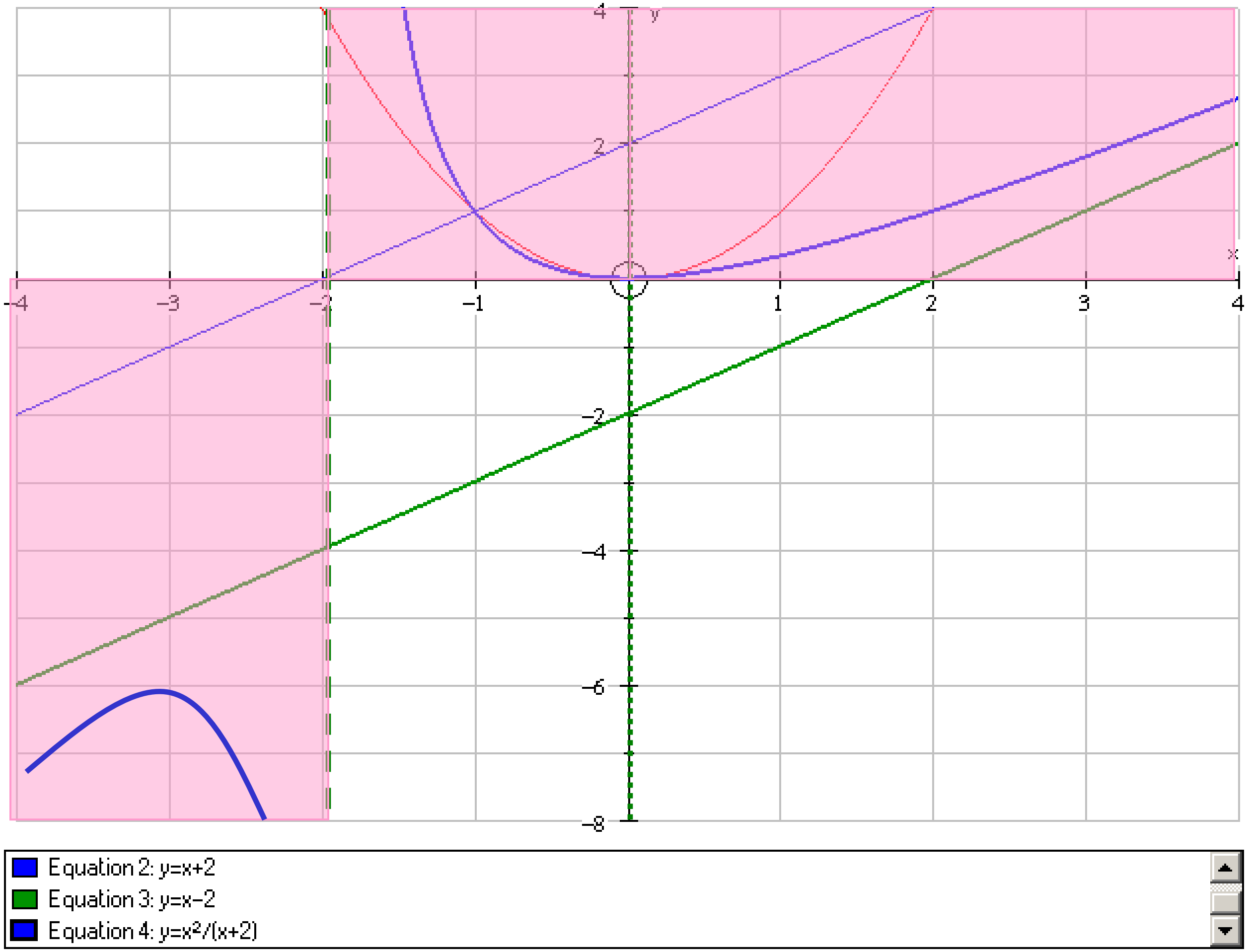
Inequalities & Graphs

e.g. (i) Solve $\dfrac{x^2}{x+2} \le 1$

Oblique asymptote: $\dfrac{x^2}{x+2} = x - 2 - \dfrac{4}{x+2}$



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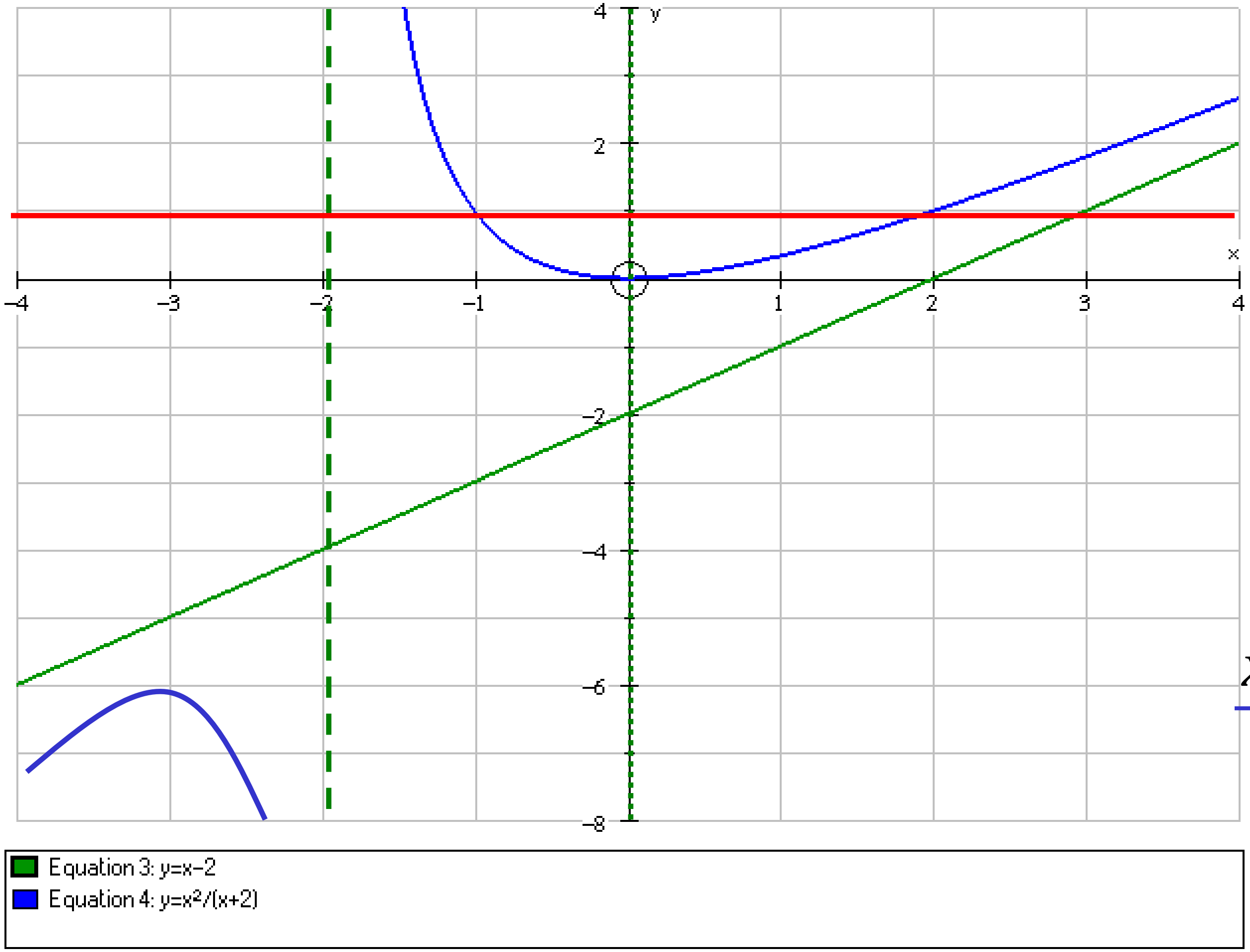


$$\frac{x^2}{x+2} = 1$$

$$x^2 = x+2$$

$$x^2 - x - 2 = 0$$

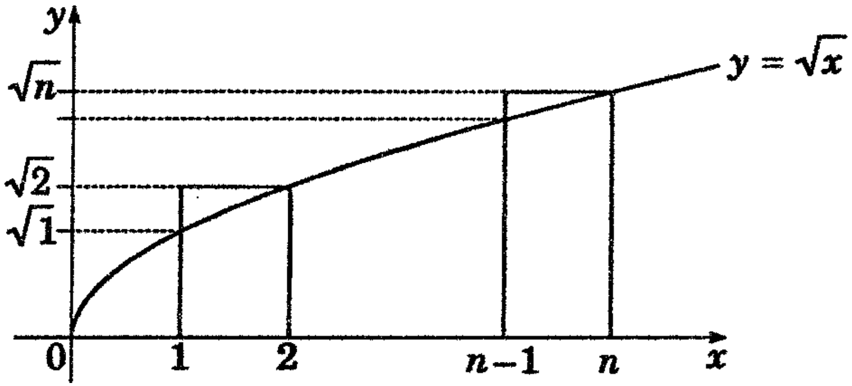
$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } x = -1$$

$$\frac{x^2}{x+2} \leq 1$$

$$x < -2 \text{ or } -1 \leq x \leq 2$$

(*ii*) (1990)



Consider the graph $y = \sqrt{x}$

a) Show that the graph is increasing for all $x \geq 0$

Curve is increasing when $\frac{dy}{dx} > 0$

$y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} > 0$ for $x > 0$

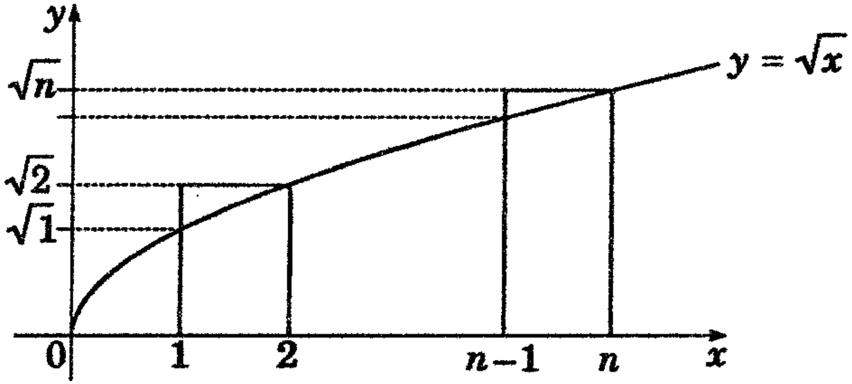
at $x = 0, y = 0$

when $x > 0, y > 0$

$\therefore$ curve is increasing for $x \geq 0$

b) Hence show that;

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}\,dx = \frac{2}{3}n\sqrt{n}$$



As $\sqrt{x}$ is increasing;

Area outer rectangles ≥ Area under curve

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}\,dx$$

$$= \left[\frac{2}{3}x\sqrt{x}\right]_0^n$$

$$= \frac{2}{3}n\sqrt{n}$$

$$\therefore \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \geq \int_0^n \sqrt{x}\,dx = \frac{2}{3}n\sqrt{n}$$

c) Use mathematical induction to show that;

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\le\frac{4n+3}{6}\sqrt{n} \text{ for all integers } n\ge 1$$

Test: $n = 1$

$$L.H.S=\sqrt{1}$$
$$=1$$

$$R.H.S=\frac{4(1)+3}{6}\sqrt{1}$$
$$=\frac{7}{6}$$

$$\therefore L.H.S\le R.H.S$$

Hence the result is true for $n = 1$

Assume the result is true for $n=k$ where k is a positive integer

i.e. $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{k}\le\frac{4k+3}{6}\sqrt{k}$

Prove the result is true for $n=k+1$

i.e. Prove $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{k+1}\le\frac{4k+7}{6}\sqrt{k+1}$

Proof:

$$
\begin{aligned}
\sqrt{1}+\sqrt{2}+\cdots+\sqrt{k+1} &= \sqrt{1}+\sqrt{2}+\cdots+\sqrt{k}+\sqrt{k+1} \\
&\le \frac{4k+3}{6}\sqrt{k}+\sqrt{k+1} \\
&= \frac{\sqrt{(4k+3)^2 k}+6\sqrt{k+1}}{6} \\
&= \frac{\sqrt{16k^3+24k^2+9k}+6\sqrt{k+1}}{6} \\
&= \frac{\sqrt{(k+1)\left(16k^2+8k+1\right)-1}+6\sqrt{k+1}}{6} \\
&= \frac{\sqrt{(k+1)(4k+1)^2-1}+6\sqrt{k+1}}{6} \\
&< \frac{\sqrt{(k+1)(4k+1)^2}+6\sqrt{k+1}}{6} \\
&= \frac{(4k+1)\sqrt{(k+1)}+6\sqrt{k+1}}{6} \\
&= \frac{(4k+7)\sqrt{(k+1)}}{6}
\end{aligned}
$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$ then it is also true for $n = 1+1$ i.e. $n=2$, and since the result is true for $n = 2$ then it is also true for $n = 2+1$ i.e. $n=3$, and so on for all positive integral values of n

d) Use b) and c) to estimate;

$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{10000}$ to the nearest hundred

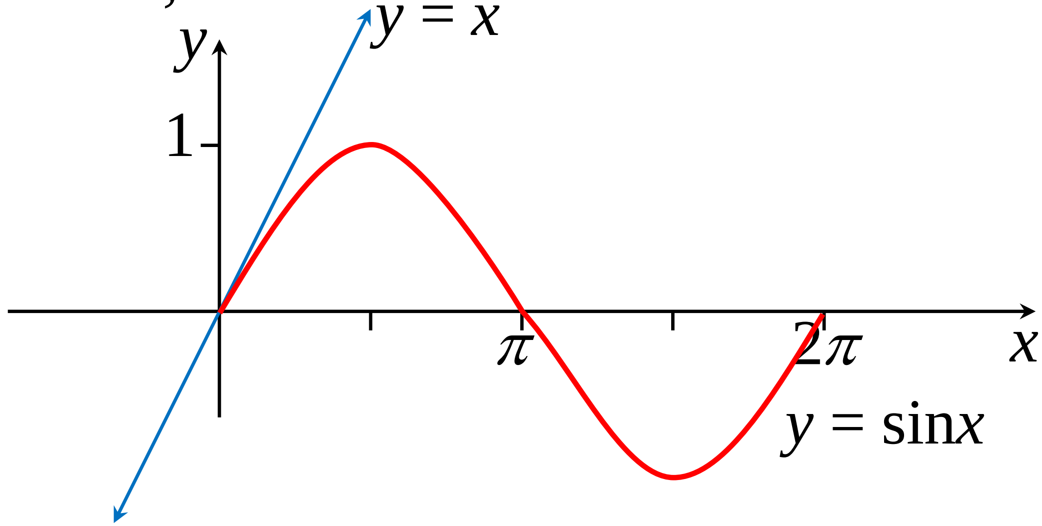
$$\frac{2}{3}n\sqrt{n} \leq \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$$

$$\frac{2}{3}(10000)\sqrt{10000} \leq \sqrt{1}+\sqrt{2}+\cdots+\sqrt{10000} \leq \frac{4(10000)+3}{6}\sqrt{10000}$$

$$666700 \leq \sqrt{1}+\sqrt{2}+\cdots+\sqrt{10000} \leq 666700$$

$$\therefore \sqrt{1}+\sqrt{2}+\cdots+\sqrt{10000} = 666700 \text{ to the nearest hundred}$$

(*iii*) Prove $x > \sin x$, for $x > 0$



$f(x) = x$

$f'(x) = 1$

$f(x) = \sin x$

$f'(x) = \cos x$

for $0 < x \leq \frac{\pi}{2}$, $\cos x < 1$

$\therefore y = x$ increases faster than $y = \sin x$

$x > \sin x$, for $0 < x \leq \frac{\pi}{2}$

for $x > \frac{\pi}{2}$, $\sin x \leq 1$

$\therefore x > \sin x$, for $x > \frac{\pi}{2}$

$\therefore x > \sin x$, for $x > 0$

Exercise 10F