Types of Proof

1. Deductive Proof

Start with known facts and deduce what you are trying to prove.

2. Inductive Proof

Assume what you are trying to prove and induce a solution.

3. Proof by Contradiction

Assume the opposite of what you are trying to prove and create a contradiction.
Contradiction means the original assumption is incorrect, therefore the opposite must be true.

Inequality Techniques

To prove $x \geq y$, it can be easier to prove $x - y \geq 0$

e.g. (i) (1995) Prove $pq \leq \frac{p^2+q^2}{2}$

$$\frac{p^2+q^2}{2} - pq = \frac{p^2-2pq+q^2}{2}$$
$$= \frac{(p-q)^2}{2}$$
$$\geq 0$$
$$\therefore \frac{p^2+q^2}{2} \geq pq$$

OR

Assume $pq > \frac{p^2+q^2}{2}$

$$2pq > p^2+q^2$$
$$0 > p^2-2pq+q^2$$
$$0 > (p-q)^2$$

But $(p-q)^2 > 0$

$$\therefore pq \leq \frac{p^2+q^2}{2}$$

$(ii)(1994)$ a) Prove $a^2+b^2+c^2>ab+bc+ac$

$$(a-b)^2>0$$

$$a^2-2ab+b^2>0$$

$$\therefore a^2+b^2>2ab$$

$$a^2+c^2>2ac$$

$$b^2+c^2>2bc$$

$$2a^2+2b^2+2c^2>2ab+2ac+2bc$$

$$\underline{a^2+b^2+c^2>ab+ac+bc}$$

b) If $a+b+c=1$, prove $ab+ac+bc<\frac{1}{3}$

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$$

$$\therefore (a+b+c)^2-2(ab+ac+bc)>ab+ac+bc$$

$$3(ab+ac+bc)<(a+b+c)^2$$

$$3(ab+ac+bc)<1$$

$$\underline{ab+ac+bc<\frac{1}{3}}$$

c) Prove $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$

$$a^2+b^2+c^2 \geq ab+ac+bc$$

$$a^2+b^2+c^2-ab-ac-bc \geq 0$$

$$(a+b+c)(a^2+b^2+c^2-ab-ac-bc) \geq 0$$

$$a^3+ab^2+ac^2-a^2b-a^2c-abc+a^2b+b^3+bc^2-ab^2-abc-b^2c$$
$$+a^2c+b^2c+c^3-abc-ac^2-bc^2 \geq 0$$

$$a^3+b^3+c^3-3abc \geq 0$$

$$\frac{1}{3}(a^3+b^3+c^3) \geq abc$$

let $a = a^{\frac{1}{3}}, b = b^{\frac{1}{3}}, c = c^{\frac{1}{3}}$

$$\frac{1}{3}(a+b+c) \geq a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

Arithmetic Mean ≥ Geometric Mean

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

d) Suppose $(1+x)(1+y)(1+z)=8$, prove $xyz \leq 1$

$$(1+x)(1+y)(1+z)=8$$

$$1+x+y+xy+z+xz+yz+xyz=8$$

$$\frac{1}{3}(x+y+z) \geq \sqrt[3]{xyz} \qquad \text{AM} \geq \text{GM}$$

$$\underline{x+y+z \geq 3\sqrt[3]{xyz}}$$

$$xy+yz+xz \geq 3\sqrt[3]{(xy)(yz)(xz)}$$

$$xy+yz+xz \geq 3\sqrt[3]{x^2y^2z^2}$$

$$\underline{xy+yz+xz \geq 3\left(\sqrt[3]{xyz}\right)^2}$$

$$1+x+y+z+xy+xz+yz+xyz=8$$

$$1+3\sqrt[3]{xyz}+3\left(\sqrt[3]{xyz}\right)^2+xyz\le 8$$

$$1+3\sqrt[3]{xyz}+3\left(\sqrt[3]{xyz}\right)^2+\left(\sqrt[3]{xyz}\right)^3\le 8$$

$$\left(1+\sqrt[3]{xyz}\right)^3\le 8$$

$$1+\sqrt[3]{xyz}\le 2$$

$$\sqrt[3]{xyz}\le 1$$

$$xyz\le 1$$

OR

AM $\geq$ GM

$$\left(1+x\right)\geq 2\sqrt{x}$$

$$\left(1+y\right)\geq 2\sqrt{y}$$

$$\left(1+z\right)\geq 2\sqrt{z}$$

$$(1+x)(1+y)\left(1+z\right)\geq 2\sqrt{x}\times 2\sqrt{y}\times 2\sqrt{z}$$

$$=8\sqrt{xyz}$$

$$\therefore 8\geq 8\sqrt{xyz}$$

$$1\geq 1\sqrt{xyz}$$

$$\sqrt{xyz}\leq 1$$

$$xyz\leq 1$$

(iii) Prove $\dfrac{9}{a+b+c} \leq \dfrac{2}{a+b} + \dfrac{2}{b+c} + \dfrac{2}{a+c} \leq \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 2\sqrt{ab} \times 2\sqrt{\frac{1}{ab}}$$
$$= 4$$

$$\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$$

$$\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}$$

$$\frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c}$$

$$\frac{2}{a}+\frac{2}{b}+\frac{2}{c} \geq \frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{a+c}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3\sqrt[3]{abc} \times 3\sqrt[3]{\frac{1}{abc}}$$
$$= 9$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c}$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c} \geq \frac{9}{2(a+b+c)}$$

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c} \geq \frac{9}{a+b+c}$$

Inequalities Sheet

Exercise 10D

Note: Cambridge 8H (Book 1); 28

$$\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$