Extension 2 HSC 2012

1/ $2z + \bar{w} = 2(5-i) + (2-3i)$
$= \underline{\underline{12 - 5i}}$

(D)

2/ $x^3 - y^3 + 3xy + 1 = 0$

$3x^2 - 3y^2\frac{dy}{dx} + (3x)(\frac{dy}{dx}) + (y)(3) = 0$

$(3x - 3y^2)\frac{dy}{dx} = -3x^2 - 3y$

$\frac{dy}{dx} = \frac{3x^2 + 3y}{3y^2 - 3x}$

at $(1,2)$, $\frac{dy}{dx} = \frac{3(1)^2 + 3(2)}{3(2)^2 - 3(1)}$
$= \underline{\underline{1}}$

(D)

3/ $i\bar{z}$ = reflect z in x axis ($\bar{z}$)
then rotate $90°$ anticlockwise ($\times i$)

$\therefore$ (A)

4/ $y = [f(x)]^2$, x intercepts $\Rightarrow$ turning pts

$\therefore$ (A)

5/ $2x^3 - 3x^2 - 5x - 1 = 0$

$\alpha\beta\gamma = \frac{1}{2}$ $\therefore$ $\frac{1}{\alpha^3\beta^3\gamma^3} = \underline{\underline{8}}$

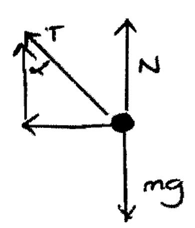
(C)

6/ $\frac{x^2}{6} - \frac{y^2}{4} = 1$

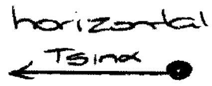
$b^2 = a^2(e^2 - 1)$
$4 = 6(e^2 - 1)$
$e^2 - 1 = \frac{2}{3}$
$e^2 = \frac{5}{3}$
$e = \sqrt{\frac{5}{3}} = \underline{\underline{\frac{\sqrt{15}}{3}}}$

(B)

7/

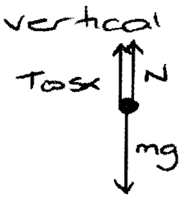


horizontal $F = mr\omega^2$



$T\sin\alpha = mr\omega^2$

vertical $F = 0$



$T\cos\alpha + N - mg = 0$
$T\cos\alpha + N = mg$

(A)

8/ $P'(x)$ has a double root at $x=1$

$\therefore P(x)$ has a triple root at $x=1$ ie $(x-1)^3$ is a possible factor.

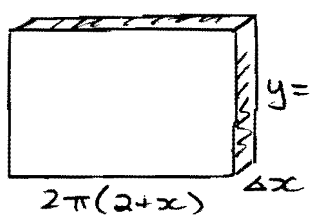
$P'(0)>0$ $\therefore P(x)$ increases to a root at $x=1$

$\therefore P(0)<0$

$A(0) = 2 > 0$ ✗
$B(0) = -2 < 0$ ✓

$\therefore$ (B)

9/



$A(x) = 2\pi(2+x) \times x(2-x)$

$V(x) = 2\pi x(2+x)(2-x)\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 2\pi x(2+x)(2-x)\Delta x$

$= 2\pi \int_0^2 x(2+x)(2-x)\,dx$ (C)

10/ (A) x = odd F

$2+\cos x$ = even F

$\therefore \dfrac{x}{2+\cos x}$ = odd F

$\int = 0$ ✗

(B) x^3 = odd F

$\sin x$ = odd F

$\therefore x^3 \sin x$ = even F

$\int_{-\pi}^{\pi} x^3 \sin x\,dx$

$= 2\int_0^{\pi} x^3 \sin x\,dx$

as $\sin x > 0$ for $0 < x < \pi$

$x^3 > 0$ for $0 < x < \pi$

then $x^3 \sin x > 0$ for $0 < x < \pi$

thus $2\int_0^{\pi} x^3 \sin x\,dx > 0$ (B)

Question 11

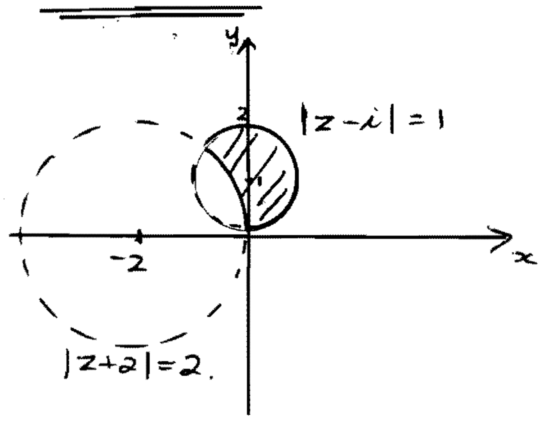
a) $\dfrac{2\sqrt{5}+i}{\sqrt{5}-i} \times \dfrac{\sqrt{5}+i}{\sqrt{5}+i}$

$= \dfrac{10 + 2\sqrt{5}i + \sqrt{5}i - 1}{5+1}$

$= \dfrac{9}{6} + \dfrac{3\sqrt{5}}{6}i$

$= \dfrac{3}{2} + \dfrac{\sqrt{5}}{2}i$

b)

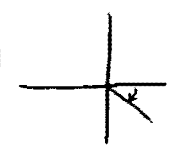


c) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$

$= \tan^{-1}(x+2) + c$

d) (i) $|\sqrt{3}-i| = \sqrt{3+1}$
$= 2$

$\arg(\sqrt{3}-i) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$
$= -\frac{\pi}{6}$



$\therefore \sqrt{3}-i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

(ii) $z^9 = 2^9\left(\cos\left(-\frac{9\pi}{6}\right) + i\sin\left(-\frac{9\pi}{6}\right)\right)$

$= 512\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

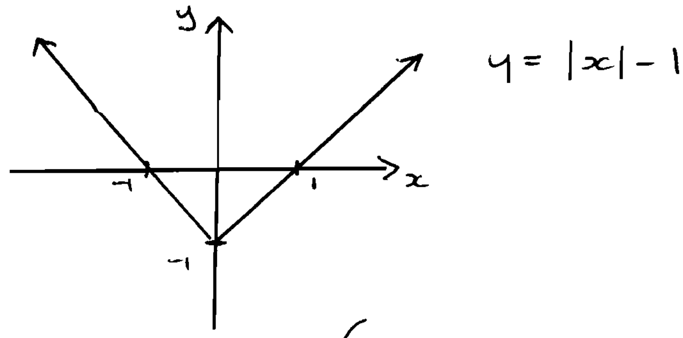
$= 512i$

e) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\left[\ln(e^{2x}+1)\right]_0^1$

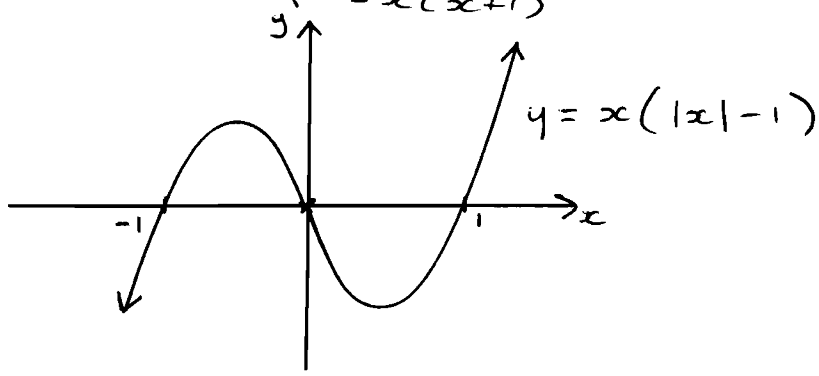
$= \frac{1}{2}\left\{\ln(e^2+1) - \ln 1\right\}$

$= \frac{1}{2}\ln(e^2+1)$

f) (i)



(ii) $x(|x|-1) = \begin{cases} x(x-1), & x \geq 0 \\ x(-x-1), & x < 0 \\ = -x(x+1) & \end{cases}$



Question 12

a) $\int \frac{d\theta}{1-\cos\theta}$

$t = \tan\frac{\theta}{2}$

$d\theta = \frac{2dt}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}$

$= \int \frac{2dt}{1+t^2-1+t^2}$

$= \int \frac{1}{t^2}\,dt$

$= -\frac{1}{t} + c$

$= -\cot\frac{\theta}{2} + c$

b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$

at P, $\frac{dy}{dx} = \frac{-b^2x_0}{a^2y_0}$

∴ tangent at P is

$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$

(ii) $m_{PN} = \frac{a^2y_0}{b^2x_0}$

$\frac{y_0 - 0}{x_0 - x} = \frac{a^2y_0}{b^2x_0}$

$a^2(x_0 - x) = b^2x_0$

$a^2x = x_0(a^2 - b^2)$

$= x_0(a^2 - a^2(1-e^2))$

$= a^2x_0(1-1+e^2)$

$= a^2x_0e^2$

$x_N = x_0e^2$

(iii) T is x intercept of tangent

$-y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$

$-a^2y_0^2 = -b^2x_0x + b^2x_0^2$

$b^2x_0x = b^2x_0^2 + a^2y_0^2$

$\frac{x_0x}{a^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$

$= 1$

$$x_T = \frac{a^2}{x_0}$$

$$\text{ON} \times \text{OT} = x_0 e^2 \times \frac{a^2}{x_0}$$

$$= a^2 e^2$$

$$= (ae)^2$$

$$= \underline{\underline{OS^2}}$$

c) $$I_n = \int_1^{e^2} (\log_e x)^n \, dx \qquad u = (\log_e x)^n \qquad v = x$$

$$du = \frac{n(\log_e x)^{n-1}}{x} dx \qquad dv = dx$$

$$= \left[x(\log_e x)^n \right]_1^{e^2} - n \int_1^{e^2} (\log_e x)^{n-1} \, dx$$

$$= e^2(\log_e e^2)^n - (\log 1)^n - nI_{n-1}$$

$$= \underline{\underline{e^2 2^n - nI_{n-1}}}$$

d) $$\overrightarrow{A_1B_1} = i\,\overrightarrow{A_1P}$$

$$w_1 - u_1 = i(z - u_1)$$

$$\underline{\underline{w_1 = u_1 + i(z - u_1)}}$$

(ii) $$\overrightarrow{A_2B_2} = -i\,\overrightarrow{A_2P}$$

$$w_2 - u_2 = -i(z - u_2)$$

$$w_2 = u_2 - i(z - u_2)$$

$$\text{midpoint of } B_1B_2 = \frac{w_1 + w_2}{2}$$

$$= \underline{\underline{\frac{u_1 + u_2}{2} + i\,\frac{u_2 - u_1}{2}}}$$

Question 13

a) $\frac{dv}{dt} = 10 - \frac{v^2}{40}$

$= \frac{400 - v^2}{40}$

$$\int_0^v \frac{dv}{400 - v^2} = \int_0^t \frac{dt}{40}$$

$$\frac{1}{40}\left[\ln\left(\frac{20+v}{20-v}\right)\right]_0^v = \frac{1}{40}\left[t\right]_0^t$$

$$\left\{\ln\left(\frac{20+v}{20-v}\right) - \ln 1\right\} = t$$

$$\ln\left(\frac{20+v}{20-v}\right) = t$$

$$\frac{20+v}{20-v} = e^t$$

$$20 + v = 20e^t - ve^t$$

$$v(e^t + 1) = 20(e^t - 1)$$

$$\underline{\underline{v = \frac{20(e^t - 1)}{e^t + 1}}}$$

(ii)

$$v\frac{dv}{dx} = \frac{400 - v^2}{40}$$

$$-\int_0^v \frac{-2v}{400 - v^2}\, dv = \int_0^x \frac{dx}{20}$$

$$-\left[\ln(400 - v^2)\right]_0^v = \frac{1}{20}\left[x\right]_0^x$$

$$-20\left(\ln(400 - v^2) - \ln 400\right) = x$$

$$x = -20 \ln\left(\frac{400 - v^2}{400}\right)$$

$$\underline{\underline{x = 20 \ln\left(\frac{400}{400 - v^2}\right)}}$$

b) (i) $\angle QPS = \angle PSR = \alpha$ (alternate $\angle$'s $=$, $PQ \parallel SR$)

$\angle QPS' = \angle SRS' = \alpha$ (corresponding $\angle$'s $=$, $PQ \parallel SR$)

$\therefore \angle PSR = \angle SRS'$

$\triangle PRS$ is isosceles (2 $=$ $\angle$'s)

$\underline{\underline{PS = PR}}$ ($=$ sides in isosceles $\triangle$)

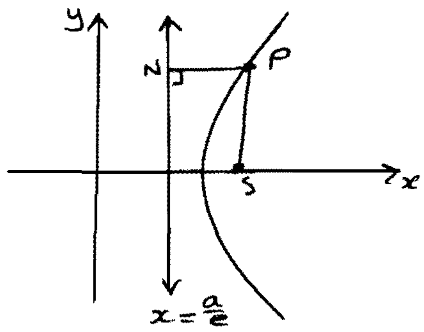
(ii) Construct $PS' \parallel PQ$

$$\frac{PS'}{PR} = \frac{QS'}{QS}$$ (ratio of intercepts of $\parallel$ lines)

$$\frac{PS'}{PS} = \frac{QS'}{QS}$$

$$\therefore \frac{PS'}{QS'} = \frac{PS}{QS}$$

c)



$$PS = ePN$$
$$= e\left(a\sec\theta - \frac{a}{e}\right)$$
$$= ae\sec\theta - a$$
$$= a(e\sec\theta - 1)$$

(ii) $$\frac{PS}{QS} = \frac{PS'}{QS'}$$

$$\frac{a(e\sec\theta - 1)}{ae - x_Q} = \frac{a(e\sec\theta + 1)}{ae + x_Q}$$

$$(ae + x_Q)(e\sec\theta - 1) = (ae - x_Q)(e\sec\theta + 1)$$

$$ae^2\sec\theta - ae + x_Q e\sec\theta - x_Q = ae^2\sec\theta + ae - x_Q e\sec\theta - x_Q$$

$$2x_Q e\sec\theta = 2ae$$

$$x_Q = \frac{a}{\sec\theta}$$

(iii) $$m_{PQ} = \frac{b\tan\theta - 0}{a\sec\theta - \frac{a}{\sec\theta}}$$

$$= \frac{b\tan\theta}{a} \times \frac{\sec\theta}{\sec^2\theta - 1}$$

$$= \frac{b\tan\theta}{a} \times \frac{\sec\theta}{\tan^2\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

$$= m_{\text{tangent}}$$

$\therefore$ PQ is tangent at P

Question 14

a) $\displaystyle\int \frac{3x^2+8}{x(x^2+4)}\,dx$

$A(x^2+4) + (Bx+C)x \equiv 3x^2+8$

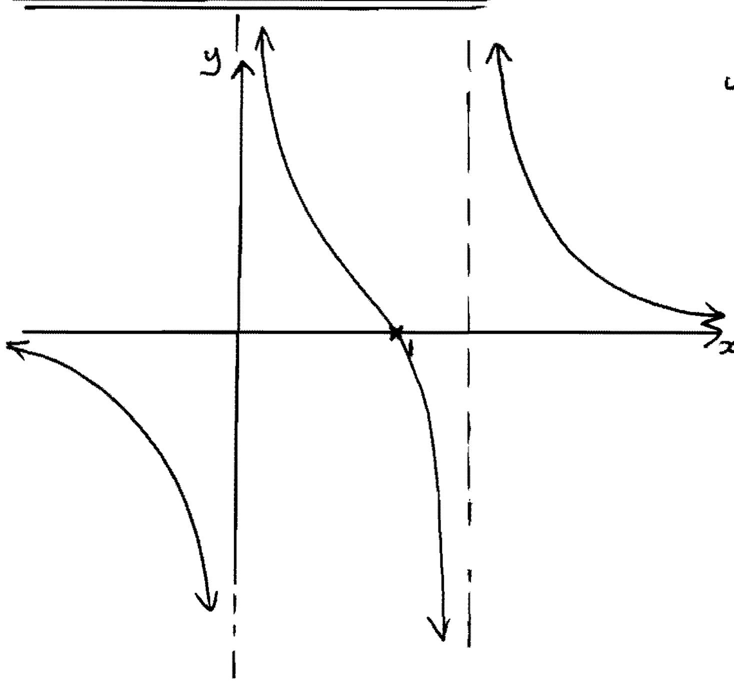
$x=0$: $4A=8$, $A=2$

$x=2i$: $-4B+2iC=-4$, $B=1$, $C=0$

$$= \int\left[\frac{2}{x} + \frac{x}{x^2+4}\right]dx$$

$$= 2\ln x + \tfrac{1}{2}\ln(x^2+4) + c$$

b)



$y = \dfrac{x-1}{x(2x-3)}$

(ii) $$\frac{x(2x-3)}{x-1} = \frac{2x^2-3x}{x-1}$$

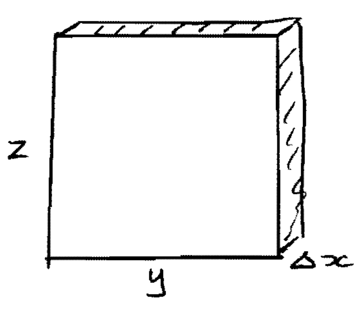
$$= \frac{2x(x-1)-x}{x-1}$$

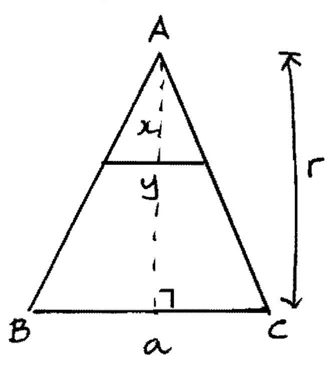
$$= \frac{2x(x-1)-1(x-1)-1}{x-1}$$

$$= 2x-1 - \frac{1}{x-1}$$

$\therefore$ l is the line $y = 2x-1$

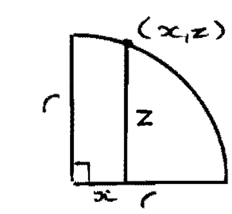
c)





$\dfrac{x}{r} = \dfrac{y}{a}$

$y = \dfrac{ax}{r}$



$z = \sqrt{r^2-x^2}$

$$A(x) = yz$$
$$= \frac{ax}{r} \times \sqrt{r^2-x^2}$$

$$\Delta V = \frac{a}{r}x\sqrt{r^2-x^2}\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{r} \frac{a}{r} x \sqrt{r^2 - x^2}\, \Delta x$$

$$= -\frac{a}{2r} \int_0^r -2x\sqrt{r^2 - x^2}\, dx$$

$$= -\frac{a}{2r} \times \frac{2}{3}\left[(r^2 - x^2)^{\frac{3}{2}}\right]_0^r$$

$$= -\frac{a}{3}(0 - r^3)$$

$$= \underline{\underline{\frac{1}{3}ar^3 \text{ units}^3}}$$

d)

$\angle PGA = \angle PEB = 90°$ (given)

$\angle PAG = \angle PBE$ (alternate segment theorem)

$\therefore \underline{\underline{\triangle APG \parallel\!| \triangle BPE}}$ (AA)

(ii) Similarly $\triangle BPF \parallel\!| \triangle APE$

$\frac{PE}{PF} = \frac{AP}{BP}$ (ratio of sides in $\parallel\!|$ $\triangle$'s)

$\frac{PG}{PE} = \frac{AP}{BP}$ (ratio of sides in $\parallel\!|$ $\triangle$'s in part i)

$\therefore \frac{PE}{PF} = \frac{PG}{PE}$

$\underline{\underline{PE^2 = PF \times PG}}$

Question 15

a) (i) $(\sqrt{a} - \sqrt{b})^2 \geq 0$

$$a^{2} - 2\sqrt{ab} + b^{2} \geq 0$$

$$a^{2} + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\therefore \sqrt{ab} \leq \frac{a+b}{2}$$

(ii) $x(y-x+1) - y = xy - x^2 + 1 - y$

$= x(y-1) - (y-1)$

$= (y-1)(x-1) \geq 0$ as $1 \leq x \leq y$

$$\therefore x(y-x+1) - y \geq 0$$

$$x(y-x+1) \geq y$$

(iii) $j(n-j+1) \geq n$

as $j \geq 1$ and $n \geq j$, $n-j+1 \geq 0$.

also $n > 0$

$$\therefore \sqrt{j(n-j+1)} \geq \sqrt{n}$$

using part (i)

$$\sqrt{j(n-j+1)} \leq \frac{j + n-j+1}{2}$$

$$= \frac{n+1}{2}$$

$$\therefore \sqrt{n} \leq \sqrt{j(n-j+1)} \leq \frac{n+1}{2}$$

(iv) $j=1 \quad \sqrt{n} \le \sqrt{n} \le \frac{n+1}{2}$

$j=2 \quad \sqrt{n} \le \sqrt{2(n-1)} \le \frac{n+1}{2}$

$j=3 \quad \sqrt{n} \le \sqrt{3(n-2)} \le \frac{n+1}{2}$

$\vdots$

$j=n \quad \sqrt{n} \le \sqrt{n(1)} \le \frac{n+1}{2}.$

multiply all together

$$(\sqrt{n})^n \le \sqrt{n \times 2(n-1) \times 3(n-2) \times \dots \times n(1)} \le \left(\frac{n+1}{2}\right)^n$$

$$(\sqrt{n})^n \le \sqrt{(n!)^2} \le \left(\frac{n+1}{2}\right)^n$$

$$\underline{\underline{(\sqrt{n})^n \le n! \le \left(\frac{n+1}{2}\right)^n}}$$

b) (i) As all coefficients are real, complex roots will appear in conjugate pairs

$\therefore$ if α, $i\alpha$ are roots

then so are $\overline{\alpha}$, $\overline{i\alpha}$

but $\overline{i\alpha} = \overline{i} \times \overline{\alpha}$

$= -i\overline{\alpha}$

$\therefore$ $\underline{\underline{\overline{\alpha} \text{ and } -i\overline{\alpha} \text{ are roots.}}}$

(ii) $z^2(z-k)^2 + (kz-1)^2$

$= z^4 - 2z^3k + z^2k^2 + z^2k^2 - 2zk + 1$

$= z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$

$= \underline{\underline{P(z)}}$

(iii) let α be a real root

$\therefore$ $i\alpha$ and $-i\overline{\alpha} = -i\alpha$ are also roots

If β is the fourth root then $i\beta$ and $-i\beta$ are also roots

However there are only four roots $\therefore$ α must be a double root.

$\therefore P(z) = (z-\alpha)^2(z-i\alpha)(z+i\alpha)$

$= (z-\alpha)^2(z^2+\alpha^2)$

Question 16

a) (i) Ways $= \dfrac{(m+n)!}{m!\,n!}$

(ii) The ten coins are broken into 4 groups

let the break between the groups be B

as there will be three breaks, the question is equivalent to

how many ways can you arrange

10 C's and 3 B's

$$\text{Ways} = \frac{13!}{10!\,3!}$$
$$= 286$$

b) (i) Let $\alpha = \tan^{-1}x$, $\beta = \tan^{-1}y$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$
$$\alpha+\beta = \tan^{-1}\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)$$
$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(ii) Prove for $n=1$

$LHS = \tan^{-1}\frac{1}{2}$ $\qquad$ $RHS = \tan^{-1}\left(\frac{1}{1+1}\right)$

$= \tan^{-1}\frac{1}{2}$

$\therefore LHS = RHS$

Hence true for $n=1$

Assume true for $n=k$, where k is positive integer

ie $\displaystyle\sum_{j=1}^{k}\tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{k}{k+1}\right)$

Prove true for $n=k+1$

ie $\displaystyle\sum_{j=1}^{k+1}\tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{k+1}{k+2}\right)$

Proof
$$\sum_{j=1}^{k+1}\tan^{-1}\left(\frac{1}{2j^2}\right) = \sum_{j=1}^{k}\tan^{-1}\left(\frac{1}{2j^2}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$
$$= \tan^{-1}\left(\frac{k}{k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$
$$= \tan^{-1}\left[\frac{\frac{k}{k+1} + \frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2k(k+1)^2+(k+1)}{2(k+1)^3-k}\right]$$

$$= \tan^{-1}\left[\frac{(k+1)(2k(k+1)+1)}{2k^3+6k^2+5k+2}\right]$$

$$= \tan^{-1}\left[\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right]$$

$$= \tan^{-1}\left(\frac{k+1}{k+2}\right)$$

Hence the result is true for $n=k+1$ if it is true for $n=k$

Since the result is true for $n=1$, then it is true for all positive integral values of ~~k~~ by induction.

(iii) $$\lim_{n\to\infty} \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right) = \lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+1}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

c) $$P(k) = \underset{\substack{\uparrow \\ \text{1st} \\ \text{picked}}}{\frac{n}{n}} \times \underset{\substack{\uparrow \\ \text{2nd} \\ \text{picked}}}{\frac{n-1}{n}} \times \underset{\substack{\uparrow \\ \text{3rd} \\ \text{picked}}}{\frac{n-2}{n}} \times \ldots \times \underset{\substack{\uparrow \\ k\text{th} \\ \text{picked}}}{\frac{n-k+1}{n}} \times \frac{k}{n}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)k}{n^{k+1}} \times \frac{(n-k)!}{(n-k)!}$$

$$= \frac{(n-1)(n-2)\ldots(n-k+1)k\,(n-k)!}{n^k\,(n-k)!}$$

$$= \frac{(n-1)!\,k}{n^k\,(n-k)!}$$

(ii) $P(k) \geqslant P(k-1)$

$$\frac{(n-1)!\,k}{n^k\,(n-k)!} \geqslant \frac{(n-1)!\,(k-1)}{n^{k-1}\,(n-k+1)!}$$

$$\frac{k}{n} \geqslant \frac{k-1}{n-k+1}$$

$$nk - k^2 + k \geqslant nk - n$$ (as all terms are >0, inequality is preserved)

$$0 \geqslant k^2 - k - n$$

$$k^2 - k - n \leqslant 0$$

(iii) $\sqrt{n+\frac{1}{4}} > k-\frac{1}{2}$

$n+\frac{1}{4} > k^2-k+\frac{1}{4}$

$n > k^2-k$

but as n and k are integers

$n \geq k^2-k+1$

$\therefore n > k^2-k+\frac{1}{4}$

$= (k-\frac{1}{2})^2$

$\therefore \underline{\underline{\sqrt{n} > k-\frac{1}{2}}}$

(iv) If $P(k) > P(k-1)$ ~~then $P(k)$ is the greatest~~

~~$\therefore$ If $P(k)$ is the greatest~~

$P(k) > P(k-1)$

$k^2-k-n < 0$ (from ii)

$0 < k < \frac{1+\sqrt{1+4n}}{2}$ ($\frac{1-\sqrt{1+4n}}{2} < 0$ and $k > 0$)

so $k < \frac{1}{2} + \frac{\sqrt{1+4n}}{2}$.

$k-\frac{1}{2} < \sqrt{\frac{1}{4}+n}$

$\therefore \sqrt{n} > k-\frac{1}{2}$ (from iii)

If $P(k-1) > P(k)$

$k^2-k-n > 0$

$k - \frac{1}{2} > \sqrt{n}$

$\therefore P(k) > P(k+1)$

$k+1-\frac{1}{2} > \sqrt{n}$

$k+\frac{1}{2} > \sqrt{n}$

If $P(k)$ is the greatest then both

$P(k) > P(k-1)$ and $P(k) > P(k+1)$ must be true

$\therefore k-\frac{1}{2} < \sqrt{n} < k+\frac{1}{2}$

<u>thus k is the nearest integer to $\sqrt{n}$.</u>