a reason why $\angle PQT = \alpha$. **1**

(ii) Prove that $PQSR$ is a cyclic quadrilateral. **2**

(c)



The diagram above shows a sketch of $y = f(x)$ where $f(x) = \sqrt{x+9}$.

(i) Copy the diagram. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$, clearly marking the x and y-intercepts. **1**

(ii) What is the domain of $f^{-1}(x)$? **1**

(iii) Find an expression for $f^{-1}(x)$. **1**

(iv) Given that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at the point P, find the x-coordinate of P. **2**

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) When an object falls from rest at $t = 0$ through a resisting liquid, the rate of change of its velocity at time t is given by $\dfrac{dv}{dt} = -k(v - 600)$, where k is a positive constant.

(i) Show that $v = 600 + Pe^{-kt}$ is a solution to the differential equation for some constant P. **1**

(ii) If the velocity of the object at $t = 3\,\text{s}$ is $25\,\text{ms}^{-1}$, find P and k. **2**

(iii) Find the velocity of the object at $t = 10\,\text{s}$. Give your answer correct to one decimal place. **1**

(iv) What is the limiting value of v as $t \to \infty$? **1**

(b) Let $(2x + y)^{12} = \sum_{k=0}^{12} T_k$ where $T_k = {}^{12}\text{C}_k \times (2x)^{12-k} \times y^k$.

(i) Show that $\dfrac{T_{k+1}}{T_k} = \dfrac{y(12 - k)}{2x(k + 1)}$. **1**

(ii) Suppose that $x = 4$ and $y = 5$ in the expansion of $(2x + y)^{12}$. Show that there are two consecutive terms that are equal, and greater in value than any of the other terms. **2**

(c) (i) Find the general solutions of the equation **3**

$$2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0\,.$$

(ii) Hence write down all the solutions in the domain $0 \le x \le \pi$. **1**

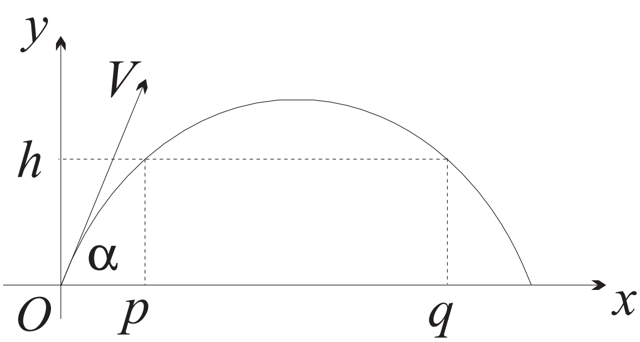
Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Using the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$, show that **2**

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

(b)



A particle is projected from a point O at an angle of elevation α with level ground at an initial velocity $V\,\text{ms}^{-1}$, as in the diagram above.

The particle just clears two vertical poles of height h metres at horizontal distances of p and q metres from O. Take acceleration due to gravity as $10\,\text{ms}^{-2}$ and ignore air resistance. You may assume the equations of motion:

$$x = Vt\cos\alpha$$

$$y = Vt\sin\alpha - 5t^2$$

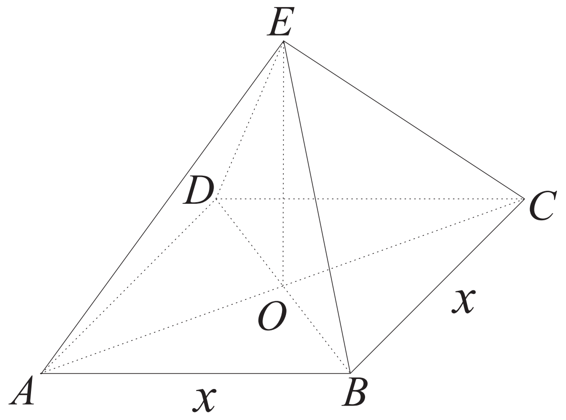
(i) Find an expression for V^2 in terms of α, p and h. **2**

(ii) Hence show that $\tan\alpha = \dfrac{h(p+q)}{pq}$. **2**

Question Seven continues on the next page

QUESTION SEVEN (Continued)

(c)



A square pyramid has its apex vertically above the centre of the base. The square base has side length x and the volume of the pyramid is V. The area of each triangular face is $\dfrac{S}{4}$ for some constant S.

(i) Show that $S^2 = x^4 + \dfrac{36V^2}{x^2}$. **2**

(ii) Prove that if V is constant and x is variable, then S has its minimum value when **2**

$$x^3 = (3\sqrt{2})V.$$

(iii) When S is at its minimum, show that each triangular face is equilateral. **2**

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0,\text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}:\ \ln x = \log_e x,\ x > 0$$

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Question 1

a) $\dfrac{(n+1)!}{n!} = n+1$

b) $\displaystyle\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$

c) $P(2) = 24$

$2^3 + 3(2)^2 + a(2) - 10 = 24$

$8 + 12 + 2a - 10 = 24$

$2a = 14$

$a = 7$

d) $y = \sin^{-1} x^3$

$\dfrac{dy}{dx} = \dfrac{3x^2}{\sqrt{1-x^6}}$

e) $x^3 - 3x^2 - 4x + 12 = 0$

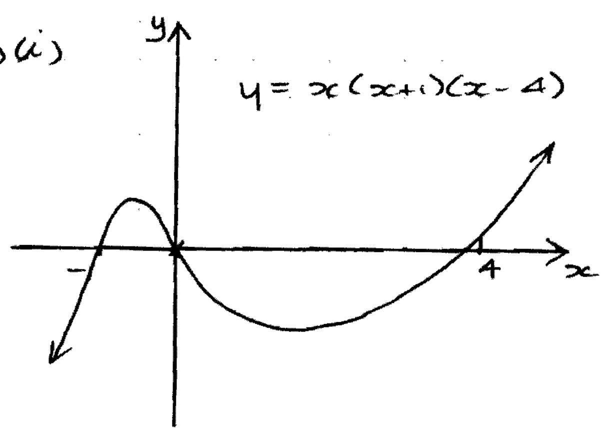
(i) $\alpha\beta + \alpha\gamma + \beta\gamma = -4$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \dfrac{-4}{-12}$

$= \dfrac{1}{3}$

f) (i)



(ii) $\dfrac{x(x+1)}{x-4} \geq 0$

$\dfrac{x(x+1)(x-4)^2}{(x-4)} \geq 0, \quad x \neq 4$

$x(x+1)(x-4) \geq 0$

$-1 \leq x \leq 0, \quad x > 4$

Question 2

a) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\frac{\sqrt{3}}{2}$

$= \dfrac{\pi}{3}$

b) $\displaystyle\lim_{x\to\infty} \frac{3-x}{2x+3} = -\frac{1}{2}$

c) $A(2,-4) \qquad B(5,2)$

$-4 : 1$

$P = \left(\dfrac{2-20}{-3}, \dfrac{-4-8}{-3}\right)$

$= (6, 4)$

d) $y = \tan^{-1}(\sin x)$

$\dfrac{dy}{dx} = \dfrac{\cos x}{1+\sin^2 x}$

at $x = \pi$, $\dfrac{dy}{dx} = \dfrac{-1}{1+0}$

$= -1$

$\therefore$ slope of tangent at $x = \pi$ is -1

e) $h = 45t - 5t^2$

(i) $45t - 5t^2 = 0$

$5t(9-t) = 0$

$t = 0$ or $t = 9$

$\therefore$ returns to ground after 9 seconds

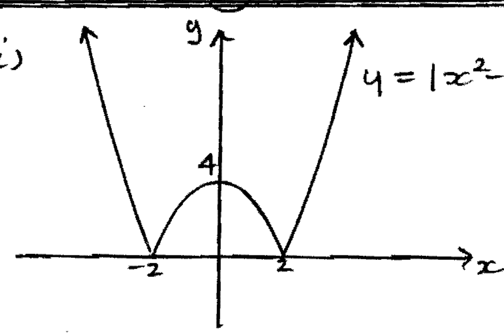
(ii) ball will be at rest at greatest height which by symmetry is after $4\frac{1}{2}$ seconds

(iii) $t = \frac{9}{2}$, $h = 45\left(\frac{9}{2}\right) - 5\left(\frac{9}{2}\right)^2$

$= \dfrac{405}{4}$

$\therefore$ greatest height is $101\frac{1}{4}$ m

f) (i)



(ii) function is not differentiable at $x = \pm 2$

Question 3

a) domain: $-1 \le \frac{x}{4} \le 1$

$-4 \le x \le 4$

range: $0 \le \frac{y}{2} \le \pi$

$0 \le y \le 2\pi$

b) $OA = OB = PB = PA = r$ (= radii)

$\therefore$ AOBP is a rhombus (4 = sides)

(ii) $OP = r$ (given)

$\therefore \triangle AOP$ is equilateral (3 = sides)

$\angle AOP = 60°$ ($\angle$ in equilateral $\triangle$)

Similarly $\angle BOP = 60°$

$\angle AOB = \angle AOP + \angle BOP$

$= 120°$

(iii) $A = \pi r^2 - 2\left[\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\right]$

$= r^2[\pi - \theta + \sin\theta]$

$= r^2\left[\pi - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right]$

$= \frac{r^2(\pi + 3\sqrt{3})}{6}$

c) $f(x) = x\log x + x - 1.1$

$f'(x) = (x)(\frac{1}{x}) + (\log x)(1) + 1 - 1.1$

$= \log x + 0.9$

$x_1 = 1 - \frac{(1)\log(1) + 1 - 1.1}{\log(1) + 0.9}$

$= 1\frac{1}{9}$

d) $T_{k+1} = {}^{12}C_k (4x^3)^{12-k}(-\frac{1}{x})^k$

$x^{36-3k} \cdot x^{-k} = x^0$

$36 - 4k = 0$

$k = 9$

$T_{10} = {}^{12}C_9\, 4^3 (-1)^9$

$= -14080$

Question 4

a) $\frac{1-t^2}{1+t^2} = \frac{3}{4}$

$4 - 4t^2 = 3 + 3t^2$

$7t^2 = 1$

$t^2 = \frac{1}{7}$

$t = \pm\frac{1}{\sqrt{7}}$

as α is acute, so is $\frac{\alpha}{2}$

$\therefore \tan\frac{\alpha}{2} = \frac{1}{\sqrt{7}}$

b) $\int_0^1 \frac{4x}{(4x+1)^2}\,dx$

$u = 4x + 1$

$du = 4dx$

$x = 0, u = 1$

$x = 1, u = 5$

$= \frac{1}{4}\int_1^5 \frac{u-1}{u^2}\,du$

$= \frac{1}{4}\int_1^5 \left[\frac{1}{u} - \frac{1}{u^2}\right]du$

$= \frac{1}{4}\left[\log u + \frac{1}{u}\right]_1^5$

$= \frac{1}{4}\left(\log 5 + \frac{1}{5} - \log 1 - 1\right)$

$= \frac{1}{4}\left(\log 5 - \frac{4}{5}\right)$

c) $V = \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$

$= 2\pi\int_0^{\frac{\pi}{4}}(1 - \cos 2x)\,dx$

$= 2\pi\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= 2\pi\left(\frac{\pi}{4} - \frac{1}{2} - 0 + 0\right)$

$= \frac{\pi^2 - \pi}{2}$ units^3

d)(i) $x = \sqrt{3}\cos 3t - \sin 3t$

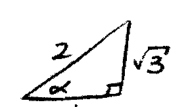
$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9x$

$\therefore$ particle moves in SHM

(ii) $\sqrt{3}\cos 3t - \sin 3t = 0$



$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$2\sin(3t - \frac{\pi}{3}) = 0$

$3t - \frac{\pi}{3} = 0$

$3t = \frac{\pi}{3}$

$t = \frac{\pi}{9}$

$\therefore$ particle first passes through origin after $\frac{\pi}{9}$ seconds.

Question 5

a) $n=1$

$LHS = \frac{1}{3} \times \frac{1}{1}$ $\quad$ $RHS = \frac{1}{2(1)+1}$

$= \frac{1}{3}$ $\quad$ $= \frac{1}{3}$

$LHS = RHS$

Hence the result is true for $n=1$

Assume the result is true for $n=k$ where k is a positive integer

ie $\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)} = \frac{k}{2k+1}$

Prove true for $n=k+1$

ie Prove

$\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+3)}\times\frac{1}{(2k+1)} = \frac{k+1}{2k+3}$

PROOF

$\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)} + \frac{1}{(2k+3)}\times\frac{1}{(2k+1)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+3)}\times\frac{1}{(2k+1)}$

$= \frac{k(2k+3)+1}{(2k+3)(2k+1)}$

$= \frac{2k^2+3k+1}{(2k+3)(2k+1)}$

$= \frac{(2k+1)(k+1)}{(2k+3)(2k+1)}$

$= \frac{k+1}{2k+3}$

Hence the result is true for $n=k+1$ if it is true for $n=k$.

Since the result is true for $n=1$ then it is true for all positive integral values of n by induction.

b) (i) $\angle PQT = \alpha$ by the alternate segment theorem.

(ii) Let $\angle RST = \beta$

$\angle TRP = \beta$ (alternate segment theorem)

$\angle QPR + \angle PQT + \angle TRP = 180$ ($\angle$ sum Δ)

$\therefore \angle QPR = 180 - \alpha - \beta$

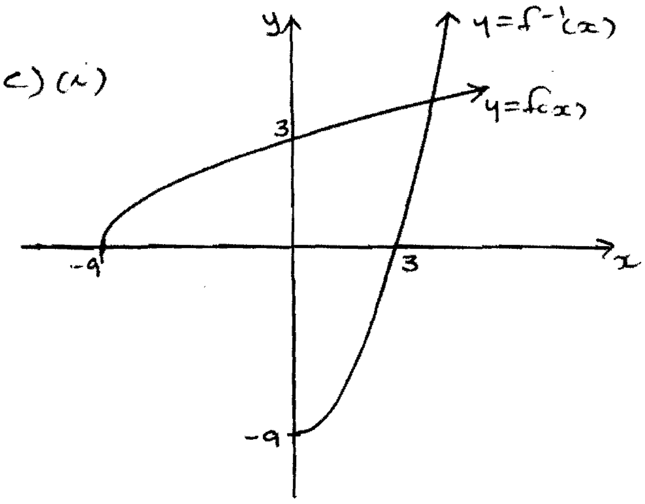
$\angle QSR = \angle QST + \angle RST$ (common $\angle$)

$\angle QSR = \alpha + \beta$

$\therefore \angle QPR + \angle QSR = 180°$

PQSR is a cyclic quadrilateral as opposite $\angle$'s supplementary

c) (i)



(ii) domain $f^{-1}(x)$: $x \geq 0$

(iii) f^{-1}: $x = \sqrt{y+9}$

$x^2 = y+9$

$y = x^2 - 9, \ x \geq 0$

(iv) meet on line $y = x$

$x = x^2 - 9$

$x^2 - x - 9 = 0$

$x = \frac{1 \pm \sqrt{37}}{2}$

but $x \geq 0$

$\therefore$ x coordinate of P is

$x = \frac{1+\sqrt{37}}{2}$

Question 6

a) (i) $v = 600 + Pe^{-kt}$

$\frac{dv}{dt} = -kPe^{-kt}$

$= -k(Pe^{-kt} + 600 - 600)$

$= -k(v - 600)$

(ii) when $t=0, v=0$

ie $0 = 600 + Pe^0$

$P = -600$

when $t=3$, $v=25$

ie $25 = 600 - 600e^{-3k}$

$600e^{-3k} = 575$

$e^{-3k} = \frac{23}{24}$

$-3k = \log\frac{23}{24}$

$k = -\frac{1}{3}\log\frac{23}{24}$

(iii) when $t=10$,

$v = 600 - 600e^{-10k}$

$= 600 - 600e^{\frac{10}{3}\log\frac{23}{24}}$

$= 600 - 600\left(\frac{23}{24}\right)^{\frac{10}{3}}$

$= 79.35716261\dots$

$= 79.4$ m/s

(iv) as $t \to \infty$, $e^{-kt} \to 0$

$\therefore$ $V \to 600$ m/s

b)(i) $$\frac{T_{k+1}}{T_k} = \frac{{}^{12}C_{k+1}(2x)^{11-k}\, y^{k+1}}{{}^{12}C_k\,(2x)^{12-k}\, y^k}$$

$$= \frac{12!}{(11-k)!(k+1)!} \times \frac{(12-k)!\,k!}{12!} \times \frac{y}{2x}$$

$$= \frac{12-k}{k+1} \times \frac{y}{2x}$$

$$= \frac{y(12-k)}{2x(k+1)}$$

(ii) If $T_{k+1} \geq T_k$ then T_{k+1} is the greatest term

ie $\frac{T_{k+1}}{T_k} \geq 1$

$$\frac{y(12-k)}{2x(k+1)} \geq 1$$

$$\frac{5(12-k)}{8(k+1)} \geq 1$$

$$60 - 5k \geq 8k + 8$$

$$13k \leq 52$$

$$k \leq 4$$

$\therefore T_4 = {}^{12}C_4\, 8^8 5^4$ and

$T_5 = {}^{12}C_5\, 8^7 5^5$ are both the greatest terms.

c) $2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0$

$$2\cos 3x(\sin 4x + 1) - (\sin 4x + 1) = 0$$

$$(\sin 4x + 1)(2\cos 3x - 1) = 0$$

$\sin 4x = -1$ or $\cos 3x = \frac{1}{2}$

$4x = \pi k + (-1)^k \sin^{-1}(-1)$

or

$3x = 2\pi k \pm \cos^{-1}\frac{1}{2}$

where k is an integer

$4x = \pi k + (-1)^k \frac{3\pi}{2}$ or $3x = 2\pi k \pm \frac{\pi}{3}$

$= 2\pi k + \frac{3\pi}{2}$ $\qquad = \frac{\pi(6k \pm 1)}{3}$

$= \frac{\pi(3+4k)}{2}$

$x = \frac{\pi(3+4k)}{8}$ or $x = \frac{\pi(6k \pm 1)}{9}$

where k is an integer

(ii) $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

Question 7

a) coefficient of x^n in $(1+2x)^{2n}$

$$= \binom{2n}{n}$$

coefficient of x^n in $(1+x)^n(1+x)^n$

$$\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]^2$$

coefficient of x^n

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$$

however

$$\binom{n}{k} = \binom{n}{n-k}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2$$

$$\therefore \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

b) $x = Vt\cos\alpha$

$$t = \frac{x}{V\cos\alpha}$$

$$y = Vt\sin\alpha - 5t^2$$

$$= x\tan\alpha - \frac{5x^2}{V^2\cos^2\alpha}$$

$$\therefore h = p\tan\alpha - \frac{5p^2}{V^2\cos^2\alpha}$$

$$\frac{5p^2}{V^2\cos^2\alpha} = p\tan\alpha - h$$

$$V^2\cos^2\alpha = \frac{5p^2}{p\tan\alpha - h}$$

$$V^2 = \frac{5p^2}{\cos^2\alpha\,(p\tan\alpha - h)}$$

(ii) similarly

$$V^2 = \frac{5q^2}{\cos^2\alpha\,(q\tan\alpha - h)}$$

$$\therefore \frac{5p^2}{\cos^2\alpha(p\tan\alpha - h)} = \frac{5q^2}{\cos^2\alpha\,(q\tan\alpha - h)}$$

$$p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$$

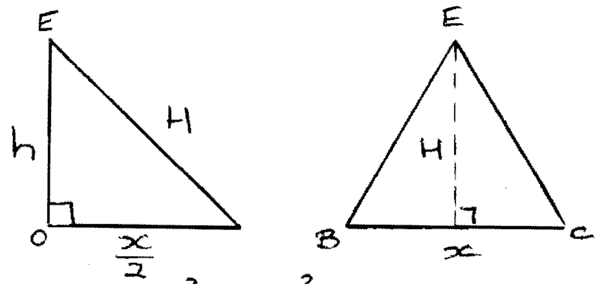
$$(p^2q - pq^2)\tan\alpha = hp^2 - hq^2$$

$$\tan\alpha = \frac{h(p+q)(p-q)}{pq(p-q)}$$

$$= \frac{h(p+q)}{pq}$$

c) $V = \frac{1}{3}x^2 h$

$h = \frac{3V}{x^2}$



$H^2 = \frac{9V^2}{x^4} + \frac{x^2}{4}$

$\frac{S}{4} = \frac{1}{2}xH$

$S = 2xH$

$S^2 = 4x^2H^2$

$= 4x^2\left(\frac{9V^2}{x^4} + \frac{x^2}{4}\right)$

$= \frac{36V^2}{x^2} + x^4$

(ii) $\frac{dS^2}{dx} = -\frac{72V^2}{x^3} + 4x^3$

stationary pts occur when $\frac{dS^2}{dx} = 0$

ie $4x^3 = \frac{72V^2}{x^3}$

$(x^3)^2 = 18V^2$

$x^3 = 3\sqrt{2}V$

x^3	$4V$	$3\sqrt{2}V$	$5V$
$\frac{dS^2}{dx}$	$-2V$ \	0 —	$\frac{28}{5}V$ /

∴ when $x^3 = 3\sqrt{2}V$, S has a minimum value

(iii) $V = \frac{x^3}{3\sqrt{2}}$

∴ $h = \frac{3x^3}{3\sqrt{2}} \times \frac{1}{x^2}$

$= \frac{x}{\sqrt{2}}$

$H^2 = \frac{x^2}{2} + \frac{x^2}{4}$

$= \frac{3x^2}{4}$

$H = \frac{\sqrt{3}x}{2}$

$\tan \angle ECB = \frac{H}{\frac{x}{2}}$

$= \frac{\sqrt{3}x}{2} \times \frac{2}{x}$

$= \sqrt{3}$

$\angle ECB = 60°$

$\angle ECB = \angle EBC$

∴ $\angle EBC = 60°$

Thus $\triangle EBC$ is equilateral.