HORNSBY GIRLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- **Reading Time – 5 minutes**
- **Working Time – 2 hours**
- **Write using a black or blue pen**
- **Approved calculators may be used**
- **A table of standard integrals is provided at the back of this paper**
- **All necessary working should be shown for every question**
- **Begin each question in a new booklet**

Total marks (84)

- **Attempt Questions 1 – 7**
- **All questions are of equal value**

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Total Marks
Attempt Questions 1–7
All Questions are of equal value

Begin each question in a new booklet, writing your student number and question number in the boxes indicated. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\lim_{x \to 0} \frac{8x}{\sin 5x}$ **1**

(b) The point $C(11,-5)$ divides the interval joining $A(-3,2)$ and B in the ratio $7:2$ internally. Find the coordinates of B. **2**

(c) Solve $\frac{2x+1}{x-3} < 3,\ x \neq 3$ **3**

(d) Evaluate $\int_1^9 \frac{dx}{x+\sqrt{x}}$ using the substitution $x = u^2$. **3**

(e) Find $\int (\tan x - 1)^2\, dx$ **3**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$. **2**

(b) Consider the function $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$.

(i) Evaluate $f(0)$. **1**

(ii) State the domain and range of $y = f(x)$. **2**

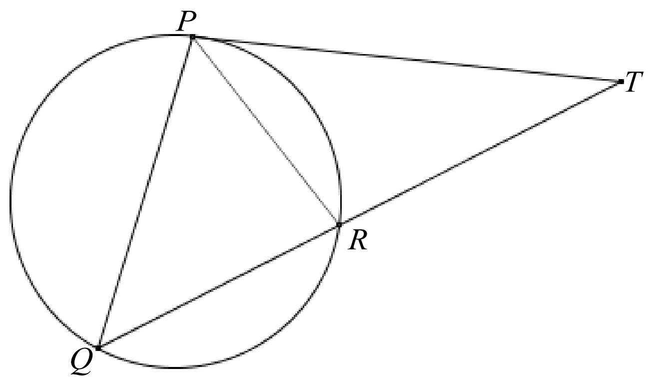
(iii) Sketch $y = f(x)$. **1**

(c) A class consists of 12 girls and 10 boys.

(i) A committee of 4 is to be chosen from the class. How many ways can this be done? **1**

(ii) How many ways could the committee be chosen if it is to be made up of 3 girls and 2 boys? **2**

(d) PT is a tangent to the circle PRQ and QR is a chord produced to intersect PT at T.



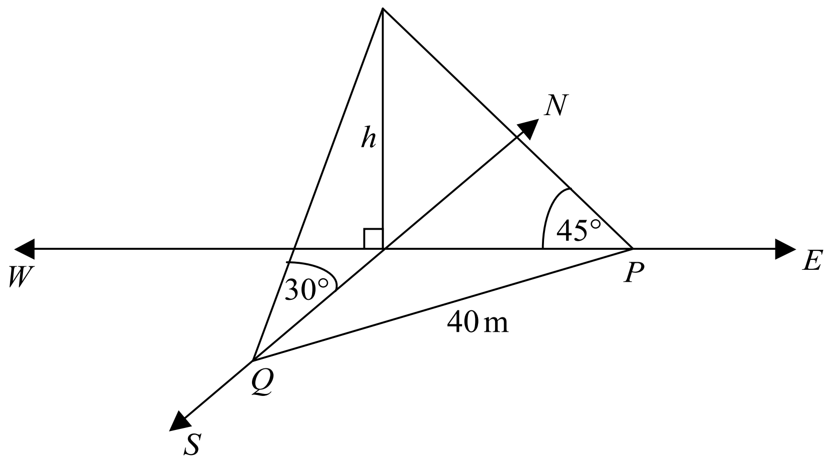
NOT TO SCALE

(i) Prove that ΔPRT and ΔQPT are similar. **2**

(ii) Hence, prove that $PT^2 = QT \times RT$. **1**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



NOT TO SCALE

A vertical tower of height h metres stands on horizontal ground. From a point P, on the ground due east of the tower, the angle of elevation of the top of the tower is $45°$. From a point Q, on the ground due south of the tower, the angle of elevation of the top of the tower is $30°$. If the distance PQ is 40 metres, find the exact height of the tower. **3**

(b) A particle P is moving along the $x-$axis with acceleration $\ddot{x} = -16x$, where x is the displacement of the particle from the origin. Initially, the particle is at the origin, moving with a velocity of 24 units per second.

(i) By using integration, show that the displacement is given by $x = 6\sin 4t$, where t is time in seconds. **3**

(ii) State the maximum distance from the origin that the particle reaches. **1**

(iii) What is the period of the motion? **1**

(iv) Sketch the graph of displacement, x, against time, t, for the first π seconds. **2**

(v) Calculate the average speed of the particle during the first π seconds. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given that $x^2 + 4x + 5 \equiv (x+a)^2 + b^2$, show that $a = 2$ and $b = \pm 1$. **2**

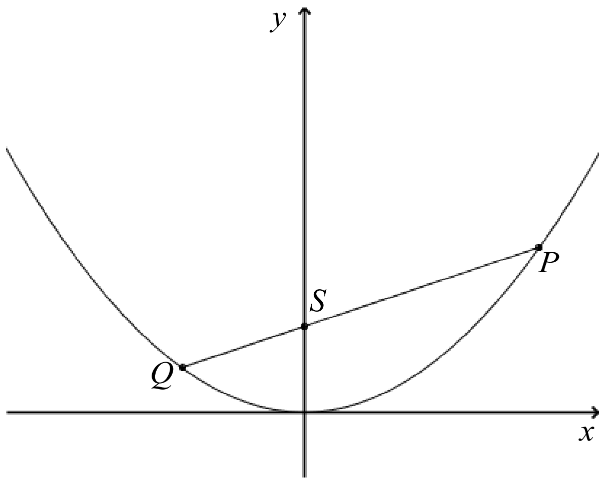
(ii) Hence, find $\int \frac{1}{x^2 + 4x + 5}\,dx$. **2**

(b) At Phillips High School in NSW there are 3 Science teachers. The probability that in a NSW a Science teacher is female is 0.6. The probability that in NSW a Science teacher (male or female) is 50 years or older is 0.2.

(i) What is the probability that at Phillips High School there is at least one female Science teacher? **2**

(ii) What is the probability that at Phillips High School all 3 Science teachers are female and younger than 50 years. **2**

(c) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$, where $a > 0$. The chord PQ passes through the focus, S.



NOT TO SCALE

(i) Show that $pq = -1$. **2**

(ii) Show that the length of chord PQ is $a\left(p + \frac{1}{p}\right)^2$. **2**

Question 5 (12 marks) Use a SEPERATE writing booklet. **Marks**

(a) A pig farm has 100 pigs. The number of pigs, N, infected with a disease at time t days is given by $N = \dfrac{100}{1+ce^{-t}}$, where c is a constant.

(i) Show that eventually all the pigs will be infected. **1**

(ii) Initially, one pig is infected. After how many days will 70 pigs be infected? **3**

(b) Prove by mathematical induction that $\sum_{k=1}^{n} k \times 2^{k-1} = 1 + (n-1)2^n$ **3**

(c) Find the roots of the equation $x^3 - 12x^2 + 30x + 8 = 0$, given that they are consecutive terms in an arithmetic series. **3**

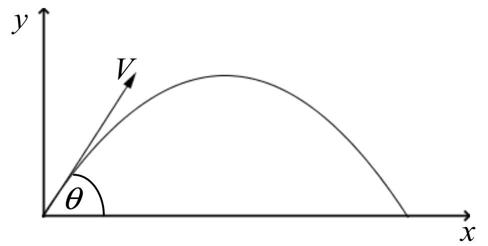
(d) The population P of a country has an annual growth rate, $\dfrac{dP}{dt} = 0.06P$. **2**

How long will it take the population of this country to double?

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle, P, is fired from the ground at $t = 0$. The particle is projected from the origin at an angle of θ to the horizontal, with a velocity of V.
The horizontal equation of motion for the particle is

$x_p = Vt\cos\theta$. **DO NOT PROVE THIS**.



NOT TO SCALE

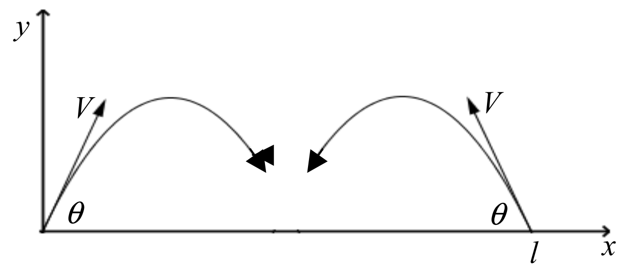
(i) Prove that the vertical equation of motion for the particle is **2**

$$y_P = Vt\sin\theta - \frac{1}{2}gt^2.$$

(ii) Show that the horizontal range of the projectile, R_P, is given by **2**

$$R_P = \frac{V^2\sin 2\theta}{g}.$$

A second particle, Q, is fired back towards the origin from the ground at a distance of l metres to the **right** of the origin at time $t = 0$, with an angle of $(180 - \theta)^\circ$ to the positive direction of the x-axis, with velocity V.



NOT TO SCALE

The equations of motion of this particle are:

$x_Q = -Vt\cos\theta + l$ and $y_Q = Vt\sin\theta - \frac{1}{2}gt^2$. **DO NOT PROVE THESE**.

(iii) Show that if the particles collide, it will occur when $t = \frac{l}{2V\cos\theta}$. **2**

(iv) For the particles to collide, it must occur while the particles are still in flight (ie above the ground). **2**

Prove that, for the particles to collide in the air, $0 < l < \frac{4v^2\cos\theta\sin\theta}{g}$.

Question 6 continues on page 9

(b) Consider $f(x) = x^3 - 3x^2 - 9x$ in the domain $x \leq -1$.

(i) Find the point(s) of intersection of $y = x$ and $y = f(x)$ in this domain **2**

(ii) Hence, find the gradient of the inverse $f^{-1}(x)$ at this point. **2**

End of Question 6

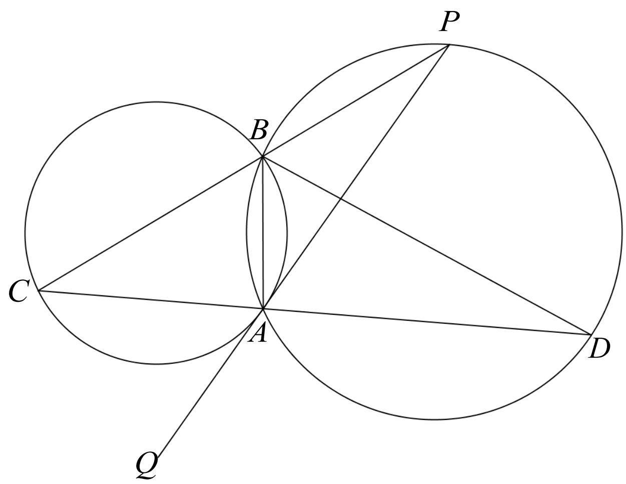
Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) It is known that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute angles.

(i) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$. **2**

(ii) Hence or otherwise, solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$. **2**

(b)



NOT TO SCALE

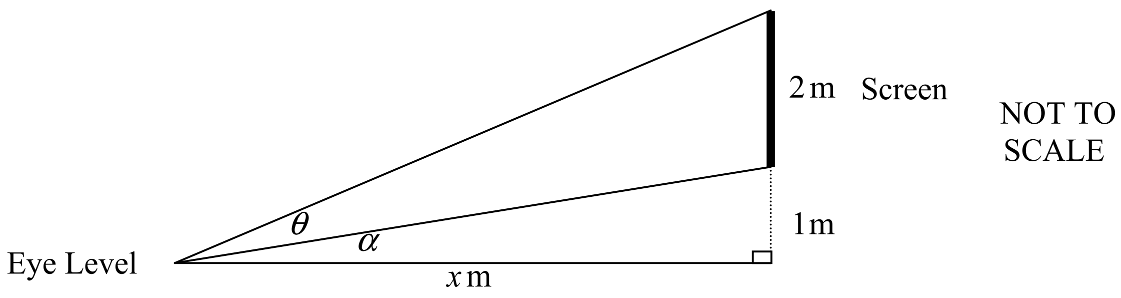
Two circles of unequal radii intersect at A and B. The tangent to the smaller circle at A cuts the larger circle at P, with PB produced cutting the smaller circle at C. The line CA produced cuts the larger circle at D.

If $\angle CAQ = \alpha$ and $\angle BAP = \beta$, show giving reasons, that $\angle ADB = \alpha - \beta$. **3**

Question 7 continues on page 11

Question 7 (continued)

(c) A projector screen on the front wall of a classroom is 2 metres high and its lower edge is 1 metre above the eye level of a seated student as indicated in the diagram. The horizontal distance of the student from the screen is x metres, the angle of elevation to the bottom of the screen is α and the viewing angle is θ. The "best" viewing angle is when θ is a maximum.



(i) Show that $\alpha = \tan^{-1}\left(\frac{1}{x}\right)$. **1**

(ii) Show that when θ is expressed as a function of x, **2**

$$\theta = \tan^{-1}\left(\frac{2x}{3+x^2}\right).$$

(iii) Hence or otherwise determine how far from the front of the room the student should sit in order to have the "best" view of the projector screen. **2**

End of paper

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Year 12 Ext 1 HGHS Solns.

Question 1

a) $\lim_{x \to 0} \frac{8x}{\sin 5x}$

$= \lim_{x \to 0} \frac{8}{5} \cdot \frac{5x}{\sin 5x}$

$= \frac{8}{5}$

b) $A(-3,2)$ $B(x,y)$

$C(11,-5)$ $7:2$ ($m:n$)

$11 = \frac{7x + 2(-3)}{9}$

$99 = 7x - 6$

$7x = 105$

$x = 15$

$-5 = \frac{7y + 2(2)}{9}$

$-45 = 7y + 4$

$-49 = 7y$

$y = -7$

$\therefore B(15,-7)$

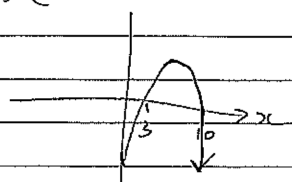
c) $\frac{2x+1}{x-3} < 3$

$(2x+1)(x-3) < 3(x-3)^2$

$(2x+1)(x+3) - 3(x-3)^2 < 0$

$(x-3)[2x+1-3x+9] < 0$

$(x-3)(10-x) < 0$



$\therefore x > 10, \quad x < 3.$

d) $I = \int_1^9 \frac{dx}{x + \sqrt{x}}$ $\quad x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

when $x = 1, u = 1$

$x = 9, u = 3.$

$\therefore I = \int_1^3 \frac{2u\,du}{u^2 + u}$

$= \int_1^3 \frac{2}{u+1}\,du$

$= 2[\ln(u+1)]_1^3$

$= 2(\ln 4 - \ln 2)$

$= 2\ln 2$

$= \ln 4$

$\doteq 1.386$ (3dp)

e) $\int (\tan x - 1)^2 dx$

$= \int \tan^2 x - 2\tan x + 1\, dx$

$= \int (\sec^2 x - 2\tan x) dx$

$= \tan x + 2\ln(\cos x) + C.$

Question 2

a) $\int_0^{\pi/4} \cos^2 x\, dx$

$= \frac{1}{2}\int_0^{\pi/4} (1 + \cos 2x) dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[(\pi/4 - 0) - \frac{1}{2}(\sin\frac{\pi}{2} - \sin 0)\right]$

$= \frac{\pi}{8} + \frac{1}{4}$

$= \frac{1}{8}(\pi + 2)$

b) $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

(i) $f(0) = 2\cos^{-1} \frac{0}{3}$

$= 2\left(\frac{\pi}{2}\right)$

$= \pi.$

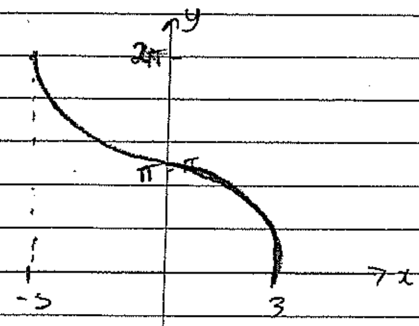
(ii) D: $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

R: $0 \le \frac{y}{2} \le \pi$

$0 \le y \le 2\pi$

iii



c) (i) ${}^{22}C_4 = 7315$ ways.

(ii) ${}^{12}C_3 \times {}^{10}C_2 = 9900$ ways

d) In ΔPRT and ΔQPT,

$\angle RTP = \angle PTQ$ (common angle)

$\angle TPR = \angle TQP$ (angle in alternate segment)

$\therefore \Delta PRT \sim \Delta QPT$ (equiangular)

(ii) $\frac{PT}{QT} = \frac{RT}{PT}$ (corresponding sides of $\Delta PRT \sim \Delta$...)

$PT^2 = QT.RT$

QUESTION 3.

a) $\tan 45 = \frac{h}{OP}$ $\tan 30 = \frac{h}{OQ}$

$OP = h$ $OQ = h\sqrt{3}$

$\therefore 40^2 = h^2 + 3h^2$

$h = 20$

b) i) $\ddot{x} = -16x$

$\frac{d(\frac{1}{2}v^2)}{dx} = -16x$

$\frac{1}{2}v^2 = -16\int x\,dx$

$v^2 = -16x^2 + C$

when $x=0$, $v=24$

$24^2 = C$

$C = 576$

$\therefore V = \sqrt{576-16x^2}$

$\frac{dt}{dx} = \frac{1}{\sqrt{576-16x^2}}$

$= \frac{1}{4\sqrt{36-x^2}}$

$t = \frac{1}{4}\int \frac{1}{\sqrt{36-x^2}}\,dx$

$t = \frac{1}{4}\sin^{-1}(\frac{x}{6})$

$4t = \sin^{-1}(\frac{x}{6})$

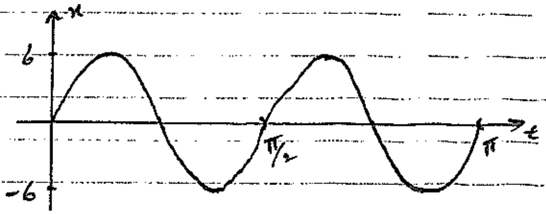
$\frac{x}{6} = \sin 4t$

$x = 6\sin 4t$

ii) 6

iii) $\frac{\pi}{2}$

iv)



v) Speed $= \frac{D}{T}$

$= \frac{48}{\pi}$

$= 15.3$

Question 4.

a) (i) $x^2+4x+4+1 = (x+2)^2+1$

$a=2,\ b^2=1$

$\therefore b = \pm 1.$

ii) $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$

$= \tan^{-1}(x+2) + C$

b) (i) P(at least one female science teacher) $= 1 - P(\text{none})$

$= 1-(0.4)^3$

$= 0.936.$

ii) P(3 female, <50) $= (0.6)^3 \times (0.8)^3$

$\doteq 0.11$ (2dp)

c) (i) Chord PQ:

$\frac{y-ap^2}{x-2ap} = \frac{aq^2-ap^2}{2aq-2ap}$

S(0,a) satisfies equation

$\frac{a-ap^2}{-2ap} = \frac{a(q-p)(q+p)}{2a(q-p)}$

$\frac{1-p^2}{-2p} = \frac{q+p}{2}$

$1-p^2 = -pq-p^2$

$-pq = 1$

$pq = -1$

(ii) $d_{PQ} = \sqrt{(2aq-2ap)^2+(aq^2-ap^2)^2}$

$= \sqrt{4a^2(q-p)^2+a^2(q-p)^2(q+p)^2}$

$= a\sqrt{4(q-p)^2+(q-p)^2(q+p)^2}$

$= a\sqrt{(q-p)^2[4+q^2+2pq+p^2]}$

$= a\sqrt{(\frac{-1}{p}-p)^2[4+(\frac{-1}{p})^2-2+p^2]}$

$= a\sqrt{(p+\frac{1}{p})^2(p^2+\frac{2\times p^2}{p^2}+(\frac{1}{p})^2)}$

$= a\sqrt{(p+\frac{1}{p})^4}$

$= a(p+\frac{1}{p})^2.$

QUESTION 5

a) i) $\lim_{t\to\infty} e^{-t} = 0$

$\therefore \lim_{t\to\infty} \frac{100}{1+ce^{-t}} = \frac{100}{1+0}$

$= 100$

ii) when $t=0$, $N=1$

$\therefore 1 = \frac{100}{1+c}$

$c = 99$

when $N = 70$

$70 = \frac{100}{1+99e^{-t}}$

$e^{-t} = \frac{1}{231}$

$t = \ln(231)$

$= 5.44\ldots$

b) $1\times2^0 + 2\times2^1 + 3\times2^2 + \ldots + n\times2^{n-1} = 1+(n-1).2^n$

• Prove true for $n=1$.

L.H.S. $= 1\times2^0 = 1$

R.H.S $= 1+0 = 1$

$\therefore$ True for $n=1$

• Assume true for $n=k+1$

i.e. $1\times2^0 + 2\times2^1 \times 3\times2^2 + \ldots + n\times2^{k-1} = 1+(k-1)$

• Prove true for $n=k+1$

i.e. $1\times2^0 + 2\times2^1 + \ldots + k\times2^{k-1} + (k+1)\times2^k = 1+k.2^{k+1}$

L.H.S. $= 1+(k-1).2^k + (k+1)\times2^k$

$= 1+k.2^k - 2^k + k.2^k + 2^k$

$= 1+2k.2^k$

$= 1+k.2^{k+1}$

$=$ R.H.S

etc.

c) Let roots be $a-d, a, a+d$

sum roots $= a-d+a+a+d = 12$

$a = 4$

prod roots $= (4-d)(4+d)\times4 = -8$

$16-d^2 = -2$

$d^2 = 18$

$d = 3\sqrt{2}$

$\therefore$ roots are 4, $4\pm3\sqrt{2}$

d) $\frac{dP}{dt} = 0.06P$

$\therefore P = P_0 e^{0.06t}$

when $2P_0 = P_0 e^{0.06t}$

$2 = e^{0.06t}$

$t = \frac{\ln 2}{0.06}$

Question 6

a) (i) $\ddot{y}_P = -g$

$\dot{y}_P = -gt + c$

when $t=0$

$\dot{y} = V\sin\theta$

$V\sin\theta = c$

$\therefore \dot{y}_P = -gt + V\sin\theta$

$y_P = -\frac{g}{2}t^2 + Vt\sin\theta + d$

when $t=0$, $y=0$

$\therefore d = 0$

$\therefore y_P = -\frac{g}{2}t^2 + Vt\sin\theta$

ii) Let $y_P = 0$

$0 = -\frac{g}{2}t^2 + Vt\sin\theta$

$0 = -t\left(\frac{g}{2}t - V\sin\theta\right)$

$\therefore \frac{g}{2}t = V\sin\theta \quad (t\neq0)$

$t = \frac{2V\sin\theta}{g}$.

Sub into x

$x = V\left(\frac{2V\sin\theta}{g}\right).\cos\theta$

$= \frac{2V^2\sin\theta\cos\theta}{g}$

$= \frac{V^2\sin2\theta}{g}$.

iii) P and Q have same y-values at time t. Need $x_Q = x_P$ at some time.

$\ell - Vt\cos\theta = Vt\cos\theta$

$\ell = 2Vt\cos\theta$

$t = \frac{\ell}{2V\cos\theta}$.

(iv) [There are a few methods]

time of flight is $\frac{2V\sin\theta}{g}$.

Need collision time to be less than this.

$0 < \frac{\ell}{2V\cos\theta} < \frac{2V\sin\theta}{g}$

$0 < \ell < \frac{4V^2\sin\theta\cos\theta}{g}$.

[$\ell > 0$, since ℓ is to right of origin].

(b) $f(x) = x^3 - 3x^2 - 9x$

Let $x = x^3 - 3x^2 - 9x$

$0 = x^3 - 3x^2 - 10x$

$0 = x(x^2 - 3x - 10)$

$0 = x(x-5)(x+2)$

$\therefore x=0, x=-2, x=5$

But $x \leq -1$

$\therefore$ Pt of intersection is $(-2,$

(ii) $f'(x) = 3x^2 - 6x - 9$

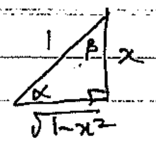
$f'(-2) = 3(-2)^2 - 6(-2) - 9$

$= 15$

$\therefore f^{-1}(x)$ has gradient $\frac{1}{15}$ at this point.

Q7 a) let $\sin^{-1}x = \alpha$

$\cos^{-1}x = \beta$



i) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x.x - \sqrt{1-x^2}.\sqrt{1-x^2}$

$= x^2-(1-x^2)$

$= 2x^2-1$

ii) $\sin(\sin^{-1}x - \cos^{-1}x) = \sin(\sin^{-1}(1-x))$

$2x^2-1 = 1-x$

$\therefore\ 2x^2+x-2=0$

$x = \dfrac{-1 \pm \sqrt{1-4(2)(-2)}}{2(2)}$

$\therefore x = \dfrac{-1\pm\sqrt{17}}{4}$ $\left(x = \dfrac{-1+\sqrt{17}}{4},\ x>0\right)$

b) $\angle CAQ = \alpha$

$= \angle ABC$ (Angle in alternate segment)

$\angle BAP = \beta$

$= \angle BCA$ (Angle in alternate segment)

$\angle APB = \angle ADB$ (Angles on same arc AB)

$\angle ABC = \angle BAP + \angle APB$ (exterior $\angle$ of $\triangle ABP$)

$= \angle BAP + \angle ADB$

$\therefore \angle ADB = \angle ABC - \angle BAP = \alpha - \beta$

c) i) $\tan\alpha = \frac{1}{x}$

$\therefore \alpha = \tan^{-1}\left(\frac{1}{x}\right)$

ii) $\tan(\theta+\alpha) = \frac{3}{x}$

$\therefore \dfrac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \dfrac{3}{x}$

$\tan\alpha = \frac{1}{x}$ $\therefore$ $x\left(\tan\theta + \frac{1}{x}\right) = 3 - \frac{3}{x}\tan\theta$

$x^2\tan\theta + 3\tan\theta = 2x$

$\tan\theta = \dfrac{2x}{x^2+3}$

ie $\theta = \tan^{-1}\left(\dfrac{2x}{x^2+3}\right)$

c) iii) $\dfrac{d\theta}{dx} = \dfrac{1}{1+\left(\frac{2x}{3+x^2}\right)^2} \times \dfrac{2(3+x^2)-2x.2x}{(3+x^2)^2}$

$= 0$ when $6+2x^2-4x^2=0$

ie $6-2x^2=0$

$x = \pm\sqrt{3}$

$\therefore x = \sqrt{3}$, $x>0$