4 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours (plus 5 minutes reading) **Exam date:** 8th August 2001

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new 4-page answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

$\boxed{3}$ (a) Find $\displaystyle\int_1^{e^2} x^2 \log_e x\,dx$.

$\boxed{3}$ (b) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \int_0^1 (1-2u^2+u^4)\,du$, where $u=\cos x$.

$\boxed{3}$ (c) Find $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$.

$\boxed{3}$ (d) (i) Show that $\displaystyle\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \tfrac{1}{2}(\pi + \ln \tfrac{27}{16})$.

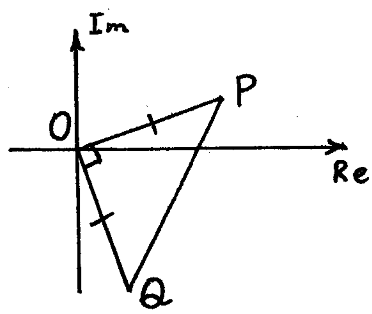
$\boxed{3}$ (ii) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+2\sin x+\cos x}\,dx$ using the substitution $t=\tan\frac{1}{2}x$.

QUESTION TWO (Start a new answer booklet)

Marks

$\boxed{2}$ (a) Evaluate $\arg\left((2+i)\overline{w}\right)$, given $w=-1-3i$.

$\boxed{2}$ (b) Write $x^2-12x+48$ as the product of two linear factors.



$\boxed{1}$ (c) In the diagram on the right, triangle POQ is right-angled and isosceles. If P represents the complex number $a+bi$, where a and b are real, find the complex number represented by Q.

(d) Sketch in the Argand diagram the locus of the complex number z given:

$\boxed{2}$ (i) $\arg(z-2) = \arg z + \frac{\pi}{2}$

$\boxed{3}$ (ii) $|z+3i| < 2|z|$

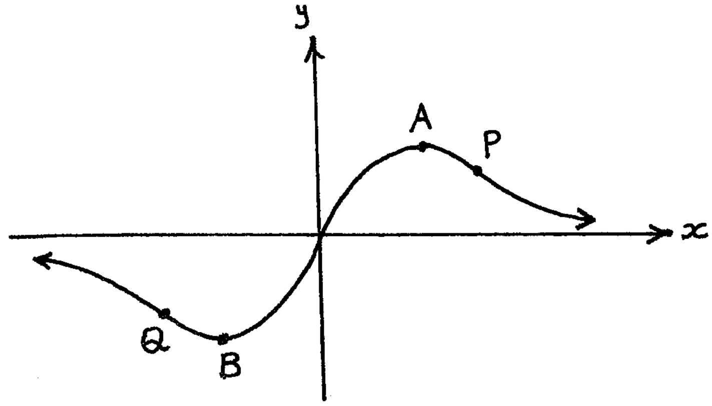
$\boxed{2}$ (e) (i) Find, in modulus-argument form, the three cube roots of -8.

$\boxed{1}$ (ii) Write the two unreal cube roots of -8 in the form $a+bi$, where a and b are real.

$\boxed{2}$ (iii) If w_1 and w_2 are the unreal cube roots of -8, show that ${w_1}^{6n} + {w_2}^{6n} = 2^{6n+1}$ for all integers n.

QUESTION THREE (Start a new answer booklet)

(a)



In the diagram above, the curve $y = \dfrac{2x}{1+x^2}$ is sketched.

Marks

2 (i) Find the coordinates of the turning points A and B.

3 (ii) Find the coordinates of the inflexion points P and Q.

(b) Draw separate sketches of:

1 (i) $y = \dfrac{|2x|}{1+x^2}$

2 (ii) $y = \dfrac{1+x^2}{2x}$

2 (iii) $y^2 = \dfrac{2x}{1+x^2}$

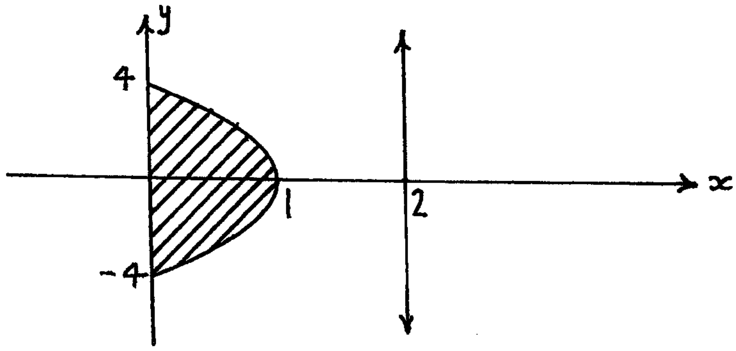
2 (iv) $y = \log_e\left(\dfrac{2x}{1+x^2}\right)$

1 (c) (i) Show that the equation $kx^3+(k-2)x=0$ can be written in the form $\dfrac{2x}{1+x^2} = kx$.

2 (ii) Using a graphical approach based on the curve $y = \dfrac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3+(k-2)x=0$ has exactly one real root.

QUESTION FOUR (Start a new answer booklet)

(a)



A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y-axis through $360°$ about the line $x = 2$.

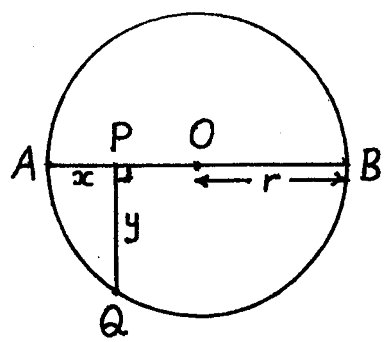
Marks

$\boxed{4}$ (i) By slicing perpendicular to the axis of rotation, find the exact volume of S.

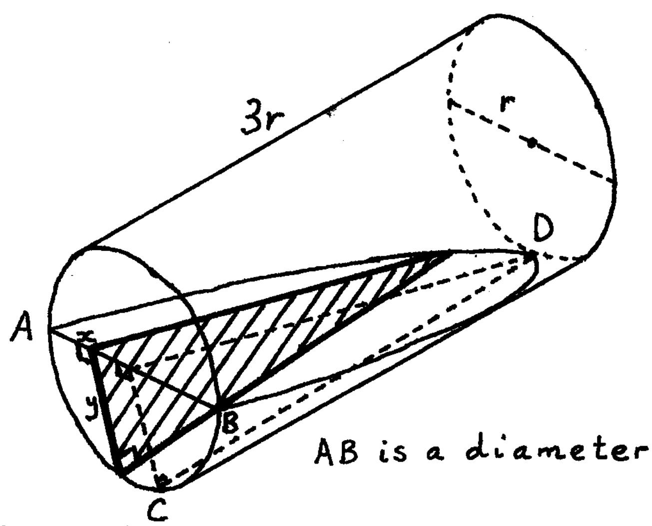
$\boxed{2}$ (ii) (α) Use the method of cylindrical shells to show that the volume of S is also given by $\displaystyle\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$.

$\boxed{3}$ (β) Confirm your answer to part (i) by calculating this definite integral using the substitution $u = 1 - x$.

$\boxed{2}$ (b) (i) In the diagram on the right, AB is a diameter of a circle whose centre is O and whose radius is r units. The point P lies on the radius OA and Q is a point on the circle such that $QP \perp OA$. Let $AP = x$ units and $PQ = y$ units. Show that $y^2 = 2rx - x^2$.



$\boxed{3}$ (ii)

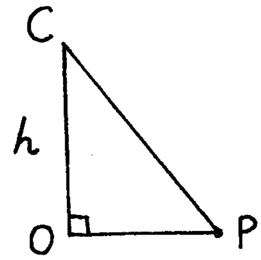


The diagram above shows a wedge $ABCD$ which is to be cut from a cylinder of radius r units and height $3r$ units. A typical slice of thickness δx is shown. Use the result in part (i) to show that the volume of the wedge is $2r^3$ cubic units.

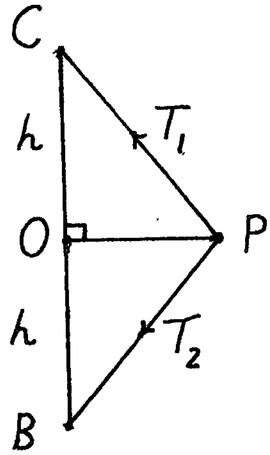
$\boxed{1}$ (iii) What percentage of the volume of the original cylinder is the volume of the wedge? (Give your answer to the nearest whole percent.)

QUESTION FIVE (Start a new answer booklet)

Marks



[2] (a) (i) The diagram on the right shows a particle P attached by an inelastic string to a fixed point C. The particle moves in uniform circular motion about a fixed point O that is at a distance h below C. Show that the angular velocity of P about O is $\sqrt{\frac{g}{h}}$, where g is acceleration due to gravity.



(ii) Suppose now, as shown in the diagram on the right, that P is attached by a second string, identical to the first, to another fixed point B which is at a distance $2h$ below C.

[2] (α) Write down two equations of motion by resolving forces vertically and horizontally at P.

[3] (β) If the angular velocity of P about O is $3\sqrt{\frac{g}{h}}$, show that the ratio $T_1 : T_2$ of the tensions in the two strings is $5 : 4$.

(b) A particle is projected from the origin with speed V at an angle α to the horizontal. The particle is subject to both gravity and an air resistance proportional to its velocity, so that its respective horizontal and vertical components of acceleration while it is rising are given by:

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

(i) Show that:

[2] (α) $\dot{x} = V\cos\alpha\, e^{-kt}$

[2] (β) $\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$

(ii) Hence show that:

[1] (α) $x = \dfrac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$

[1] (β) $y = \left(\dfrac{g}{k^2} + \dfrac{V\sin\alpha}{k}\right)\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$

[2] (iii) When the particle reaches its greatest height, show that it has travelled a horizontal distance of $\dfrac{V^2\sin 2\alpha}{2(g + Vk\sin\alpha)}$.

QUESTION SIX (Start a new answer booklet)

(a) The polynomial $P(x) = x^3 + cx + d$, where c and d are real and non-zero, has zeroes $a + ib$, $a - ib$ and k, where a and b are real and non-zero, and $k < 0$. It is known that the graph of $y = P(x)$ has two turning points.

Marks

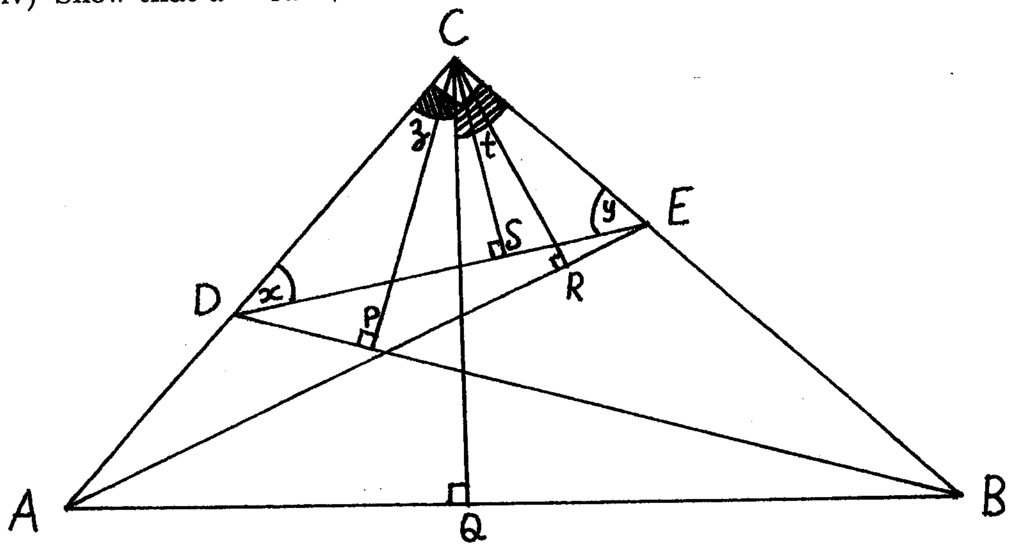
[1] (i) By considering $P'(x)$, show that $c < 0$.

[2] (ii) Sketch the graph of $y = P(x)$.

[1] (iii) Deduce that $a > 0$.

[3] (iv) Show that $d = 8a^3 + 2ac$.

(b)



In the diagram above, $\triangle ABC$ is right-angled at C. D and E are arbitrary points on AC and BC respectively. Perpendiculars with feet P, Q, R and S are dropped from C to DB, AB, AE and DE respectively.
Let $\angle CDS = x$, $\angle CES = y$, $\angle ACQ = z$ and $\angle QCB = t$.

Make a large copy of the diagram in your answer booklet.

[1] (i) Show that $x + y = 90°$.

[2] (ii) Explain why the quadrilateral $CDPS$ is cyclic, and hence give a reason why $\angle CPS = x$.

[2] (iii) Using similar arguments to part (ii), deduce that $\angle CRS = y$, $\angle ARQ = z$ and $\angle QPB = t$.

[3] (iv) Hence prove that $PQRS$ is a cyclic quadrilateral.
(Hint: Let $\angle SPB = u$ and $\angle SRA = v$.)

QUESTION SEVEN (Start a new answer booklet)

(a) Let $I_n = \int_0^1 (1-x^2)^n \, dx$ and $J_n = \int_0^1 x^2(1-x^2)^n \, dx$.

Marks

2 (i) Apply integration by parts to I_n to show that $I_n = 2n\, J_{n-1}$.

2 (ii) Hence show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$.

2 (iii) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \dfrac{1}{2n+3} I_n$.

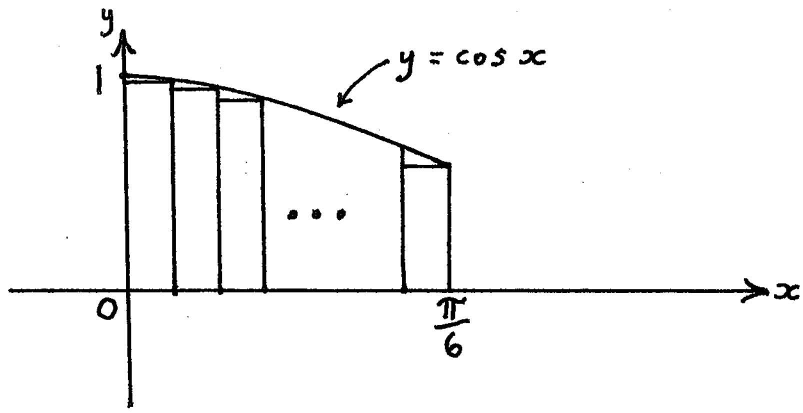
1 (iv) Hence write down a reduction formula for J_n in terms of J_{n-1}.

4 (b) (i) Prove by mathematical induction that for all positive integers n:

$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \frac{\sin\frac{1}{2}(2n-1)\theta}{2\sin\frac{1}{2}\theta}$$

(Hint: You may use the identity $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.)

(ii)



In the above diagram, n rectangles of equal width are constructed under the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{6}$.

2 (α) Use the result in part (i) to show that the sum S_n of the areas of the n rectangles is given by:

$$S_n = \frac{\pi}{12n}\left((\sqrt{3}-1) + \frac{\sin(2n-1)\frac{\pi}{12n}}{\sin\frac{\pi}{12n}}\right)$$

2 (β) Hence find $\lim_{n\to\infty} S_n$. (Hint: Let $n = \dfrac{1}{h}$.)

QUESTION EIGHT (Start a new answer booklet)

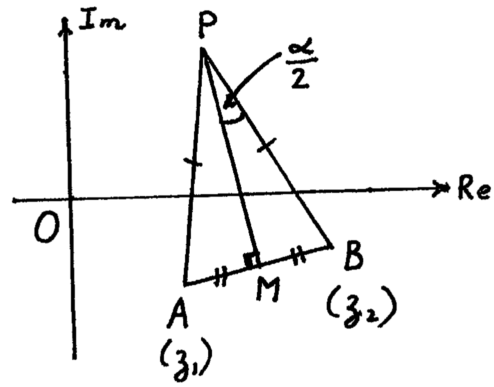
(a) Suppose that $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear points in the real number plane, and T is the point $(ux_1 + vx_2 + wx_3, uy_1 + vy_2 + wy_3)$, where u, v and w are non-zero and $u + v + w = 1$.

Marks

2 (i) Show that $x_1y_2 + x_2y_3 + x_3y_1 = x_1y_3 + x_2y_1 + x_3y_2$.

3 (ii) Hence, or otherwise, show that T lies on the line through A, B and C.

(b) The diagram on the right shows an isosceles triangle ABP in the Argand diagram, with base AB and $\angle APB = \alpha$. PM is the perpendicular bisector of AB and so bisects $\angle APB$. Suppose that A and B represent the complex numbers z_1 and z_2 respectively.



(i) Find the complex number represented by:

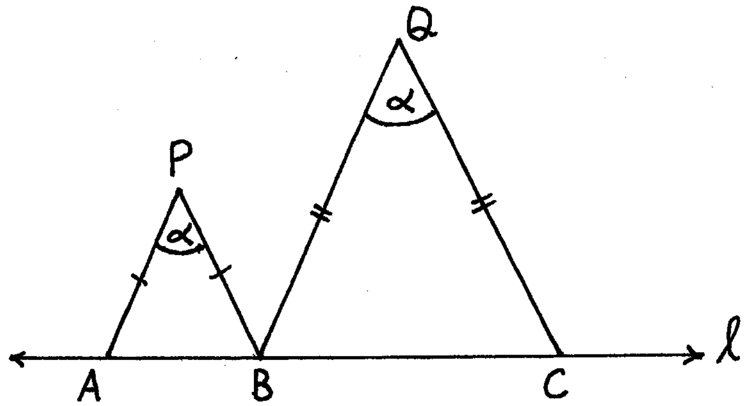
1 (α) the vector AM,

1 (β) the vector MP.

2 (ii) Hence show that P represents the complex number:

$$\tfrac{1}{2}(1 - i\cot\tfrac{1}{2}\alpha)z_1 + \tfrac{1}{2}(1 + i\cot\tfrac{1}{2}\alpha)z_2$$

6 (c)



In the diagram above, A, B and C are arbitrary points on the line ℓ, and α is a given fixed angle. Isosceles triangles APB and BQC are constructed with bases AB and BC respectively and equal angles of α at P and Q. Suppose that a third isosceles triangle PRQ is constructed with base PQ and $\angle PRQ = \alpha$. If the cyclic orientation PRQ is anticlockwise, prove that R lies on ℓ. (You may use the results in parts (a) and (b).)

DS

SOLUTIONS : 4U TRIAL, 2001 TOTAL = 120

(1)(a) Let $u = \log_e x$

$\therefore u' = \frac{1}{x}$

Let $v' = x^2$

$\therefore v = \frac{x^3}{3}$

$\therefore \int_1^{e^2} x^2 \log_e x \, dx$

$= \left[\frac{x^3}{3}\log_e x\right]_1^{e^2} - \int_1^{e^2} \frac{x^2}{3}\,dx$

$= \frac{e^6}{3}.2 - \left(\frac{e^6}{9} - \frac{1}{9}\right)$

$= \frac{5e^6+1}{9}$

(b) $\sin^5 x = \sin x(\sin^2 x)^2$

$= \sin x(1-\cos^2 x)^2$

$= \sin x(1-2\cos^2 x + \cos^4 x)$

Let $u = \cos x$

$\therefore du = -\sin x \, dx$

x	0	$\frac{\pi}{2}$
u	1	0

$\therefore \int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \int_0^{\frac{\pi}{2}} \sin x(1-2\cos^2 x+\cos^4 x)\,dx$

$= \int_1^0 (1-2u^2+u^4).-du$

$= \int_0^1 (1-2u^2+u^4)\,du$

(c) $\int \frac{2x+1}{x^2+2x+2}\,dx$

$= \int \frac{(2x+2)-1}{x^2+2x+2}\,dx$

$= \int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{1}{(x+1)^2+1}\,dx$

$= \ln(x^2+2x+2) - \tan^{-1}(x+1) + c$

[no penalty for omission of c]

(d)(i)

Let $\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$

$\therefore A(1+x^2) + (Bx+C)(1+2x) = 5-5x^2$

Let $x = -\frac{1}{2}$

$\therefore \frac{5A}{4} = 5.\frac{3}{4}$

$\therefore A = 3$

Let $x=0$

$\therefore 3 + C = 5$

$\therefore C = 2$

Let $x=1$

$\therefore 6 + (B+2).3 = 0$

$\therefore B = -4$

$\therefore \int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx$

$= \int_0^1 \frac{3}{1+2x}\,dx + \int_0^1 \frac{-4x+2}{1+x^2}\,dx$

$= \left[\frac{3}{2}\ln(1+2x) - 2\ln(1+x^2) + 2\tan^{-1}x\right]_0^1$

$= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}$

$= \frac{1}{2}(3\ln 3 - 4\ln 2 + \pi)$

$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$

(ii) $dx = \frac{2}{1+t^2}\,dt$

x	0	$\frac{\pi}{2}$
t	0	1

$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+2\sin x+\cos x}\,dx$

$= \int_0^1 \frac{1-t^2}{1+t^2} \cdot \frac{1+t^2}{1+t^2+4t+1-t^2} \cdot \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{2(1-t^2)}{(1+t^2)(4t+2)}\,dt$

$= \frac{1}{5}\int_0^1 \frac{5-5t^2}{(1+t^2)(1+2t)}\,dt$

$= \frac{1}{10}\left(\pi + \ln\frac{27}{16}\right)$ [using (i)]

(2)(a) $\arg((2+i)\bar{w})$

$= \arg((2+i)(-1+3i))$

$= \arg(-5+5i)$

$= \frac{3\pi}{4}$

(b) $x^2 - 12x + 48$

$= (x-6)^2 + 12$

$= (x-6)^2 - (2\sqrt{3}i)^2$

$= (x-6-2\sqrt{3}i)(x-6+2\sqrt{3}i)$

(c) Q represents $-i(a+bi)$

$= b - ai$

(d)(ii) Let $z = x+iy$

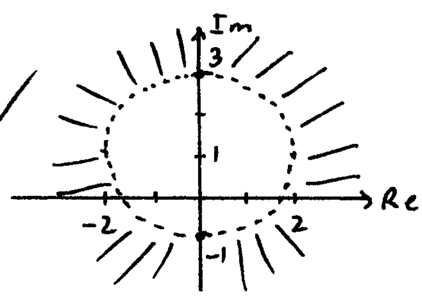
$\therefore |x + i(y+3)| < 2|x+iy|$

$x^2 + (y+3)^2 < 4(x^2+y^2)$

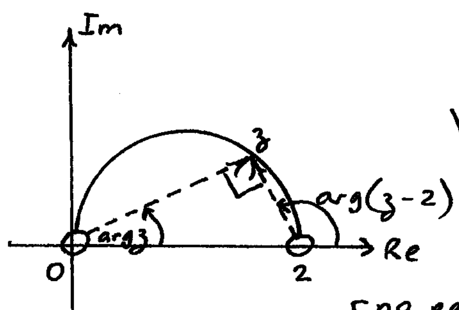
$6y + 9 < 3x^2 + 3y^2$

$x^2 + y^2 - 2y > 3$

$x^2 + (y-1)^2 > 4$



(i) $\arg(z-2) - \arg z = \frac{\pi}{2}$



[no penalty for omission of open circles]

(e)(i) Let $(r \operatorname{cis}\theta)^3 = -8$

$\therefore r^3 \operatorname{cis} 3\theta = 8 \operatorname{cis}(\pi + 2k\pi)$, where k is an integer

$\therefore r = 2$ and $\theta = (2k+1)\frac{\pi}{3}$

Choose $k = -1, 0, 1$

$\therefore$ the 3 cube roots of -8 are $2\operatorname{cis}\left(\frac{-\pi}{3}\right)$, $2\operatorname{cis}\frac{\pi}{3}$ and $2\operatorname{cis}\pi$.

(ii) $2\operatorname{cis}\left(\frac{-\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 1 - \sqrt{3}i$

and $2\operatorname{cis}\frac{\pi}{3} = 1 + \sqrt{3}i$

(iii) Let $w_1 = 2\operatorname{cis}\frac{\pi}{3}$

and $w_2 = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$.

$\therefore w_1^{6n} + w_2^{6n}$

$= \left(2\operatorname{cis}\frac{\pi}{3}\right)^{6n} + \left(2\operatorname{cis}\left(\frac{-\pi}{3}\right)\right)^{6n}$

$= 2^{6n}\operatorname{cis}(2n\pi) + 2^{6n}\operatorname{cis}(-2n\pi)$

$= 2^{6n}.1 + 2^{6n}.1$

(since $\cos 2n\pi = 1$ and $\sin 2n\pi = 0$ when n is an integer)

$= 2.2^{6n}$

$= 2^{6n+1}$

or (Better)

$w_1^{6n} + w_2^{6n} = (w_1^6)^n + (w_2^6)^n$

$= 64^n + 64^n$ (since $w_1^3 = w_2^3 = -$

$= 2 \times 2^{6n}$

$= 2^{6n+1}$

(3) (a) (i) By the quotient rule,

$$y' = \frac{(1+x^2).2 - 2x.2x}{(1+x^2)^2}$$

$$= \frac{2-2x^2}{(1+x^2)^2}$$

$y'=0$ when $x = \pm 1$, so the turning points are $A(1,1)$ and $B(-1,-1)$.

(ii) By the quotient rule,

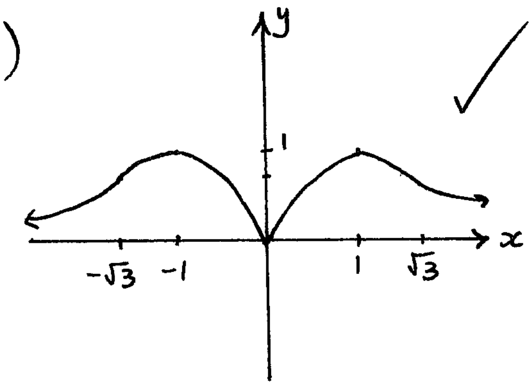
$$y'' = \frac{(1+x^2)^2.(-4x) - 2(1-x^2).4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)(-4x-4x^3-8x+8x^3)}{(1+x^2)^4}$$

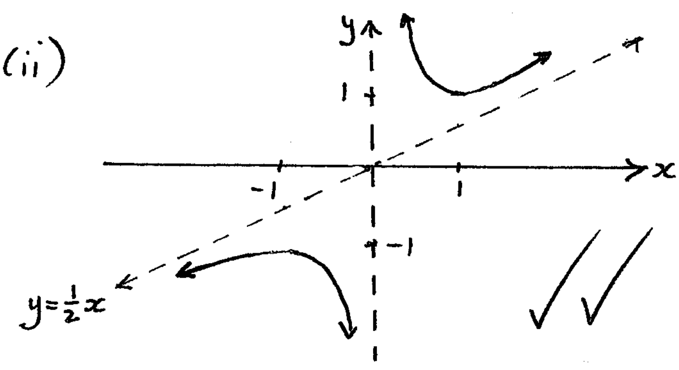
$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

$y''=0$ when $x = 0$ or $\pm\sqrt{3}$. It follows that $P = (\sqrt{3}, \frac{\sqrt{3}}{2})$ and $Q = (-\sqrt{3}, -\frac{\sqrt{3}}{2})$.

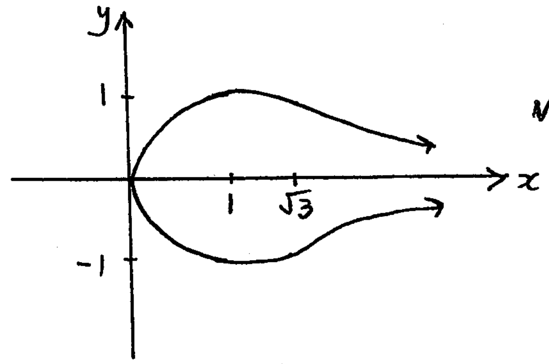
(b) (i)



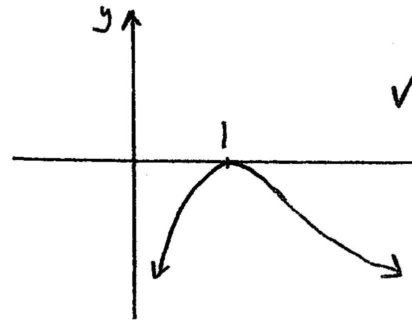
(ii)



(iii)



(iv)



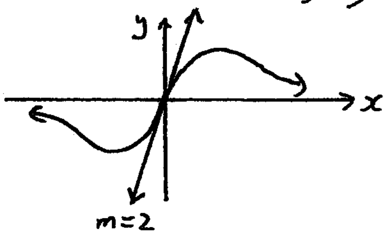
Don't penalize incorrect concavity as $x \to \infty$.

(c) (i)

$$kx^3 + (k-2)x = 0$$
$$2x = kx^3 + kx$$
$$2x = kx(1+x^2)$$
$$\frac{2x}{1+x^2} = kx$$

(ii) Graphical Solution

The curve $y = \frac{2x}{1+x^2}$ has gradient 2 at $(0,0)$.



So $y = kx$ and the curve will intersect exactly once for $k \geq 2$ or $k \leq 0$.

Algebraic Solution (Also worth 2 mark

$$x(kx^2 + k - 2) = 0$$

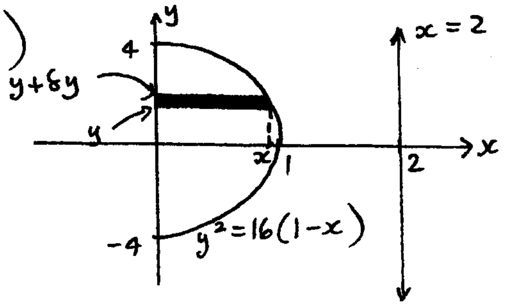
$\therefore x = 0$ or $x^2 = \frac{2-k}{k}$

For one real root, $\frac{2-k}{k} \leq 0$

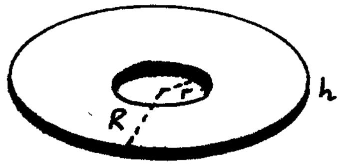
i.e. $\frac{k-2}{k} \geq 0$

$(\times k^2)$ $k(k-2) \geq 0$ $\therefore k \geq 2$ or $k <$

(4)(a)(i)



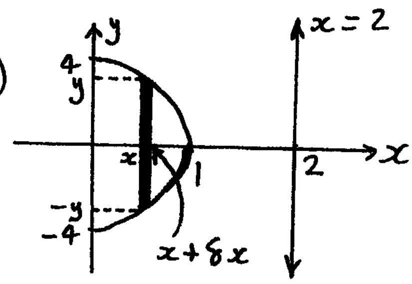
Typical slice:



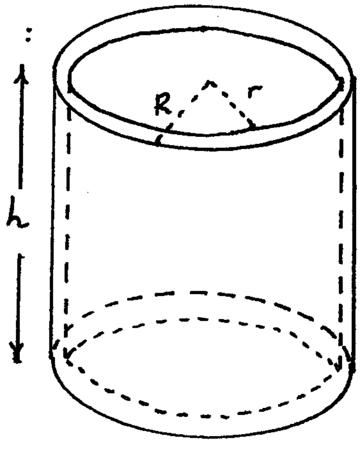
$$\begin{cases} R = 2 \\ r = 2-x \\ h = \delta y \end{cases}$$

$$\delta V \doteqdot (\pi R^2 - \pi r^2)h$$
$$= \pi (R-r)(R+r)h$$
$$= \pi . x . (4-x) . \delta y$$
$$\therefore V = 2\pi \int_0^4 (4x - x^2)\,dy$$
$$= 2\pi \int_0^4 \left[4\left(1-\frac{y^2}{16}\right) - \left(1-\frac{y^2}{16}\right)^2\right] dy$$
$$= 2\pi \int_0^4 \left(3 - \frac{y^2}{8} - \frac{y^4}{256}\right) dy$$
$$= 2\pi \left[3y - \frac{y^3}{24} - \frac{y^5}{1280}\right]_0^4$$
$$= \frac{256\pi}{15} \text{ units}^3$$

(ii)(α)



Typical shell:



$R = 2-x$

$r = 2-(x+\delta x)$

$h = 2y$

$$\delta V \doteqdot \pi(R-r)(R+r)h$$
$$= \pi . \delta x . (4 - 2x - \delta x) . 2y$$
$$\doteqdot 4\pi y(2-x)\,\delta x$$
$$\therefore V = \int_0^1 4\pi y (2-x)\,dx$$
$$= \int_0^1 4\pi . 4\sqrt{1-x} . (2-x)\,dx$$
$$= \int_0^1 16\pi (2-x)\sqrt{1-x}\,dx$$

(β) Let $u = 1-x$

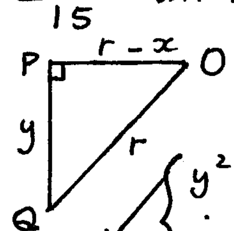
$\therefore \quad x = 1-u$

$\therefore \quad dx = -du$

x	0	1
u	1	0

$$V = 16\pi \int_1^0 (u+1) . u^{\frac{1}{2}} . -du$$
$$= 16\pi \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$
$$= 16\pi \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$
$$= 16\pi\left(\frac{2}{5} + \frac{2}{3}\right)$$
$$= \frac{256\pi}{15} \text{ units}^3$$

(b)(i)



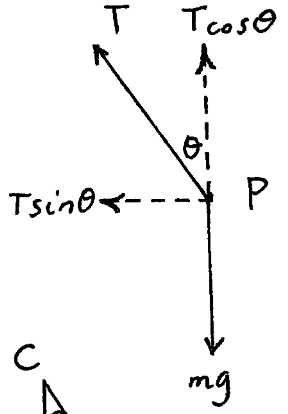
By Pythagoras,

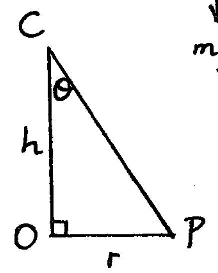
$$y^2 + (r-x)^2 = r^2$$
$$y^2 + r^2 - 2xr + x^2 = r$$
$$\therefore \quad y^2 = 2rx - x^2$$

(ii) $$\delta V \doteqdot \frac{1}{2} . y . 3y . \delta x$$
$$\therefore V = 2\int_0^r \frac{3}{2} y^2\,dx$$
$$= 3\int_0^r (2rx - x^2)\,d$$
$$= 3\left[rx^2 - \frac{1}{3}x^3\right]_0^r$$
$$= 3\left(r^3 - \frac{1}{3}r^3\right)$$
$$= 2r^3 \text{ units}^3$$

(iii) $$\frac{2r^3}{\pi . r^2 . 3r} \times 100\,\%$$
$$= \frac{200}{3\pi}\,\%$$
$$\doteqdot 21\%$$

(5)(a)(i)





Resolving forces vertically at P:

$$T\cos\theta = mg \quad (1)$$

Resolving forces horizontally at P:

$$T\sin\theta = mrw^2 \quad (2)$$

(2) ÷ (1):

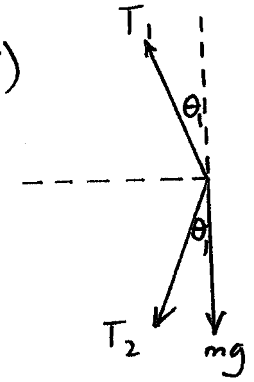
$$\tan\theta = \frac{rw^2}{g}$$

$$\therefore \frac{r}{h} = \frac{rw^2}{g}$$

$$w^2 = \frac{g}{h}$$

$$w = \sqrt{\frac{g}{h}}$$

(ii)



Resolving forces vertically at P:

$$T_1\cos\theta_1 - T_2\cos\theta_1 = mg \quad (3)$$

Resolving horizontally at P:

$$T_1\sin\theta_1 + T_2\sin\theta_1 = mrw^2 \quad (4)$$

(4) ÷ (3):

$$\frac{T_1 + T_2}{T_1 - T_2}\tan\theta_1 = \frac{Rw^2}{g} \quad (5)$$

Now, $\tan\theta_1 = \frac{R}{h}$ and $w = 3\sqrt{\frac{g}{h}}$,

so (5) can be written:

$$\frac{T_1 + T_2}{T_1 - T_2} \cdot \frac{R}{h} = \frac{R}{g} \cdot \frac{9g}{h}$$

$$\therefore \frac{T_1 + T_2}{T_1 - T_2} = 9$$

$$T_1 + T_2 = 9T_1 - 9T_2$$

$$10T_2 = 8T_1$$

$$\therefore \frac{T_1}{T_2} = \frac{10}{8}$$

$$= \frac{5}{4}$$

(b) (i) (α) $\dfrac{d\dot{x}}{dt} = -k\dot{x}$

$\therefore \dfrac{dt}{d\dot{x}} = \dfrac{-1}{k\dot{x}}$

$\therefore t = -\dfrac{1}{k}\ln\dot{x} + c_1$

when $t=0$, $\dot{x} = V\cos\alpha$

$\therefore c_1 = \dfrac{1}{k}\ln(V\cos\alpha)$

$\therefore t = \dfrac{1}{k}\ln\left(\dfrac{V\cos\alpha}{\dot{x}}\right)$

$\therefore e^{kt} = \dfrac{V\cos\alpha}{\dot{x}}$

$\therefore \dot{x} = V\cos\alpha \cdot e^{-kt}$

(β) $\dfrac{d\dot{y}}{dt} = -g - k\dot{y}$

$\therefore \dfrac{dt}{d\dot{y}} = \dfrac{-1}{g + k\dot{y}}$

$\therefore t = -\dfrac{1}{k}\ln(g + k\dot{y}) + c_2$

When $t=0$, $\dot{y} = V\sin\alpha$

$\therefore c_2 = \dfrac{1}{k}\ln(g + kV\sin\alpha)$

$\therefore t = \dfrac{1}{k}\ln\left(\dfrac{g + kV\sin\alpha}{g + k\dot{y}}\right)$

$\therefore e^{kt} = \dfrac{g + kV\sin\alpha}{g + k\dot{y}}$

$\therefore e^{-kt}(g + kV\sin\alpha) = g + k\dot{y}$

$\therefore \dot{y} = \left(\dfrac{g}{k} + V\sin\alpha\right)e^{-kt} - \dfrac{g}{k}$

(ii) (α) $x = V\cos\alpha \displaystyle\int e^{-kt}\,dt$

$= -\dfrac{V\cos\alpha}{k}e^{-kt} + c_3$

When $t=0$, $x=0$

$\therefore c_3 = \dfrac{V\cos\alpha}{k}$

$\therefore x = \dfrac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$

(β) $y = \left(+\dfrac{g}{k} + V\sin\alpha\right)\displaystyle\int e^{-kt}\,dt - \int \dfrac{g}{k}\,dt$

$= -\dfrac{1}{k}\left(\dfrac{g}{k} + V\sin\alpha\right)e^{-kt} - \dfrac{g}{k}t + c_4$

When $t=0$, $y=0$

$\therefore c_4 = \dfrac{1}{k}\left(\dfrac{g}{k} + V\sin\alpha\right)$

$\therefore y = \dfrac{1}{k}\left(\dfrac{g}{k} + V\sin\alpha\right) - \dfrac{1}{k}\left(\dfrac{g}{k} + V\sin\alpha\right)e^{-kt} - \dfrac{g}{k}t$

$\therefore y = \left(\dfrac{g}{k^2} + \dfrac{V\sin\alpha}{k}\right)\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$

(iii) when $\dot{y} = 0$,

$e^{-kt} = \dfrac{\frac{g}{k}}{\frac{g}{k} + V\sin\alpha}$

$= \dfrac{g}{g + Vk\sin\alpha}$

$\therefore x = \dfrac{V\cos\alpha}{k}\left(1 - \dfrac{g}{g + Vk\sin\alpha}\right)$

$= \dfrac{V\cos\alpha}{k} \cdot \dfrac{g + Vk\sin\alpha - g}{g + Vk\sin\alpha}$

$= \dfrac{V^2\cos\alpha\sin\alpha}{g + Vk\sin\alpha}$

$= \dfrac{V^2\sin 2\alpha}{2(g + Vk\sin\alpha)}$

(6)(a)(i) $P'(x) = 3x^2 + c$

Let $P'(x) = 0$

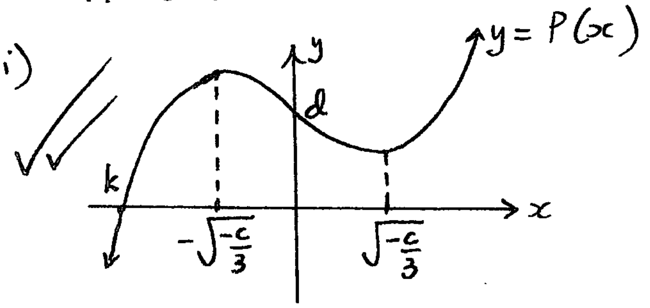
$\therefore x^2 = -\frac{c}{3}$

This equation has two real roots if there are two turning points

$\therefore -\frac{c}{3} > 0$

$\therefore c < 0$

(ii)



(iii) Sum of roots of $P(x)$ is zero.

$\therefore 2a + k = 0$

$\therefore a = \frac{-k}{2}$

But $k < 0$, so $a > 0$.

(iv) The 3 relationships between the coefficients and zeroes of $P(x)$ are:

$$\begin{cases} 2a + k = 0 & (1) \\ a^2 + b^2 + 2ak = c & (2) \\ k(a^2+b^2) = -d & (3) \end{cases}$$

From (1), $k = -2a$

Substitute into (2) and (3):

$\therefore -3a^2 + b^2 = c$ (4)

and $2a(a^2+b^2) = d$ (5)

From (4), $b^2 = 3a^2 + c$

Substitute into (5):

$2a(a^2 + 3a^2 + c) = d$

$\therefore d = 8a^3 + 2ac$

(b)(i) In ΔCDE,

$x + y = 90°$ ($\angle$ sum of Δ)

(ii) Since $\angle CPD = \angle CSD = 90°$, quadrilateral $CDPS$ is cyclic (converse of angles in a semicircle)

$\therefore \angle CDS = \angle CPS = x$

($\angle$'s at circumference standing on the same arc).

(iii) Similarly, quadrilaterals $CERS$, $CBQP$ and $CAQR$ are cyclic.

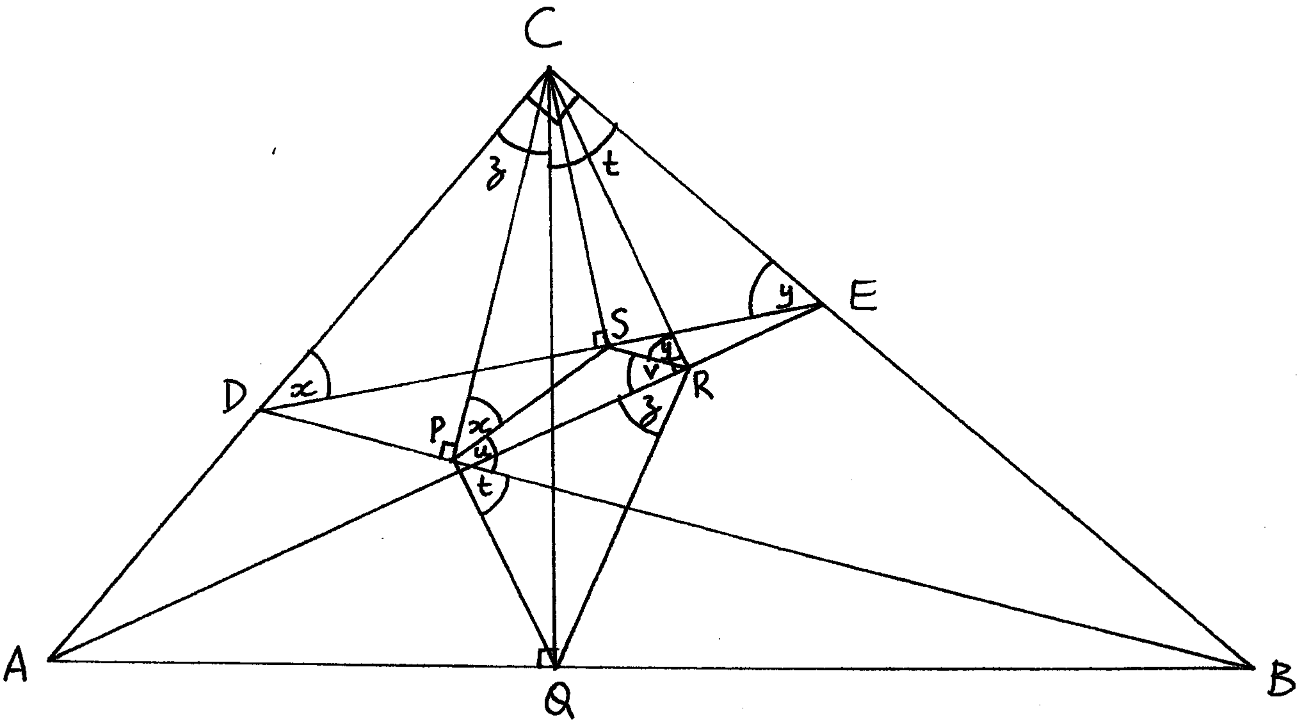
$\therefore \angle CES = \angle CRS = y$

and $\angle QCB = \angle QPB = t$

and $\angle ACQ = \angle ARQ = z$

($\angle$'s at the circumference of their respective circles standing on the same arc).

(iv)



Let $\angle SPB = u$ and $\angle SRA = v$.

$x + u = 90^\circ$ and $y + v = 90^\circ$ (adjacent complementary angles)

$\therefore \quad x + y + u + v = 180^\circ$

But from part (i), $x + y = 90^\circ$.

$\therefore \quad u + v = 90^\circ$

Also, $z + t = 90^\circ$ (since $z + t = \angle ACB$),

so $\quad u + v + z + t = 180^\circ$

i.e. $(u + t) + (v + z) = 180^\circ$

i.e. $\angle SPQ + \angle SRQ = 180^\circ$.

$\therefore$ Quadrilateral $PQRS$ is cyclic (opposite angles supplementary)

(7) (a) (i)

$$I_n = \int_0^1 (1-x^2)^n \, dx$$

$$= \left[x(1-x^2)^n\right]_0^1 - \int_0^1 -2nx^2(1-x^2)^{n-1} dx$$

Let $u = (1-x^2)^n$

$\therefore \; u' = -2nx(1-x^2)^{n-1}$

Let $v' = 1$

$\therefore \; v = x$

$$= 0 + 2n \int_0^1 x^2 (1-x^2)^{n-1} dx$$

$$= 2n \cdot J_{n-1}$$

(ii) Continuing from (i), $I_n = 2n \int_0^1 x^2 (1-x^2)^{n-1} dx$

$$= -2n \int_0^1 \left[(1-x^2)-1\right](1-x^2)^{n-1} dx$$

$$= -2n\left(I_n - I_{n-1}\right)$$

$$\therefore \; I_n(2n+1) = 2n \cdot I_{n-1}$$

$$\therefore \; I_n = \frac{2n}{2n+1} \cdot I_{n-1}$$

(iii) $J_n = \int_0^1 x^2 (1-x^2)^n dx$

$$= -\int_0^1 \left[(1-x^2)-1\right](1-x^2)^n dx$$

$$= -\left(I_{n+1} - I_n\right)$$

$$= I_n - I_{n+1}$$

$$= I_n - \frac{2n+2}{2n+3} I_n \quad \text{(using (ii))}$$

$$= \frac{2n+3-2n-2}{2n+3} \cdot I_n$$

$$= \frac{1}{2n+3} \cdot I_n$$

(iv) Combining (ii) and (i):

$$J_n = \frac{1}{2n+3} \cdot I_n$$

$$= \frac{2n}{2n+3} \cdot J_{n-1}$$

(7)(b)(i) When $n=1$, $LHS = \frac{1}{2}$

$$RHS = \frac{\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \frac{1}{2}$$

$\therefore$ the statement is true for $n=1$.

Suppose it's true for $n=k$, where k is a positive integer.

i.e. suppose $\frac{1}{2}+\cos\theta+\ldots+\cos(k-1)\theta = \dfrac{\sin\frac{1}{2}(2k-1)\theta}{2\sin\frac{1}{2}\theta}$

Prove it's true for $n=k+1$.

i.e. prove that $\frac{1}{2}+\cos\theta+\ldots+\cos(k-1)\theta+\cos k\theta = \dfrac{\sin\frac{1}{2}(2k+1)\theta}{2\sin\frac{1}{2}\theta}$

$$LHS = \frac{\sin\frac{1}{2}(2k-1)\theta}{2\sin\frac{1}{2}\theta} + \cos k\theta \quad \text{(by the assumption)}$$

$$= \frac{\sin\frac{1}{2}(2k-1)\theta + 2\sin\frac{1}{2}\theta\cos k\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\sin\frac{1}{2}(2k-1)\theta + \sin(k\theta+\frac{1}{2}\theta)-\sin(k\theta-\frac{1}{2}\theta)}{2\sin\frac{1}{2}\theta}$$

(using the identity $2\cos A\sin B = \sin(A+B) - \sin(A-B)$)

$$= \frac{\cancel{\sin\frac{1}{2}(2k-1)\theta} - \cancel{\sin\frac{1}{2}(2k-1)\theta} + \sin\frac{1}{2}(2k+1)\theta}{2\sin\frac{1}{2}\theta}$$

$= RHS$

So the statement is true for $n=k+1$ if it's true for $n=k$. But it's true for $n=1$, so by mathematical induction it's true for all positive integer values of n.

(7)(b)(ii)(α) The width of each rectangle is $\frac{\pi}{6n}$.

$$\therefore\ S_n = \frac{\pi}{6n}\left[\cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \dots + \cos\frac{(n-1)\pi}{6n} + \cos\frac{\pi}{6}\right]$$

$$= \frac{\pi}{6n}\left[\frac{\sqrt{3}}{2} + \left(\frac{1}{2} + \cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \dots + \cos\frac{(n-1)\pi}{6n}\right) - \frac{1}{2}\right]$$

$$= \frac{\pi}{6n}\left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + \frac{\sin\frac{1}{2}(2n-1)\frac{\pi}{6n}}{2\sin\frac{1}{2}\cdot\frac{\pi}{6n}}\right]$$, using part (i) with $\theta = \frac{\pi}{6n}$,

$$= \frac{\pi}{12n}\left[(\sqrt{3}-1) + \frac{\sin(2n-1)\frac{\pi}{12n}}{\sin\frac{\pi}{12n}}\right]$$, as required.

(β) $$\lim_{n\to\infty} S_n = \lim_{h\to 0} S_{\frac{1}{h}}$$

$$= \lim_{h\to 0}\left[\frac{h\pi(\sqrt{3}-1)}{12} + \frac{\frac{\pi h}{12}}{\sin\frac{\pi h}{12}}\cdot\sin\left(\frac{2}{h}-1\right)\frac{\pi h}{12}\right]$$

$$= \lim_{h\to 0}\left[\frac{h\pi(\sqrt{3}-1)}{12} + \frac{\frac{\pi h}{12}}{\sin\frac{\pi h}{12}}\cdot\sin\left(\frac{\pi}{6} - \frac{\pi h}{12}\right)\right]$$

$$= 0 + 1\cdot\sin\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{1}{2}$$

(8) (a)(i) $m_{AB} = m_{AC}$

$$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$$

$$\therefore \quad x_2 y_3 - x_2 y_1 - x_1 y_3 + \cancel{x_1 y_1} = x_3 y_2 - x_1 y_2 - x_3 y_1 + \cancel{x_1 y_1}$$

$$\therefore \quad x_1 y_2 + x_2 y_3 + x_3 y_1 = x_1 y_3 + x_2 y_1 + x_3 y_2 ,$$

as required.

(ii) $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$, $m_{AT} = \dfrac{(u-1) y_1 + v y_2 + w y_3}{(u-1) x_1 + v x_2 + w x_3}$

So the condition for T to lie on the line through A, B and C is :

$$m_{AB} = m_{AT}$$

i.e.

$$(u-1) x_1 y_2 + \cancel{v x_2 y_2} + w x_3 y_2 - \cancel{(u-1)} x_1 y_1 - v x_2 y_1 - w x_3 y_1$$
$$= (u-1) x_2 y_1 + \cancel{v x_2 y_2} + w x_2 y_3 - \cancel{(u-1)} x_1 y_1 - v x_1 y_2 - w x_1 y$$

i.e. $(1-u-v) x_1 y_2 + w x_2 y_3 + w x_3 y_1 = w x_1 y_3 + (1-u-v) x_2 y_1 + w x_3 y$

i.e. $w(x_1 y_2 + x_2 y_3 + x_3 y_1) = w(x_1 y_3 + x_2 y_1 + x_3 y_2)$ [since $u + v + w = 1$

i.e. $x_1 y_2 + x_2 y_3 + x_3 y_1 = x_1 y_3 + x_2 y_1 + x_3 y_2$ [since $w \neq 0$

which is the condition established in part (i),

so T lies on the line through A, B and C.

(b)(i)(α) $\overrightarrow{AB}$ represents $z_2 - z_1$, so $\overrightarrow{AM}$ represents $\frac{1}{2}(z_2 - z_1)$

(β) $MP = AM \cot\frac{\alpha}{2}$

$\therefore$ $\overrightarrow{MP}$ represents $\frac{1}{2}(z_2 - z_1) . i \cot\frac{\alpha}{2}$

(ii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP}$,

so P represents $z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_2 - z_1) . i \cot\frac{\alpha}{2}$

$$= z_1\left(1 - \tfrac{1}{2} - \tfrac{1}{2} i \cot\tfrac{\alpha}{2}\right) + z_2\left(\tfrac{1}{2} + \tfrac{1}{2} i \cot\tfrac{\alpha}{2}\right)$$

$$= \tfrac{1}{2}\left(1 - i\cot\tfrac{\alpha}{2}\right) z_1 + \tfrac{1}{2}\left(1 + i \cot\tfrac{\alpha}{2}\right) z_2$$

(c) Suppose we consider the diagram to be drawn in the Argand diagram, and let A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.

Then from (b),

P represents the complex number $\frac{1}{2}\left(1-i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1+i\cot\frac{\alpha}{2}\right)z_2$

and Q " " " " $\frac{1}{2}\left(1-i\cot\frac{\alpha}{2}\right)z_2 + \frac{1}{2}\left(1+i\cot\frac{\alpha}{2}\right)z_3$.

Now apply the same result to ΔPRQ, noting that Q – R – P is the clockwise cyclic orientation corresponding to A – P – B and B – Q – C.

$\therefore$ The point R represents the complex number

$$\frac{1}{2}(1-ic)\left[\frac{1}{2}(1-ic)z_2 + \frac{1}{2}(1+ic)z_3\right] + \frac{1}{2}(1+ic)\left[\frac{1}{2}(1-ic)z_1 + \frac{1}{2}(1+ic)z_2\right]$$

$\left(\text{where } c = \cot\frac{\alpha}{2}\right)$

$$= \frac{1}{4}(1-ic)^2 z_2 + \frac{1}{4}(1-ic)(1+ic)z_3 + \frac{1}{4}(1+ic)(1-ic)z_1 + \frac{1}{4}(1+ic)^2 z_2$$

$$= \frac{1}{4}\left(1-c^2-\cancel{2ci}\right)z_2 + \frac{1}{4}(1+c^2)z_3 + \frac{1}{4}(1+c^2)z_1 + \frac{1}{4}\left(1-c^2+\cancel{2ci}\right)z_2$$

$$= \frac{1}{2}(1-c^2)z_2 + \frac{1}{4}(1+c^2)z_3 + \frac{1}{4}(1+c^2)z_1$$

which is of the form $uz_1 + vz_2 + wz_3$, where

$$u+v+w = \frac{1}{4}(1+c^2) + \frac{1}{2}(1-c^2) + \frac{1}{4}(1+c^2)$$

$$= 1.$$

So by part (a), R lies on the line through A, B and C.