Sydney Grammar School

2011 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 2nd August 2011

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist
- SGS booklets — 8 per boy
- Candidature — 85 boys

Examiner
DS

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\displaystyle\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Find $\displaystyle\int \frac{1}{\sqrt{x^2-12x+61}}\,dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx$. **3**

(d) Use the substitution $x=\sqrt{2}\sin\theta$ to find the exact value of $\displaystyle\int_0^1 \frac{x^2}{\sqrt{2-x^2}}\,dx$. **4**

(e) Find $\displaystyle\int \frac{x(x+9)}{(x+3)(x^2+9)}\,dx$. **4**

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Express $\dfrac{23-14i}{3-4i}$ in the form $a+bi$, where a and b are real. **2**

(b) Find the two square roots of $-16+30i$. **2**

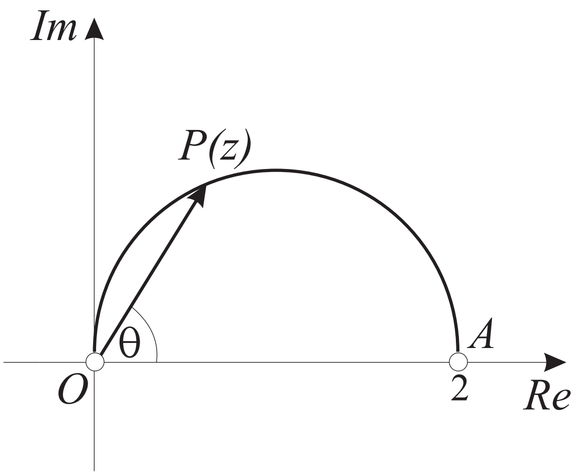
(c) Let $w=-\sqrt{3}+i$.

(i) Express w in modulus–argument form. **2**

(ii) Show that $w^9+512i=0$. **2**

(d) Shade the region in the complex plane where $|z+2|\le 2$ and $-\frac{\pi}{6}\le \arg(z+3)\le\frac{\pi}{3}$ are simultaneously satisfied. **3**

(e)



The diagram above shows the semicircular locus of the point P that represents the complex number z.
Let $\arg z=\theta$, as shown on the diagram.

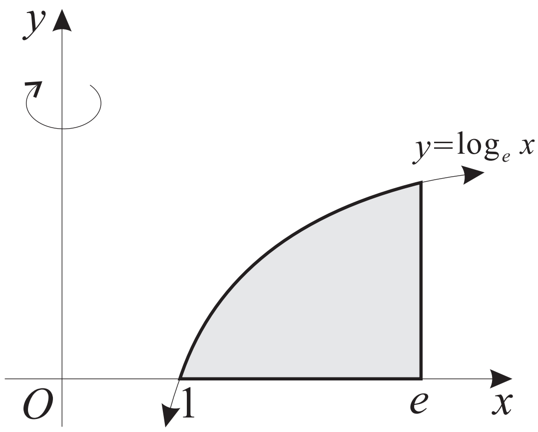
(i) Copy the diagram and on it show a vector representing $z-2$. **1**

(ii) Explain why $\left|\dfrac{z-2}{z}\right|=\tan\theta$. **1**

(iii) Show that $\arg\left(\dfrac{z-2}{z}\right)=\dfrac{\pi}{2}$. **2**

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $y = \log_e x$, the x-axis and the vertical line $x = e$. The region is rotated about the y-axis to form a solid.

(i) Find the volume of the solid by slicing perpendicular to the axis of rotation. **3**

(ii) Find the volume of the solid by the method of cylindrical shells. **4**

(b) It is known that $5 + 6i$ is a zero of the polynomial $P(x) = 2x^3 - 19x^2 + 112x + d$, where d is real.

(i) What are the other two zeroes of $P(x)$? **2**

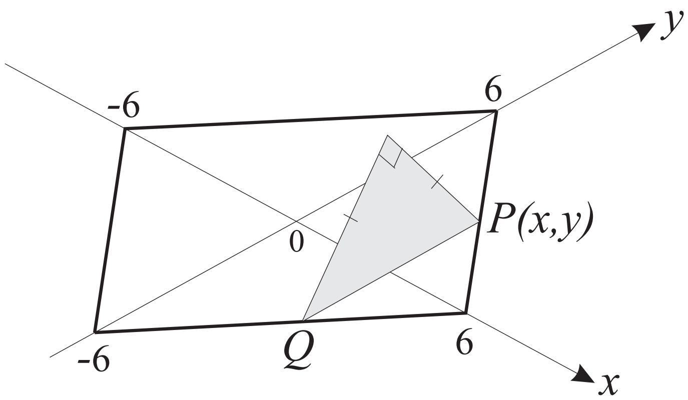
(ii) Find the value of d. **2**

(c) The polynomial equation $2x^3 - x^2 + 5 = 0$ has roots α, β and γ. Find a polynomial equation with integer coefficients whose roots are α^3, β^3 and γ^3. **4**

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a)



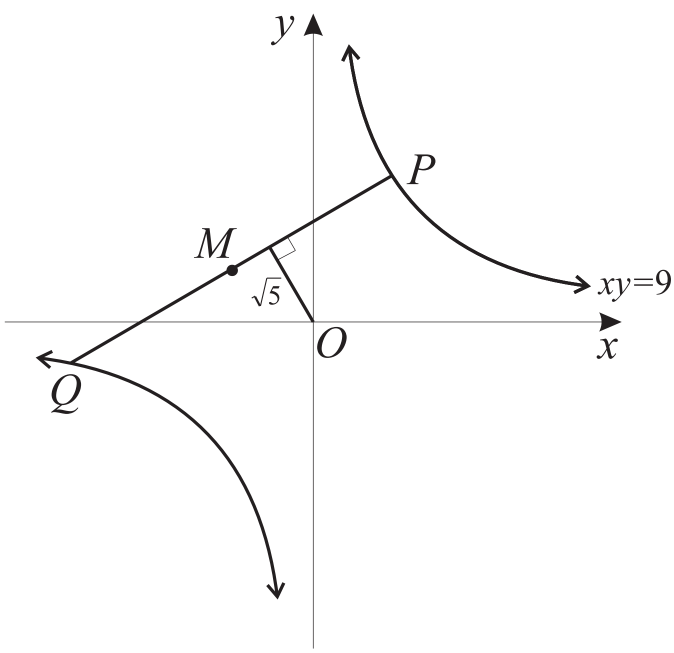
The diagram above shows the horizontal square base of a solid. Vertical cross-sections of the solid perpendicular to the x-axis are right-angled isosceles triangles with hypotenuse in the base.

(i) Find, as a function of x, the area of the typical cross-section standing on the interval PQ. **2**

(ii) Find the volume of the solid. **2**

QUESTION FOUR (Continued)

(b)



In the diagram above, $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ are variable points on the rectangular hyperbola $xy = 9$. The perpendicular distance from the origin to the chord PQ is $\sqrt{5}$ units. Let M be the midpoint of the chord PQ.

(i) Show that the chord PQ has equation $x + pqy = 3(p + q)$. **2**

(ii) Using the perpendicular distance formula, or otherwise, show that **1**

$$9(p+q)^2 = 5(1 + p^2q^2).$$

(iii) Show that the locus of M has Cartesian equation $y^2 = \dfrac{5x^2}{4x^2 - 5}$. **3**

(c) Suppose that $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. **5**

So $H(1) = 1$, $H(2) = 1 + \frac{1}{2}$, $H(3) = 1 + \frac{1}{2} + \frac{1}{3}$, and so on.

Prove by mathematical induction that

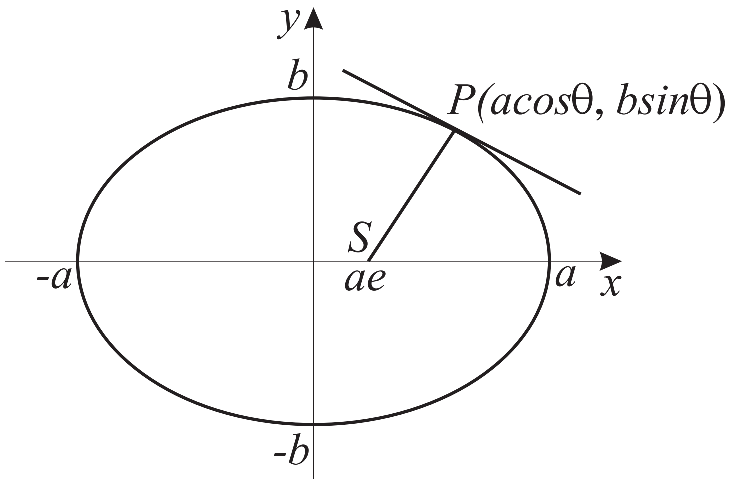
$$n + H(1) + H(2) + H(3) + \cdots + H(n-1) = nH(n)$$

for $n = 2, 3, 4, \ldots$.

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $1 + 2x - x^2 > \dfrac{2}{x}$. **4**

(b)



The diagram above shows the variable point $P(a\cos\theta, b\sin\theta)$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(i) Find the gradient of the tangent at P. **1**

(ii) Show that the product of the gradient of the interval SP and the gradient of the tangent at P is **2**

$$\frac{\cos\theta(1 - e^2)}{e - \cos\theta}.$$

(iii) Prove that SP is never perpendicular to the tangent at P, provided that $\theta \neq 0$ or π. **2**

(c) (i) Use de Moivre's theorem to find expressions for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$. **2**

(ii) Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. **1**

(iii) By letting $\theta = \frac{\pi}{12}$ in part (ii), show that $\tan\frac{\pi}{12}$ is a root of the equation **1**

$$x^3 - 3x^2 - 3x + 1 = 0.$$

(iv) <u>Hence</u> find the exact value of $\tan\frac{\pi}{12}$. **2**

Exam continues overleaf ...

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

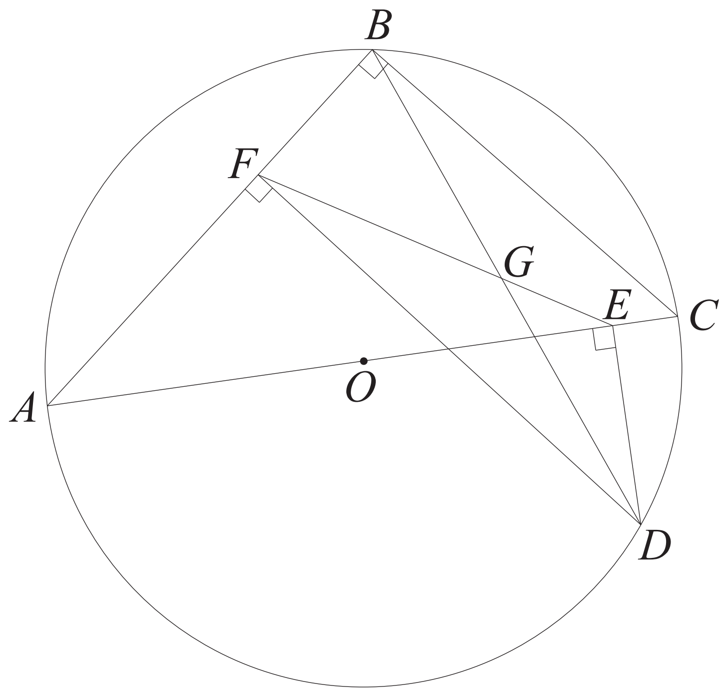
(a) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.

(i) Use integration by parts to show that $I_n = (n-1)\displaystyle\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$. **2**

(ii) Hence show that $I_n = \dfrac{n-1}{n} I_{n-2}$, for $n = 2, 3, 4, \ldots$. **1**

(iii) Find the exact value of $I_9 \times I_{10}$. **2**

(b)



In the diagram above, triangle ABC is right-angled at B. Its circumcircle is drawn, with centre O. A point D is chosen on the circumcircle, then DE and DF are drawn perpendicular to AC and AB respectively. The point G is the intersection of DB and EF.

NOTE: You do not have to copy the diagram. It has been reproduced for you on a tear-off sheet at the end of the paper. Insert the tear-off sheet into your answer booklet.

(i) Explain why $ADEF$ is a cyclic quadrilateral. **1**

(ii) Let $\angle DAE = \theta$.
Prove that $\triangle FGB$ is isosceles. **2**

(iii) Prove that $ODEG$ is a cyclic quadrilateral. **2**

(iv) Deduce that OG is perpendicular to BD. **1**

Exam continues next page . . .

QUESTION SIX (Continued)

(c) Let $P(x)$ be a polynomial of degree n, where n is odd.
It is known that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

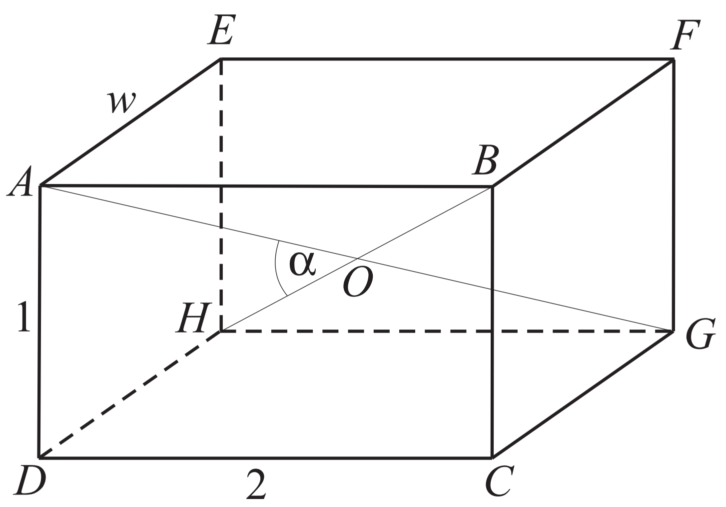
(i) Write down the zeroes of the polynomial $(x+1)P(x) - x$. **1**

(ii) Let A be the leading coefficient of the polynomial $(x+1)P(x) - x$. Factorise the polynomial, and hence find A. **2**

(iii) Find $P(n+1)$. **1**

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a)



In the rectangular prism above $DC = 2$, $AD = 1$ and $AE = w$. Angle α is the acute angle between the diagonals AG and BH, which intersect at O. Let r be the ratio of the volume of the prism to its surface area.

(i) Show that $AG^2 = 5 + w^2$. **1**

(ii) Show that $\cos\alpha = \dfrac{|3 - w^2|}{5 + w^2}$. **2**

(iii) Show that $r < \frac{1}{3}$ for all possible values of w. **2**

(iv) If $r \geq \frac{1}{4}$, prove that $\alpha \leq \cos^{-1}\frac{1}{9}$. **2**

QUESTION SEVEN (Continued)

(b) A particle of mass 2 kg experiences a resistive force, in Newtons, of 10% of the square of its velocity v metres per second when it moves through the air. The particle is projected vertically upwards from a point A with velocity u metres per second. The highest point reached is B, directly above A. Assume that $g = 10\,\text{ms}^{-2}$, and take upwards as the positive direction.

(i) Show that the acceleration of the particle as it rises is given by **1**

$$\ddot{x} = -\,\frac{v^2 + 200}{20}\,.$$

(ii) Show that the distance x metres of the particle from A as it rises is given by **2**

$$x = 10\log_e\left(\frac{200 + u^2}{200 + v^2}\right).$$

(iii) Show that the time t seconds that the particle takes to reach a velocity of v metres per second is given by **2**

$$t = \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right).$$

(iv) Now suppose that we take two of the 2 kg particles described above. **3**
One of the particles is projected upwards from A with initial velocity $10\sqrt{2}\,\text{ms}^{-1}$, then, $\frac{3\sqrt{2}}{5}$ seconds later, the other particle is projected upwards from A with initial velocity $30\sqrt{2}\,\text{ms}^{-1}$. Will the second particle catch up to the first particle before the first particle reaches its maximum height? You must explain your reasoning and show your working.

Exam continues next page ...

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) Show that $\dfrac{1+\cos\alpha}{\sin\alpha} = \tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right)$. **2**

(b) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin x}\,dx$, where $0 < \alpha < \frac{\pi}{2}$.

(i) Use the substitution $t = \tan\frac{x}{2}$ to show that **3**

$$I = \int_0^1 \frac{2}{(t+\cos\alpha)^2 + \sin^2\alpha}\,dt\,.$$

(ii) Use the further substitution $t + \cos\alpha = \sin\alpha\tan u$ and the result in part (a) above to show that $I = \dfrac{\alpha}{\sin\alpha}$. **4**

(c) (i) Find, in modulus–argument form, the roots of the equation $z^{2n+1} = 1$. **2**

(ii) Hence factorise $z^{2n} + z^{2n-1} + \cdots + z^2 + z + 1$ into quadratic factors with real coefficients. **2**

(iii) Deduce that **2**

$$2^n \sin\tfrac{\pi}{2n+1}\sin\tfrac{2\pi}{2n+1}\cdots\sin\tfrac{n\pi}{2n+1} = \sqrt{2n+1}\,.$$

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

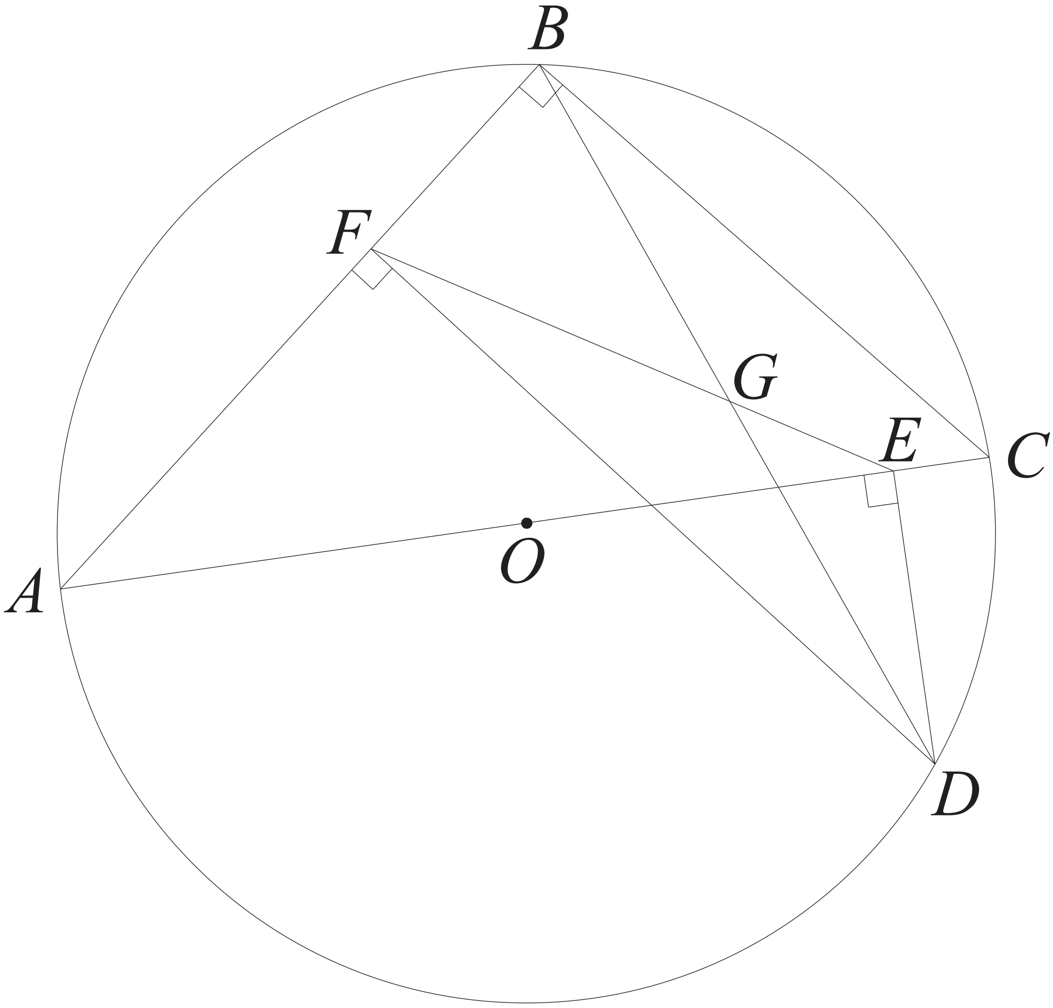
$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

CANDIDATE NUMBER: ....

DETACH THIS SHEET AND BUNDLE IT WITH THE REST OF QUESTION SIX.

<u>QUESTION SIX</u>

(b)



Sydney Grammar Extension 2 2011

Question 1

a) $\int_0^1 xe^{-x^2}\,dx$

$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$

$= -\frac{1}{2}(e^{-1} - 1)$

$= \frac{1}{2}(1 - e^{-1})$

b) $\int \frac{dx}{\sqrt{x^2 - 12x + 61}}$

$= \int \frac{dx}{\sqrt{(x-6)^2 + 25}}$

$= \log(x - 6 + \sqrt{x^2 - 12x + 61}) + c$

c) $\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x(1 + \tan^2 x)\tan x\,dx$

$= \int_0^{\frac{\pi}{4}} (\tan x + \tan^3 x)\sec^2 x\,dx$

$= \left[\frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{2} + \frac{1}{4} - 0\right)$

$= \frac{3}{4}$

d) $\int_0^1 \frac{x^2}{\sqrt{2 - x^2}}\,dx$

$x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\,d\theta$

$x = 0,\ \theta = 0$

$x = 1,\ \theta = \frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \frac{2\sin\theta \cdot \sqrt{2}\cos\theta\,d\theta}{\sqrt{2}\cos\theta}$

$= 2\int_0^{\frac{\pi}{4}} \sin\theta\,d\theta$

$= -2\left[\cos\theta\right]_0^{\frac{\pi}{4}}$

$= -2\left(\frac{1}{\sqrt{2}} - 1\right)$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right)$

e) $\int \frac{x(x+9)}{(x+3)(x^2+9)}\,dx$

$A(x^2+9) + (Bx + C)(x+3) \equiv x(x+9)$

$x = -3$: $18A = -18$, $A = -1$

$x = 0$: $9A + 3C = 0$, $3C = 9$, $C = 3$

$x = 1$: $10A + 4B + 4C = 10$, $4B = 8$, $B = 2$

$\int \frac{x(x+9)}{(x+3)(x^2+9)}\,dx$

$= \int \left[\frac{-1}{x+3} + \frac{2x}{x^2+9} + \frac{3}{x^2+9}\right]dx$

$= -\log(x+3) + \log(x^2+9) + \tan^{-1}\frac{x}{3} + c$

$= \log\left(\frac{x^2+9}{x+3}\right) + \tan^{-1}\frac{x}{3} + c$

Question 2

a) $\frac{23 - 14i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i}$

$= \frac{69 + 92i - 42i + 56}{9 + 16}$

$= \frac{125 + 50i}{25}$

$= 5 + 2i$

b) $a^2 - b^2 = -16$ $\quad 2ab = 30$

$a^2 - \frac{225}{a^2} = -16$ $\quad b = \frac{15}{a}$

$a^4 + 16a^2 - 225 = 0$

$(a^2 + 25)(a^2 - 9) = 0$

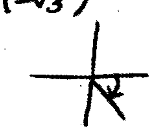
$a^2 = -25$ or $a^2 = 9$

no real solution $\quad a = \pm 3$

$\therefore \sqrt{-16 + 30i} = \pm(3 + 5i)$

c)(i) $|w| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\arg w = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$



$\therefore w = 2\text{cis}\left(-\frac{\pi}{6}\right)$

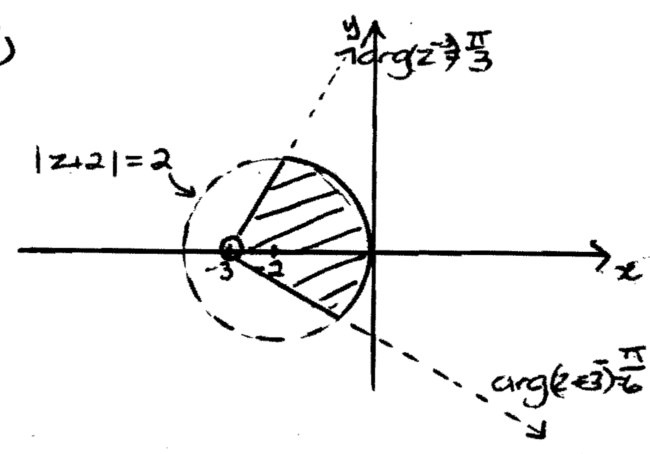
(ii) $w^9 + 512i = 2^9\text{cis}\left(-\frac{9\pi}{6}\right) + 512i$

$= 512\text{cis}\left(-\frac{3\pi}{2}\right) + 512i$

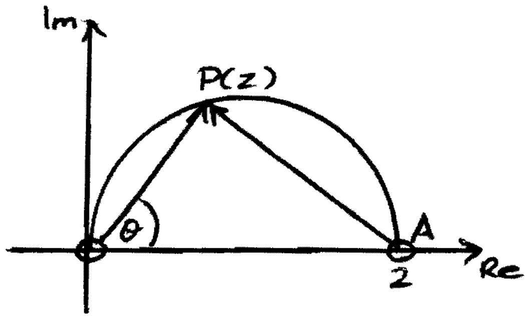
$= -512i + 512i$

$= 0$

d)



e)(i)



(ii) $\angle OPA = 90^\circ$ ($\angle$ in semicircle)

$\therefore \triangle OPA$ is right angled

$\tan\theta = \frac{AP}{OP}$

$= \frac{|z-2|}{|z|}$

$= \left|\frac{z-2}{z}\right|$

(iii) $\theta + \angle OPA = \angle PAR$ (exterior $\angle$ $\triangle$)

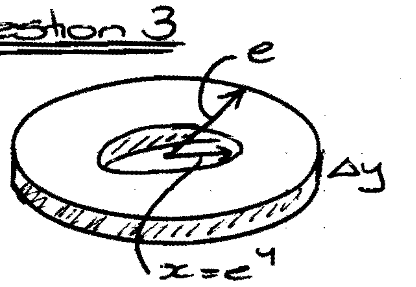
$\arg z + \frac{\pi}{2} = \arg(z-2)$

$\arg(z-2) - \arg z = \frac{\pi}{2}$

$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{2}$

Question 3

a)(i)



$A(y) = \pi(e^2 - e^{2y})$

$\Delta V = \pi(e^2 - e^{2y})\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(e^2 - e^{2y})\Delta y$

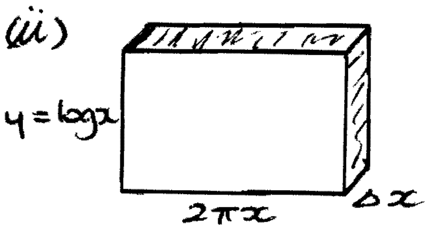
$= \pi\int_0^1 (e^2 - e^{2y})\,dy$

$= \pi\left[e^2 y - \frac{1}{2}e^{2y}\right]_0^1$

$= \pi(e^2 - \frac{1}{2}e^2 - 0 + \frac{1}{2})$

$= \frac{1}{2}\pi(e^2+1)$ units^3

(ii)



$A(x) = 2\pi x \log x$

$\Delta V = 2\pi x \log x \,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=1}^{e} 2\pi x \log x \,\Delta x$

$= 2\pi\int_1^e x\log x\,dx$

$u = \log x$ $\quad$ $v = \frac{1}{2}x^2$

$du = \frac{dx}{x}$ $\quad$ $dv = x\,dx$

$V = 2\pi\left[\frac{1}{2}x^2\log x\right]_1^e - \pi\int_1^e x\,dx$

$= \pi\left[x^2\log x - \frac{1}{2}x^2\right]_1^e$

$= \pi(e^2 - \frac{1}{2}e^2 - 0 + \frac{1}{2})$

$= \frac{1}{2}\pi(e^2+1)$ units^3

b) $\alpha + \beta + \gamma = \frac{19}{2}$

$\alpha = 5+6i$, $\beta = 5-6i$

as roots appear in conjugate pairs

$\therefore 10 + \gamma = \frac{19}{2}$

$\gamma = -\frac{1}{2}$

other two zeroes are $5-6i$ and $-\frac{1}{2}$

(ii) $\alpha\beta\gamma = -\frac{d}{2}$

$(5+6i)(5-6i)(-\frac{1}{2}) = -\frac{d}{2}$

$d = 25+36$

$d = 61$

c) let $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$

$2y - y^{\frac{2}{3}} + 5 = 0$

$y^{\frac{2}{3}} = 2y + 5$

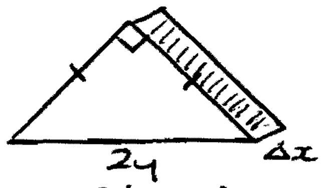
$y^2 = (2y+5)^3$

$= 8y^3 + 60y^2 + 150y + 125$

$8y^3 + 59y^2 + 150y + 125 = 0$

Question 4

a)



$2y = 2(6-x)$

$A(x) = \frac{1}{2}\times\frac{1}{2}(2y)^2$

$= y^2$

$\underline{A(x) = (6-x)^2}$

(ii) $\Delta V = (6-x)^2 \Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=-6}^{6} (6-x)^2 \Delta x$

$= 2\int_0^6 (6-x)^2\,dx$

$= -\frac{2}{3}\left[(6-x)^3\right]_0^6$

$= -\frac{2}{3}(0 - 6^3)$

$= \underline{144 \text{ units}^3}$

b) $m_{PQ} = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}$

$= \dfrac{q-p}{pq(p-q)}$

$= \dfrac{-1}{pq}$

$y - \frac{3}{p} = \frac{-1}{pq}(x - 3p)$

$pqy - 3q = -x + 3p$

$\underline{x + pqy = 3(p+q)}$

(ii) $\dfrac{|0 + 0 - 3(p+q)|}{\sqrt{1^2 + p^2q^2}} = \sqrt{5}$

$\dfrac{9(p+q)^2}{1 + p^2q^2} = 5$

$\underline{9(p+q)^2 = 5(1 + p^2q^2)}$

(iii) $M\left(\dfrac{3p+3q}{2}, \dfrac{\frac{3}{p}+\frac{3}{q}}{2}\right)$

$= \left(\frac{3}{2}(p+q), \frac{3}{2}\frac{p+q}{pq}\right)$

$y^2 = \frac{9}{4} \times \frac{(p+q)^2}{p^2q^2}$ ~~$\frac{5x^2}{4x^2-5} = \frac{45}{4}(p+q)^2$~~

$= \frac{5(1+p^2q^2)}{4p^2q^2}$ ×

$= \frac{5}{4p^2q^2} + \frac{5}{4}$

$\frac{5x^2}{4x^2-5} = \frac{5}{4} + \frac{25}{4} \times \frac{1}{4x^2-5}$

$4x^2 - 5 = 4 \times \frac{9}{4}(p+q)^2 - 5$

$= 9(p+q)^2 - 5$

$= 5 + 5p^2q^2 - 5$

$= 5p^2q^2$

$\therefore \frac{5x^2}{4x^2-5} = \frac{5}{4} + \frac{25}{4} \times \frac{1}{5p^2q^2}$

$= \frac{5}{4} + \frac{5}{4p^2q^2}$

$= \underline{y^2}$

c) $\underline{n=2}$

LHS $= 2 + H(1)$
$= 2 + 1$
$= 3$

RHS $= 2H(2)$
$= 2(1 + \frac{1}{2})$
$= 3$

$\therefore$ LHS = RHS

Hence the result is true for $n=2$

Assume the result is true for $n=k$, where k is integer ≥ 2

ie $k + H(1) + H(2) + \dots + H(k-1) = kH(k)$

Prove true for $n = k+1$

ie Prove

$k+1 + H(1) + H(2) + \dots + H(k) = (k+1)H(k+1)$

Proof

$k+1 + H(1) + H(2) + \dots + H(k-1) + H(k)$

$= kH(k) + 1 + H(k)$

$= (k+1)H(k) + 1$

$= (k+1)\left[H(k) + \frac{1}{k+1}\right]$

$= (k+1)\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1}\right]$

$= (k+1)H(k+1)$

Hence the result is true for $n = k+1$, if it is true for $n=k$.

Since the result is true for $n=2$, then it is true for all integers ≥ 2 by induction.

Question 5

a) $1 + 2x - x^2 > \frac{2}{x}$

$x \neq 0$

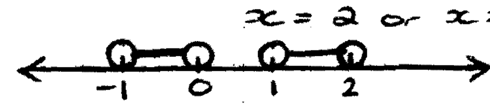
$1 + 2x - x^2 = \frac{2}{x}$

$x^3 - 2x^2 - x + 2 = 0$

$x^2(x-2) - (x-2) = 0$

$(x-2)(x^2-1) = 0$

$x = 2$ or $x = \pm 1$



$\underline{-1 < x < 0, \quad 1 < x < 2}$

b) $x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$$\therefore m_T = \frac{-b\cos\theta}{a\sin\theta}$$

(ii) $m_T \times m_{PS}$

$$= \frac{-b\cos\theta}{a\sin\theta} \times \frac{b\sin\theta}{a\cos\theta - ae}$$

$$= \frac{-b^2\cos\theta}{a^2(\cos\theta - e)}$$

$$= \frac{a^2(1-e^2)\cos\theta}{a^2(\cos\theta - e)}$$

$$= \frac{(1-e^2)\cos\theta}{(e - \cos\theta)}$$

(iii) $\perp$ if $m_T \times m_{PS} = -1$

$$(1-e^2)\cos\theta = \cos\theta - e$$

$$\cos\theta - e^2\cos\theta = \cos\theta - e$$

$$e^2\cos\theta = e$$

$$\cos\theta = \frac{1}{e}$$

for an ellipse $0<e<1$

$\therefore \frac{1}{e} > 1$

but $\cos\theta < 1$

Thus SP can never be perpendicular to tangent.

c) $\cos 3\theta + i\sin 3\theta$

$= (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

(ii) $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$

$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(iii) let $\theta = \frac{\pi}{12}$

$$\tan\frac{\pi}{4} = \frac{3\tan\frac{\pi}{12} - \tan^3\frac{\pi}{12}}{1 - 3\tan^2\frac{\pi}{12}}$$

$$1 =$$

$$1 - 3\tan^2\frac{\pi}{12} = 3\tan\frac{\pi}{12} - \tan^3\frac{\pi}{12}$$

$$\tan^3\frac{\pi}{12} - 3\tan^2\frac{\pi}{12} - 3\tan\frac{\pi}{12} + 1 = 0$$

$\therefore \tan\frac{\pi}{12}$ is a root of

$$x^3 - 3x^2 - 3x + 1 = 0$$

(iv) $(x+1)(x^2 - 4x + 1) = 0$

$\therefore \tan\frac{\pi}{12}$ is a root of $x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{12}}{2}$$

$$x = 2 \pm \sqrt{3}$$

but $\tan\frac{\pi}{12} < \tan\frac{\pi}{4} = 1$

$$\therefore \tan\frac{\pi}{12} = 2 - \sqrt{3}$$

Question 6

a) $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$

$u = \sin^{n-1}\theta \qquad v = -\cos\theta$

$du = (n-1)\sin^{n-2}\theta\cos\theta \, d\theta \qquad dv = \sin\theta \, d\theta$

$$I_n = \left[-\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta \, d\theta$$

(ii) $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta) \, d\theta$

$$= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2}\theta - \sin^n\theta) \, d\theta$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$n I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

(iii) $I_9 \times I_{10} = \frac{8}{9} I_7 \times \frac{9}{10} I_8$

$$= \frac{8}{10} \times \frac{6}{7} I_5 \times \frac{7}{8} I_6$$

$$= \frac{6}{10} \times \frac{4}{5} I_3 \times \frac{5}{6} I_4$$

$$= \frac{4}{10} \times \frac{2}{3} I_1 \times \frac{3}{4} I_2$$

$$= \frac{2}{10} \times I_1 \times \frac{1}{2} I_0$$

$$= \frac{1}{10} \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \times \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{10}\left[-\cos\theta\right]_0^{\frac{\pi}{2}} \times \left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{10} \times 1 \times \frac{\pi}{2}$$

$$= \frac{\pi}{20}$$

b) $\angle AFD = \angle AED = 90^\circ$ (given)

$\therefore$ ADEF is cyclic quadrilateral as $\angle$'s in same segment =

further AD is diameter as $\angle$ in semicircle $= 90^\circ$.

(ii) $\angle DAE = \angle DFE = \theta$ ($\angle$'s in same segment)

$\angle BFD = \angle DFE + \angle EFB$ (common $\angle$)

$90 = \theta + \angle EFB$

$\angle EFB = 90 - \theta$

$\angle DAC = \angle DBC = \theta$ ($\angle$'s in same segment)

$\angle CBF = \angle DBC + \angle DBF$ (common $\angle$)

$90 = \theta + \angle DBF$

$\angle DBF = 90 - \theta = \angle EFB$

$\therefore$ ΔFGB is isosceles (2 = $\angle$'s)

(iii) $\angle BGF + \angle EFB + \angle DBF = 180$ ($\angle$ sum ΔFGB)

$\angle BGF + 90 - \theta + 90 - \theta = 180$

$\angle BGF = 2\theta$

$\angle DGE = \angle BGF = 2\theta$ (vertically opposite $\angle$'s =)

$\angle COD = 2\angle CAD$ ($\angle$ at centre twice $\angle$ at circumference standing on same arc)

$= 2\theta$

$\therefore \angle COD = \angle DGE$

$\therefore$ ODEG is cyclic quadrilateral as $\angle$'s in same segment =

(iv) $\angle OGD = \angle OED = 90^\circ$ ($\angle$'s in same segment =)

ie $OG \perp BD$

c) (i) $Q(x) = (x+1)P(x) - x$

$Q(k) = (k+1)\cdot\frac{k}{k+1} - k$

$= 0$

$\therefore$ zeroes of $Q(x)$ are $k = 0, 1, 2, \dots n$.

(ii) $(x+1)P(x) - x$

$= Ax(x-1)(x-2)\dots(x-n)$

let $x = -1$

$1 = A(-1)(-2)(-3)\dots(-1-n)$

$= A(-1)^{n+1}(1)(2)(3)\dots(n+1)$

$= A(n+1)!$ $\quad$ ($-1^{n+1} = 1$ as n is odd)

$A = \frac{1}{(n+1)!}$

(iii) let $x = n+1$

$(n+2)P(n+1) - (n+1)$

$= \frac{(n+1)(n)(n-1)\dots(1)}{(n+1)!}$

$= \frac{(n+1)!}{(n+1)!}$

$= 1$

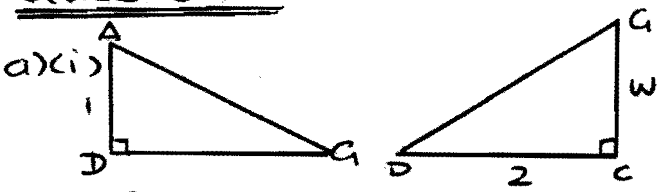
$(n+2)P(n+1) = n+2$

$P(n+1) = \frac{n+2}{n+2}$

$= 1$

Question 7

a) (i)

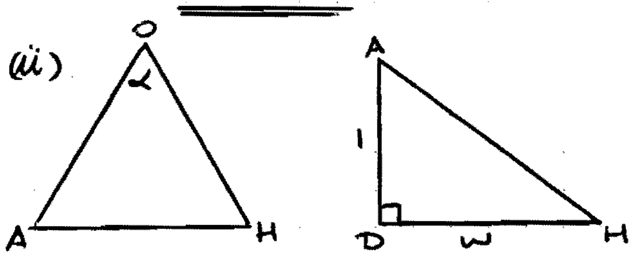


$DG^2 = w^2 + 4$

$AG^2 = DG^2 + 1$

$= w^2 + 5$

(ii)



$AH^2 = w^2 + 1$

$OA^2 = OH^2 = (\frac{1}{2}AG)^2$

$= \frac{1}{4}(w^2+5)$

$\cos\alpha = \left|\frac{OA^2 + OH^2 - AH^2}{2\cdot OA \cdot OH}\right|$

note α is acute $\therefore \cos\alpha > 0$.

$= \frac{\left|\frac{1}{4}(w^2+5) + \frac{1}{4}(w^2+5) - (w^2+1)\right|}{2\cdot\frac{1}{4}(w^2+5)}$

$= \frac{\left|-\frac{1}{2}w^2 + \frac{3}{2}\right|}{\frac{1}{2}(w^2+5)}$

$= \frac{|3 - w^2|}{w^2+5}$

(iii) $V = 1 \times 2 \times w$

$= 2w$

$S = 2(2w + w + 2)$

$= 6w + 4$

$$r = \frac{V}{5A}$$
$$= \frac{2w}{6w+4}$$
$$< \frac{2w}{6w}$$
$$= \frac{1}{3}$$
$\therefore$ $r < \frac{1}{3}$ for all $w > 0$

(iv) $r \geq \frac{1}{4}$

$$\frac{w}{3w+2} \geq \frac{1}{4}$$
$$4w \geq 3w + 2$$
$$w \geq 2$$

$$\frac{|3-w|}{5+w^2} = \left|\frac{3-w}{5+w^2}\right|$$
$$= \left|-1 + \frac{8}{5+w^2}\right|$$
$$= \left|1 - \frac{8}{5+w^2}\right|$$

minimum value will occur when $\frac{8}{5+w^2}$ is a maximum

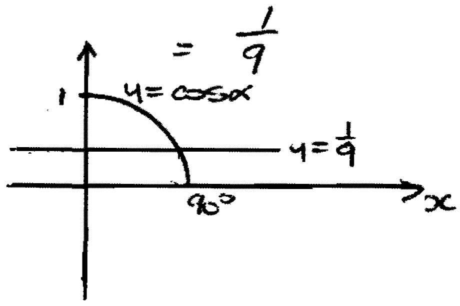
ie $5+w^2$ is a minimum

but $w \geq 2$

$\therefore$ minimum $5+w^2 = 9$

ie $\cos\alpha \geq \left|1 - \frac{8}{9}\right|$

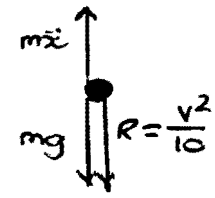
$$= \frac{1}{9}$$



as α is acute.

$$\cos\alpha \geq \frac{1}{9}$$
$$\alpha \leq \cos^{-1}\frac{1}{9}$$

b)



$$m\ddot{x} = -mg - \frac{v^2}{10}$$
$m = 2$, $g = 10$
$$\ddot{x} = -10 - \frac{v^2}{20}$$
$$= -\frac{(200+v^2)}{20}$$

(ii) $$v\frac{dv}{dx} = -\frac{(200+v^2)}{20}$$
$$\int_0^x dx = -\int_u^v \frac{20v}{200+v^2}\,dv$$
$$x = -10\left[\log(200+v^2)\right]_u^v$$
$$= -10\log\left(\frac{200+v^2}{200+u^2}\right)$$
$$= 10\log\left(\frac{200+u^2}{200+v^2}\right)$$

(iii) $$\frac{dv}{dt} = -\frac{(200+v^2)}{20}$$
$$\int_0^t dt = -\int_u^v \frac{20\,dv}{200+v^2}$$
$$t = -\frac{20}{10\sqrt{2}}\left[\tan^{-1}\frac{v}{10\sqrt{2}}\right]_u^v$$
$$= -\frac{2}{\sqrt{2}}\left(\tan^{-1}\frac{v}{10\sqrt{2}} - \tan^{-1}\frac{u}{10\sqrt{2}}\right)$$
$$= \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)$$

(iv) P_1: greatest height occurs when $v = 0$

$u = 10\sqrt{2}$
$$x = 10\log\left(\frac{200+(10\sqrt{2})^2}{200}\right)$$
$$= 10\log\left(\frac{200+200}{200}\right)$$
$$= 10\log 2 \text{ m}$$
$$t = \sqrt{2}\left(\tan^{-1}\frac{10\sqrt{2}}{10\sqrt{2}} - \tan^{-1}0\right)$$
$$= \sqrt{2}\left(\frac{\pi}{4}\right)$$
$$= \frac{\pi\sqrt{2}}{4}$$

$\therefore$ Particle 1 takes $\frac{\pi\sqrt{2}}{4}$ seconds ($\doteqdot 1.11$ seconds) to reach B

$\underline{\underline{P_2}}$: when $x = 10\log 2$, $u = 30\sqrt{2}$

$$10\log 2 = 10\log\left(\frac{200+(30\sqrt{2})^2}{200+v^2}\right)$$

$$\frac{200+1800}{200+v^2} = 2$$

$$200+v^2 = 1000$$

$$v^2 = 800$$

$$v = 20\sqrt{2}$$

$$t = \sqrt{2}\left(\tan^{-1}\frac{30\sqrt{2}}{10\sqrt{2}} - \tan^{-1}\frac{20\sqrt{2}}{10\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\tan^{-1}3 - \tan^{-1}2\right)$$

$$\doteqdot 0.2006\ldots$$

but particle 2 starts $\frac{3\sqrt{2}}{5}$ seconds

$\therefore$ time to reach B

$$= \sqrt{2}(\tan^{-1}3 - \tan^{-1}2) + \frac{3\sqrt{2}}{5}$$

$$\doteqdot 1.049\ldots$$

Thus P_2 reaches B first

It must overtake P_1 on the way up.

Question 8

a) $$\frac{1+\cos\alpha}{\sin\alpha} = \frac{1+\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$$

$$= \frac{1+t^2+1-t^2}{2t}$$

$$= \frac{2}{2t}$$

$$= \frac{1}{t}$$

$$= \cot\frac{\alpha}{2}$$

$$= \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

b) $$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos\alpha\sin x}$$

$t = \tan\frac{x}{2}$

$x = 0, t = 0$

$x = \frac{\pi}{2}, t = 1$

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\cos\alpha\left(\frac{2t}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2dt}{1+t^2+2t\cos\alpha}$$

$$= \int_0^1 \frac{2dt}{\sin^2\alpha+\cos^2\alpha+t^2+2t\cos\alpha}$$

$$= \int_0^1 \frac{2dt}{\cos^2\alpha+2t\cos\alpha+t^2+\sin^2\alpha}$$

$$= \int_0^1 \frac{2dt}{(\cos\alpha+t)^2+\sin^2\alpha}$$

(ii) $t+\cos\alpha = \sin\alpha\tan u$

$dt = \sin\alpha\sec^2 u\,du$

when $t=0$, $\cos\alpha = \sin\alpha\tan u$

$\tan u = \cot\alpha$

$u = \left(\frac{\pi}{2}-\alpha\right)$

$t=1$, $1+\cos\alpha = \sin\alpha\tan u$

$\tan u = \frac{1+\cos\alpha}{\sin\alpha}$

$u = \left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$

$$I = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2\sin\alpha\sec^2 u\,du}{\sin^2\alpha\tan^2 u+\sin^2\alpha}$$

$$= \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2\sec^2 u\,du}{\sin\alpha(\tan^2 u+1)}$$

$$= \frac{2}{\sin\alpha}\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} du$$

$$= \frac{2}{\sin\alpha}\Big[u\Big]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}}$$

$$= \frac{2}{\sin\alpha}\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right)$$

$$= \frac{2}{\sin\alpha}\left(\frac{\alpha}{2}\right)$$

$$= \frac{\alpha}{\sin\alpha}$$

c)(i) $z^{2n+1} = 1$

$z = \text{cis}\left(\frac{2\pi k}{2n+1}\right) \quad k=0,1,\ldots,2n$

$z = \text{cis}\,0, \text{cis}\frac{2\pi}{2n+1}, \text{cis}\frac{4\pi}{2n+1}, \ldots, \text{cis}\frac{4\pi n}{2n+1}$

(ii) $k = 0, \pm 1, \pm 2, \ldots, \pm n$
ie conjugate pairs

each quadratic factor can be written as

$(z^2 - 2z\cos\frac{2\pi k}{2n+1} + 1)$

$\therefore z^{2n+1} - 1 = (z-1)(1+z+z^2 \ldots + z^{2n})$

$= (z-1)(z^2 - 2z\cos\frac{2\pi}{2n+1} + 1)(z^2 - 2z\cos\frac{4\pi}{2n+1} + 1) \ldots (z^2 - 2z\cos\frac{2\pi n}{2n+1} + 1)$

$\therefore 1 + z + z^2 + \ldots + z^{2n}$

$= (z^2 - 2z\cos\frac{2\pi}{2n+1} + 1)(z^2 - 2z\cos\frac{4\pi}{2n+1} + 1) \ldots (z^2 - 2z\cos\frac{2\pi n}{n+1} + 1)$

(iii) let $z = 1$

$1 + 1 + 1^2 + \ldots + 1^{2n}$

$= (2 - 2\cos\frac{2\pi}{2n+1})(2 - 2\cos\frac{4\pi}{2n+1}) \ldots (2 - 2\cos\frac{2\pi n}{2n+1})$

$2n+1 = 2^n(1 - \cos\frac{2\pi}{2n+1})(1 - \cos\frac{4\pi}{2n+1}) \ldots (1 - \cos\frac{2\pi n}{2n+1})$

~~$= 2^n(1 - \frac{1}{2}(1-$~~

$= 2^n(2\sin^2\frac{\pi}{2n+1})(2\sin^2\frac{2\pi}{2n+1}) \ldots (2\sin^2\frac{n\pi}{2n+1})$

$= 2^{2n}(\sin^2\frac{\pi}{2n+1})(\sin^2\frac{2\pi}{2n+1}) \ldots (\sin^2\frac{n\pi}{2n+1})$

$\therefore \sqrt{2n+1}$

$= 2^n \sin\frac{\pi}{2n+1} \sin\frac{2\pi}{2n+1} \ldots \sin\frac{n\pi}{2n+1}$