QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$. [2]

(b) Find $\displaystyle\int \tan^3 x \sec^2 x\,dx$. [2]

(c) Find $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$. [3]

(d) (i) Find the values of A and B such that $\dfrac{3x^2-10}{x^2-4x+4} = 3 + \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$. [2]

(ii) Find $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$. [2]

(e) Use integration by parts twice, to show that $\displaystyle\int_1^e \sin(\ln x)\,dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$. [4]

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Simplify $|\cos\theta + i\sin\theta|$. [1]

(b) Express $\dfrac{i^5(1-i)}{2+i}$ in the form $a+ib$ where a and b are rational. [2]

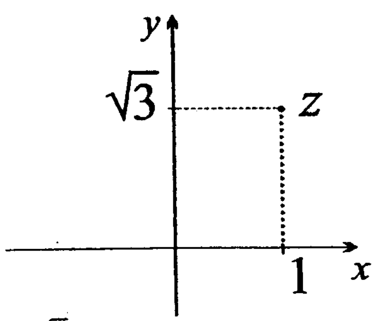
(c) By drawing a diagram, or otherwise, find the solutions of $z^5 = -1$. [2]

(d) Graph the region in the Argand diagram which simultaneously satisfies [3]

$$1 \le |z-i| \le 2 \text{ and } \operatorname{Im} z \ge 0.$$

(e) Find the complex number ϕ if $1+i$ is a root of the equation $z^2 + \phi z - i = 0$. [2]

(f)



Suppose that $z = 1 + \sqrt{3}i$ and $\omega = (\operatorname{cis}\alpha)z$ where $-\pi < \alpha \le \pi$.

(i) Find the argument of z. [1]

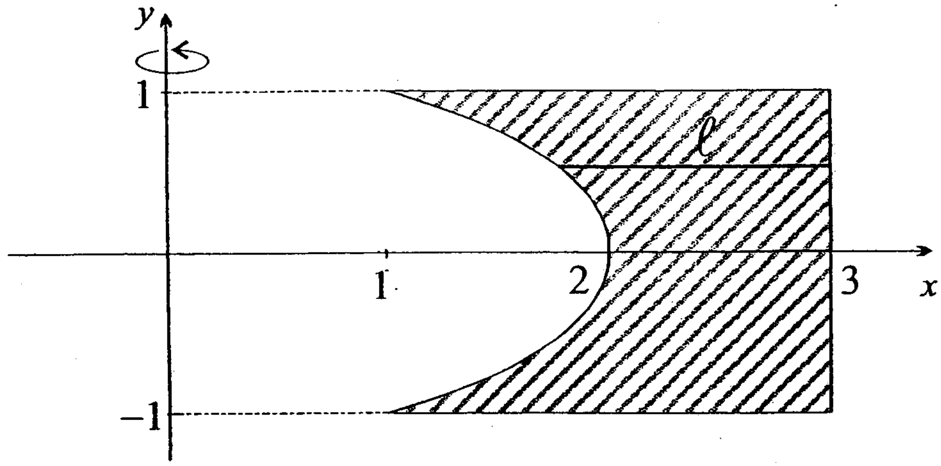
(ii) Find the value of α if ω is purely imaginary and $\operatorname{Im}(\omega) > 0$. [2]

(iii) Find the value of $\arg(z+\omega)$ if ω is purely imaginary and $\operatorname{Im}(\omega) > 0$. [2]

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $x = 2 - y^2$ and the lines $x = 3, y = 1$ and $y = -1$. This region is rotated about the y-axis to form a solid. The interval ℓ at height y sweeps out an annulus.

(i) Show that the annulus at height y has area equal to **2**

$$\pi(5 + 4y^2 - y^4).$$

(ii) Find the volume of the solid. **2**

(b) Consider the function $f(x) = \dfrac{1}{1 + x^3}$.

(i) Show that there is a horizontal point of inflexion at $x = 0$. **2**

(ii) Find the vertical asymptote and the horizotal asymptote. **2**

(iii) Sketch $y = f(x)$ showing the features from parts (a) and (b) and the y-intercept. **2**

(iv) On a separate diagram sketch $y = |f(x)|$. **1**

(v) On a separate diagram sketch $y^2 = f(x)$. **2**

(vi) On a separate diagram sketch $y = e^{f(x)}$. **2**

Exam continues overleaf ...

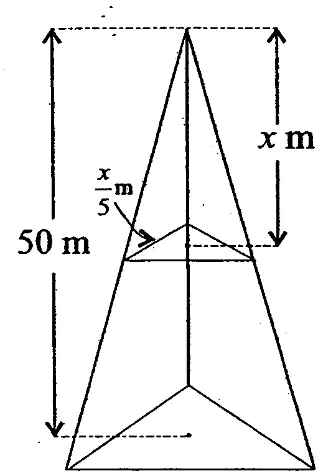
QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the polynomial equation $x^3 - 3x^2 + x - 5 = 0$ which has roots α, β and γ.

(i) Show that $\alpha + \beta = 3 - \gamma$. **1**

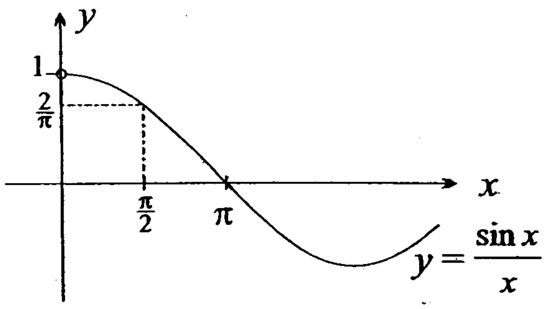
(ii) Write down similar expressions for $\alpha + \gamma$ and $\beta + \gamma$ and hence find a polynomial equation which has the roots $\alpha + \beta, \alpha + \gamma$ and $\beta + \gamma$. **2**

(b)



The diagram above shows a monument 50 metres high. A horizontal cross section x metres from the top is an equilateral triangle with sides $\frac{x}{5}$ metres. Use integration to find the volume of the monument.

(c)



Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = \dfrac{\sin x}{x}$ and the lines $y = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the y-axis.

(d) An hyperbola is defined parametrically by $x = 3\sec\theta$ and $y = 4\tan\theta$.

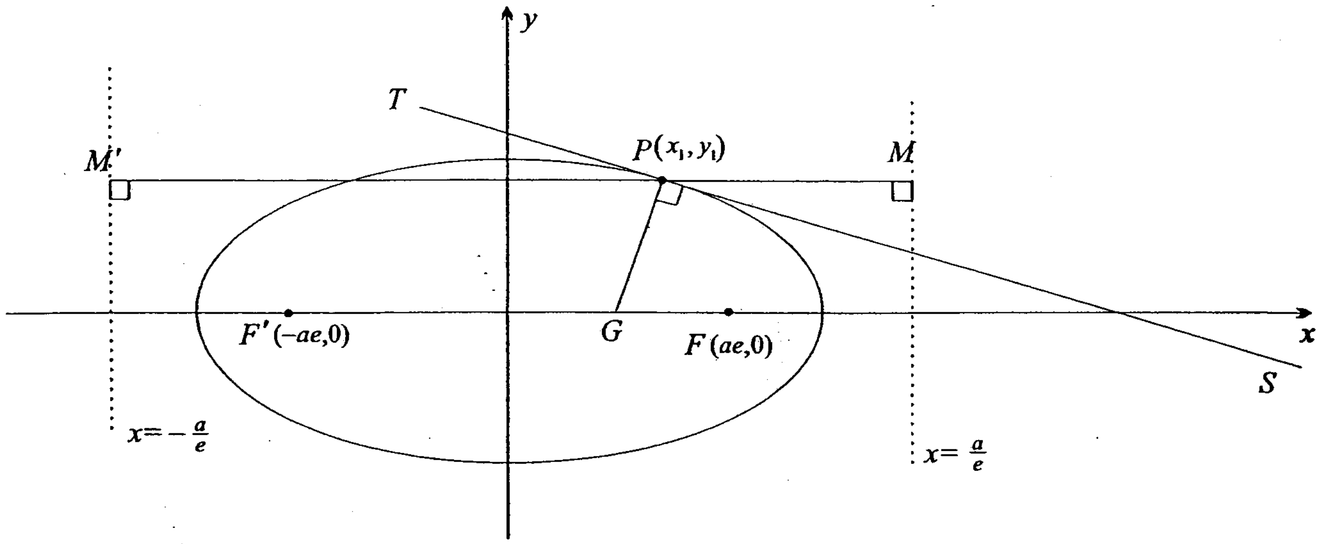
(i) Write the equation of the curve in Cartesian form and show that the eccentricity is $\frac{5}{3}$. **2**

(ii) Sketch the curve showing its x-intercepts, foci, directrices and asymptotes. **4**

Exam continues next page ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $F(ae, 0)$ and $F'(-ae, 0)$.
$P(x_1, y_1)$ is any point on the ellipse.
Let M and M' be the feet of the perpendiculars from P to the directrices $x = \dfrac{a}{e}$ and $x = -\dfrac{a}{e}$.
Line TS is a tangent to the ellipse at P and G is the point where the normal at P meets the x-axis.

(i) Show that the equation of the normal at P is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$. **3**

(ii) Show that the point G has co-ordinates $(e^2x_1, 0)$. **3**

(iii) Show that the distance PF is $a - ex_1$. **2**

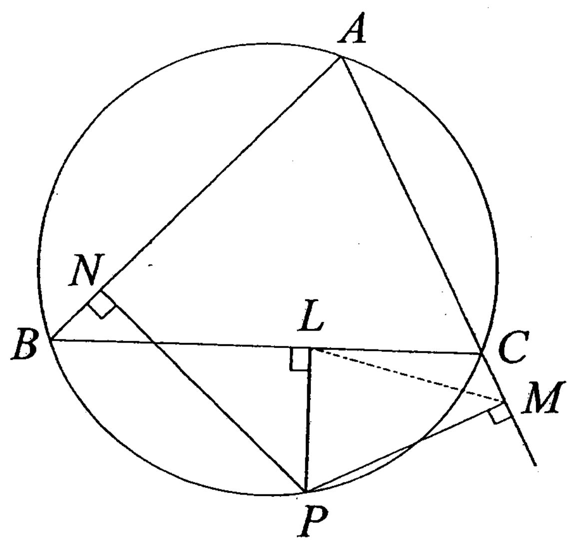
(iv) Show that $\dfrac{PF}{FG} = \dfrac{PF'}{F'G}$. **2**

(b) (i) Show that $1 - \cos 2\theta - i\sin 2\theta = 2\sin\theta(\sin\theta - i\cos\theta)$. **2**

(ii) Given that $\dfrac{z-1}{z} = \operatorname{cis}\frac{2\pi}{5}$, show that $z = \dfrac{1}{2}(1 + i\cot\frac{\pi}{5})$. **3**

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, ABC is a triangle with the circumcircle through points A, B and C drawn. P is another point on the minor arc BC. Points L, M and N are the feet of the perpendiculars from P to the sides BC, CA and AB respectively.

(i) Copy the diagram and explain why P, L, N and B are concyclic. **1**

(ii) Explain why P, L, C and M are concyclic. **1**

(iii) Let $\angle PLM = \alpha$.

(α) Show that $\angle ABP = \alpha$. **2**

(β) Hence show that M, L and N are collinear. **2**

(b) A particle of unit mass is thrown vertically downwards with an initial velocity of v_0. It experiences a resistive force of magnitude kv^2 where v is its velocity. Taking downwards as the positive direction, the equation of motion of the particle is given by

$$\ddot{x} = g - kv^2 .$$

Let V be the terminal velocity of the particle.

(i) Explain why $V = \sqrt{\dfrac{g}{k}}$. **1**

(ii) Show that $v^2 = V^2 + (v_0^2 - V^2)e^{-2kx}$. **4**

(c) Let $z = x + iy$ be any non-zero complex number such that $z + \dfrac{1}{z} = k$, where k is a real number.

(i) Prove that either $y = 0$ or $x^2 + y^2 = 1$. **2**

(ii) Show that if $y = 0$ then $|k| \geq 2$. **2**

Exam continues next page ...

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down $\cos 2\theta$ in terms of $\tan\theta$. $\boxed{1}$

(ii) Show that $\cos 4\theta = \dfrac{1 - 6\tan^2\theta + \tan^4\theta}{1 + 2\tan^2\theta + \tan^4\theta}$. $\boxed{3}$

(iii) Deduce that $\tan^2\frac{\pi}{8} + \tan^2\frac{3\pi}{8} = 6$. $\boxed{3}$

(b) Consider the equation $z^7 = 1$.

This equation has seven roots $1, \rho, \rho^2, \ldots, \rho^6$, where $\rho = \operatorname{cis}\frac{2\pi}{7}$.

Let $\alpha = \rho + \rho^2 + \rho^4$ and $\theta = \rho^3 + \rho^5 + \rho^6$.

(i) Express ρ^9 as a lower positive power of ρ. $\boxed{1}$

(ii) Simplify $\alpha + \theta$. $\boxed{2}$

(iii) Simplify $\alpha\theta$. $\boxed{2}$

(iv) Form a quadratic equation with α and θ as roots. $\boxed{1}$

(v) Deduce that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$. $\boxed{2}$

Exam continues overleaf ...

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) The sequence $a_1, a_2, \ldots, a_n$ is defined by $a_n = \dfrac{(2n)!}{2^n n!}$. **4**

Show by induction on n that a_n is an odd positive integer.

(b) Suppose that $y = f(x)$ is an increasing function for $x \geq 1$.
Suppose also that $f(x) \geq 0$ for $x \geq 1$.

(i) Explain, with the aid of a diagram, why **2**

$$f(1) + f(2) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + f(3) + \cdots + f(n).$$

(ii) Show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$. **2**

(iii) Use parts (i) and (ii) to deduce that, for $n > 1$:

(α) $n! > \dfrac{n^n}{e^{n-1}}$ **3**

(β) $n! < \dfrac{n^{n+1}}{e^{n-1}}$ **2**

(iv) Find $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$. (You may assume that $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$.) **2**

END OF EXAMINATION

Question One

(a) $\int_0^4 (2x+1)^{-\frac{1}{2}}\,dx$

$= \left[\dfrac{(2x+1)^{\frac{1}{2}}}{2\times\frac{1}{2}}\right]_0^4$ ✓

$= \sqrt{9} - \sqrt{1}$ ✓

$= 2.$

(b) $I = \int \tan^3 x \sec^2 x\,dx$

Let $u = \tan x$

$du = \sec^2 x\,dx$ ✓

$I = \int u^3\,du$

$= \dfrac{u^4}{4} + C$ ✓

$= \dfrac{\tan^4 x}{4} + C$

(c) $\int \dfrac{x}{x^2-4x+8}\,dx$

$= \frac{1}{2}\int \dfrac{2x-4+4}{x^2-4x+8}\,dx$ ✓

$= \frac{1}{2}\int \dfrac{2x-4}{x^2-4x+8}\,dx + \int \dfrac{2}{(x-2)^2+4}\,dx$

$= \frac{1}{2}\ln(x^2-4x+8) + 2\times\frac{1}{2}\tan^{-1}\dfrac{x-2}{2} + C$

$= \frac{1}{2}\ln(x^2-4x+8) + \tan^{-1}\dfrac{x-2}{2} + C$

✓ ✓

(d) (i) $\dfrac{3x^2-10}{x^2-4x+4} = 3 + \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$

$3x^2-10 = 3(x-2)^2 + A(x-2) + B$

subst $x=2$: $12-10 = B$ ✓

$B = 2$

subst $x=0$: $-10 = 12 + B - 2A$

$-22 = 2 - 2A$

$-24 = -2A$ ✓

$A = 12$

$\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx = \int 3 + \frac{12}{x-2} + \frac{2}{(x-2)^2}$

$= 3x + 12\ln|x-2| - \dfrac{4}{x-1} + C$ ✓ ✓

(e) $\displaystyle\int_1^e \sin(\ln x)\frac{d}{dx}(x)\,dx$

$= \displaystyle x\sin(\ln x)\Big]_1^e - \int_1^e x\cdot\cos(\ln x)\cdot\frac{1}{x}\,dx$ ✓

$= \displaystyle x\sin(\ln x)\Big]_1^e - \int_1^e \cos\ln x\,\frac{d}{dx}(x)\,dx$

$= \displaystyle e\sin 1 - 0 - \left\{\left[x\cos(\ln x)\right]_1^e - \int_1^e x\cdot -\sin(\ln x)\cdot\frac{1}{x}\,dx\right.$ ✓

$= \displaystyle e\sin 1 - \left\{e\cos 1 - 1 + \int_1^e \sin(\ln x)\,dx\right.$

So $\displaystyle 2\int_1^e \sin(\ln x)\,dx = e\sin 1 - e\cos 1 + 1$ ✓

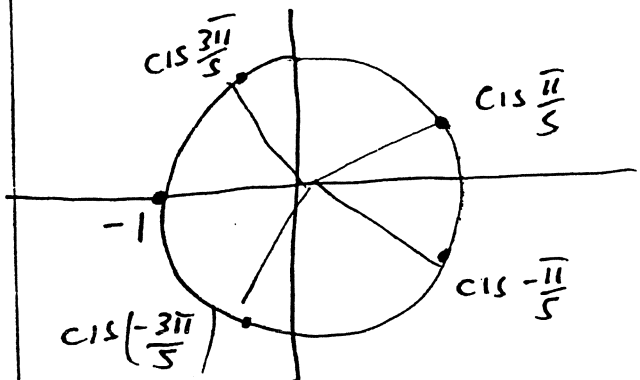
$\displaystyle\int_1^e \sin(\ln x)\,dx = \frac{e}{2}\left[\sin 1 - \cos 1\right] + \frac{1}{2}$ ✓

Question Two

(a) $|\cos\theta + i\sin\theta| = 1$ ✓

(b)
$$\frac{i^5(1-i)}{2+i} = \frac{i(1-i)}{2+i}$$
$$= \frac{i+1}{2+i}$$ ✓
$$= \frac{(1+i)(2-i)}{4+1}$$
$$= \frac{2-i+2i+1}{5}$$
$$= \frac{3}{5} + \frac{1}{5}i$$ ✓

(c)

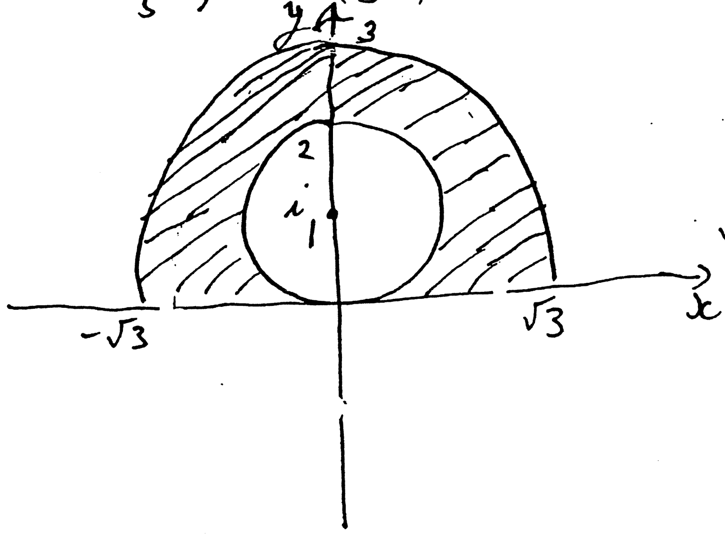


Clearly $z=-1$ is a root. The other four roots are equally spaced around the circle. ✓

Solutions to $z^5 = -1$ are ✓

-1, cis $\frac{\pi}{5}$, cis$\left(-\frac{\pi}{5}\right)$, cis $\frac{3\pi}{5}$ and cis$\left(-\frac{3\pi}{5}\right)$

(d)



✓ annulus (0,1) centre
✓ above x axis
✓ correct shading (radius between 1 and 2)

(e) $1+i$ is a root

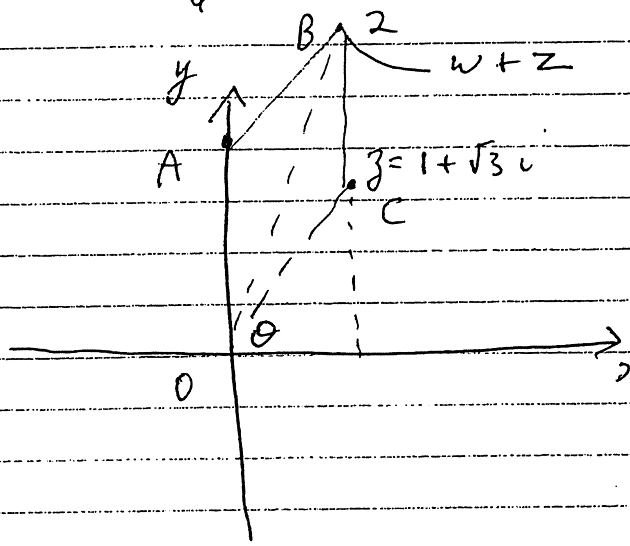
$\therefore (1+i)^2 + \phi(1+i) - i = 0$ ✓

$1+2i-1+\phi(1+i) - i = 0$

$\phi = \frac{-i}{1+i} \times \frac{1-i}{1-i}$ ✓

$\phi = \underline{-1-i}$ ✓

(f)



(i) $\tan\theta = \sqrt{3}$ $\quad$ $\arg(z) = \frac{\pi}{3}$ ✓

$\theta = \frac{\pi}{3}$

(ii) Multiplication by $cis\,\alpha$ is a rotation anticlockwise through α $\quad \alpha + \frac{\pi}{3} = \frac{\pi}{2}$ ✓

So if w is purely imaginary and $Im(z) > 0$ then $\alpha = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ✓

(iii) OABC is a rhombus

Diagonals bisect vertex ✓

So $\arg(z+w) = \frac{\pi}{3} + \frac{\pi}{12}$

$= \frac{5\pi}{12}$ ✓

(a) (i) $A(y) = \pi(r_1^2 - r_2^2)$

$= \pi\left(3^2 - (2-y^2)^2\right)$ ✓

$= \pi\left(9 - 4 + 4y^2 - y^4\right)$

$= \pi\left(5 + 4y^2 - y^4\right)$ ✓

(ii) $V = 2\pi \int_0^1 5 + 4y^2 - y^4 \, dy$ ✓

$= 2\pi \left[5y + \frac{4y^3}{3} - \frac{y^5}{5}\right]_0^1$

$= 2\pi \left[5 + \frac{4}{3} - \frac{1}{5}\right]$

$= \frac{2\pi[75 + 20 - 3]}{15} = \frac{184\pi}{15}$ ✓

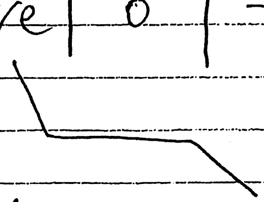
(b) (i) $f(x) = (1+x^3)^{-1}$

$f'(x) = -\frac{3x^2}{(1+x^3)^2}$ ✓

Stat point where $x = 0$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y'	$-ve$	0	$-ve$

✓



Horizontal point of inflexion where $x = 0$, $y = 1$

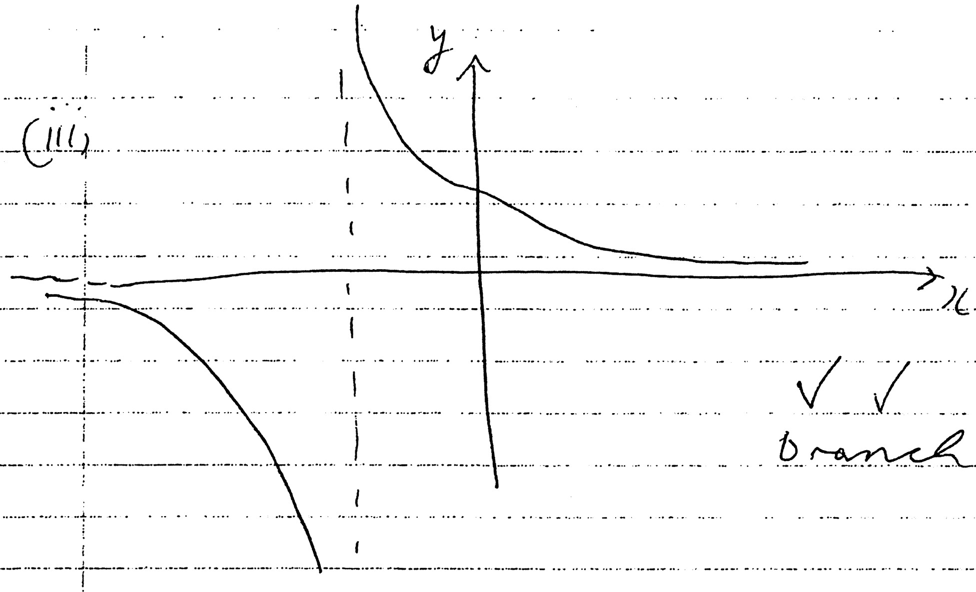
(ii) Vertical asymptote at $x = -1$ ✓

As $x \to \infty$, $y \to 0^+$

$x \to -\infty$ $y \to 0^-$ ✓

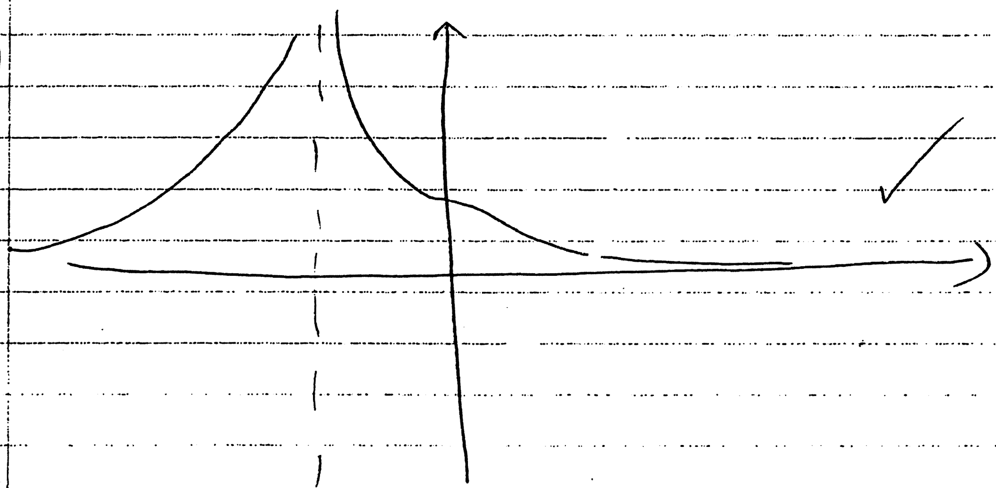
Horizontal asymptote at $y = 0$.

(iii)



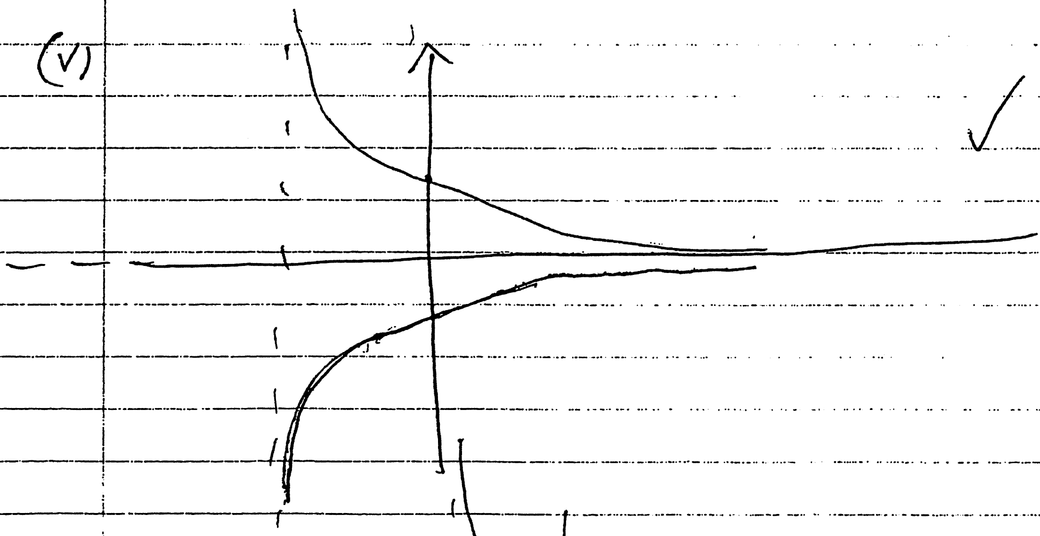
✓ ✓ for each branch correct.

(iv)



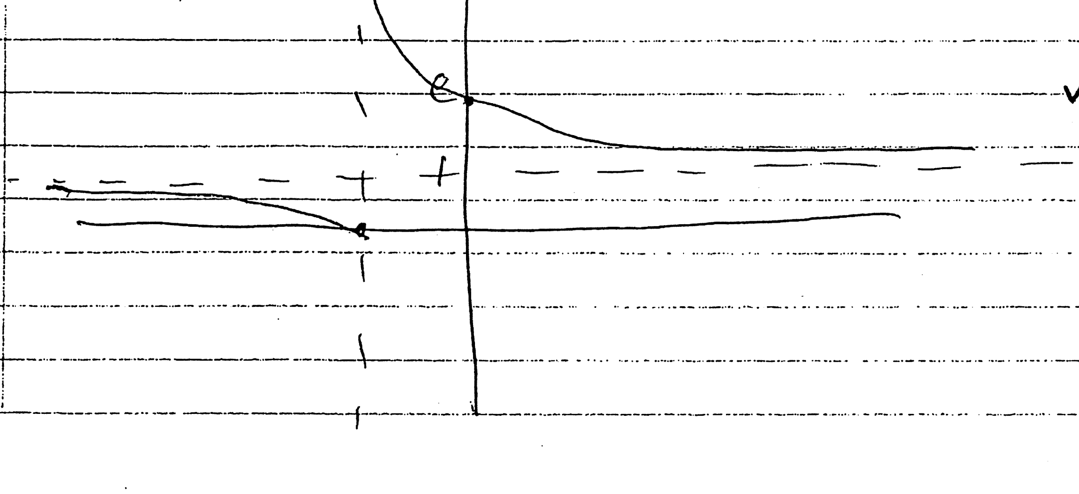
✓

(v)



correct domain ✓ ✓ reflection.

(vi)



✓ ✓

Question Four

(a) $x^3 - 3x^2 + x - 5 = 0$

(i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha + \beta + \gamma = 3$

$\alpha + \beta = 3 - \gamma$

(ii) $\beta + \gamma = 3 - \alpha$ and $\alpha + \gamma = 3 - \beta$

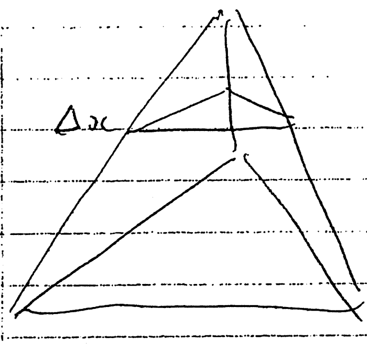
The polynomial equation has roots $3-\alpha,\ 3-\beta,\ 3-\gamma$.

Transformation is $y = 3 - x$

$x = 3 - y$

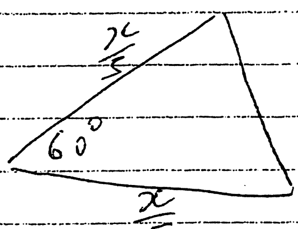
New equation is $(3-y)^3 - 3(3-y)^2 + (3-y) - 5 = 0$

(b)



$$V = \lim_{\Delta x \to 0} \sum A(x)\,\Delta x$$

$$V = \int_0^{50} \frac{1}{2}\left(\frac{x}{5}\right)^2 \cdot \frac{\sqrt{3}}{2}\,dx$$



$$= \frac{\sqrt{3}}{100}\left[\frac{x^3}{3}\right]_0^{50}$$

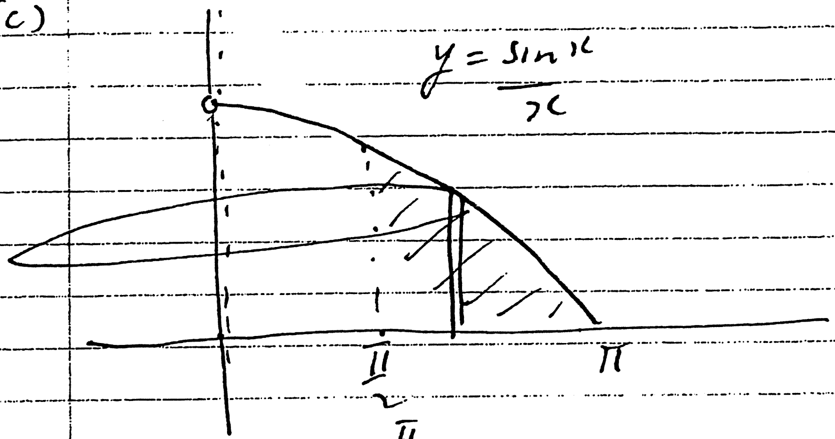
$$= \frac{\sqrt{3}}{100} \times \frac{50^3}{3}$$

$$= \frac{1250\sqrt{3}}{6} \text{ units}^3$$

$A(x) = \frac{1}{2}\left(\frac{x}{5}\right)^2 \sin 60^\circ$

$= \frac{x^2}{50} \cdot \frac{\sqrt{3}}{2}$

(c)



$\Delta V = 2\pi\, x\, f(x)\quad x.$ ✓

$$V = 2\pi \int_{\frac{\pi}{2}}^{\pi} x \cdot \frac{\sin x}{x}\, dx$$

$$= 2\pi \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx \quad ✓$$

$$= 2\pi \left[-\cos x \right]_{\frac{\pi}{2}}^{\pi} \quad ✓$$

$$= 2\pi \left[--1 - 0 \right] = 2\pi.$$

(d) $x = 3\sec\theta \qquad y = 4\tan\theta$

(i)
$$\frac{x^2}{9} - \frac{y^2}{1} = 1 \quad ✓$$

$a^2 = 9, \qquad b^2 = 1$

$b^2 = a^2(e^2 - 1)$

$16 = 9(e^2 - 1)$

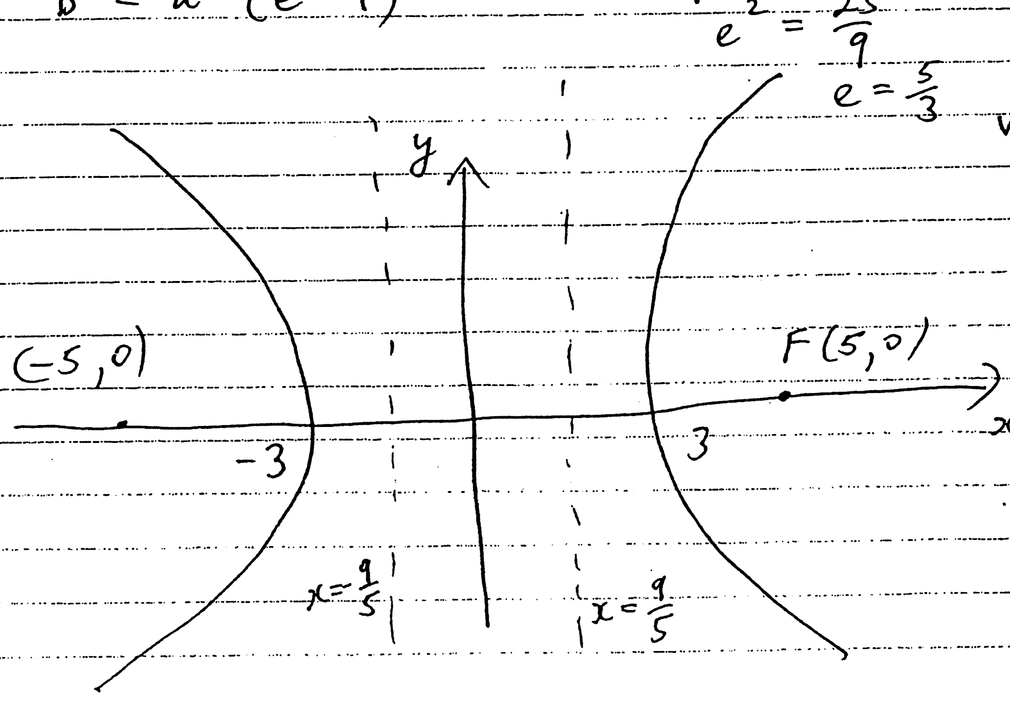
$\frac{16}{9} + 1 = e^2$

$e^2 = \frac{25}{9}$

$e = \frac{5}{3}$ ✓

(ii)

x-ints ✓
~~foci~~ ✓
directrices ✓
asymptotes ✓



Question Five

(a) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(i) Differentiate implicitly wrt x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

At $P(x_1, y_1)$, gradient $= -\dfrac{b^2 x_1}{a^2 y_1}$

So gradient of normal is $\dfrac{a^2 y_1}{b^2 x_1}$

Eqn of normal is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$\frac{y}{y_1} - 1 = \frac{a^2}{b^2}\left(\frac{x}{x_1} - 1\right)$$

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

(ii) When $y = 0$, $x = \dfrac{a^2 - b^2}{a^2} x_1$

But $\dfrac{a^2 - b^2}{a^2} = e^2$

So $G = (e^2 x_1, 0)$

(iii) Now $PF = e\,PM$ (defn of ellipse)

$$PF = e\left(\frac{a}{e} - x_1\right)$$

$$= a - e x_1$$

(iv) $\dfrac{PF}{FG} = \dfrac{a - ex_1}{ae - e^2x_1} = \dfrac{1}{e}$ ✓

$\dfrac{PF'}{F'G} = \dfrac{a + ex_1}{a + e^2x_1} = \dfrac{1}{e}$ ✓

So $\dfrac{PF}{FG} = \dfrac{PF'}{F'G}$

(b) (i) $LHS = 1 - \cos 2\alpha - i \sin 2\alpha$

$= 1 - (1 - 2\sin^2\alpha) - 2i \sin\alpha \cos\alpha$ ✓

$= 2\sin^2\alpha - 2i \sin\alpha \cos\alpha$

$= 2\sin\alpha(\sin\alpha - i\cos\alpha)$ ✓

$= RHS$

(ii) $\dfrac{z-1}{z} = \text{cis}\,\dfrac{2\pi}{5}$

$z - 1 = z\cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$

$z\left(1 - \cos\dfrac{2\pi}{5} - i\sin\dfrac{2\pi}{5}\right) = 1$

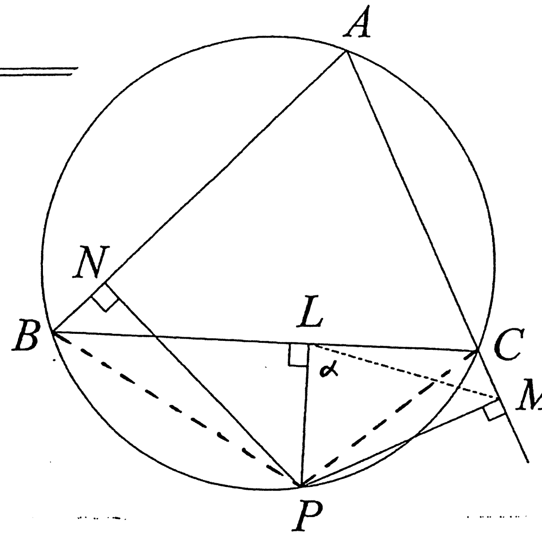
$z \times 2\sin\dfrac{\pi}{5}\left(\sin\dfrac{\pi}{5} - i\cos\dfrac{\pi}{5}\right) = 1$ by part (i) ✓

$2z\sin\dfrac{\pi}{5} = \sin\dfrac{\pi}{5} + i\cos\dfrac{\pi}{5}$ ✓

$z = \dfrac{1}{2}\left(1 + i\cot\dfrac{\pi}{5}\right)$ ✓

Q6

a)



(i) $\angle BLP = \angle BNP$ (given)

So B, L, N and P are concyclic by converse of angles standing on the same arc

(ii) $\angle PLC + \angle PMC = 180°$ (given)

So P, L, C and M are concyclic by converse of opposite angles of a cyclic quad.

(iii) (α) $\angle PCM = \angle PLM$ (angles standing on the same arc PM)
$= \alpha$

$\angle ABP = \angle PCM$ (opposite interior angle of cyclic quad ABPC)
$= \alpha$

(β) $\angle NBP = \alpha$ (same as $\angle ABP$)

So $\angle NLP = 180° - \alpha$ (opp angles of cyclic quad BNLP)

So $\angle NLM = \angle NLP + \angle MLP$
$= 180° - \alpha + \alpha$
$= 180°$

$\therefore$ N, L and M are collinear.

b) (i) Terminal velocity when $\ddot{x} = 0$

$$g - kV^2 = 0$$

$$V^2 = \frac{g}{k}$$

So $V = \sqrt{\frac{g}{k}}$ (since $V_0 > 0$ initially and so cannot change sign)

(b) (ii) $v\frac{dv}{dx} = g - kv^2$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$\frac{dx}{dv} = \frac{-1}{2k}\frac{-2kv}{g-kv^2}$$

$$x = \frac{-1}{2k}(g-kv^2) + c$$

When $x=0$, $v=V_0$: $\quad c = \frac{1}{2k}(g-kV^2)$

$$x = \frac{-1}{2k}\ln\left(\frac{g-kv^2}{g-kV_0^2}\right)$$

$$\frac{g-kv^2}{g-kV_0^2} = e^{-2kx}$$

$$g-kv^2 = (g-kV_0^2)e^{-2kx}$$

$$-kv^2 = -g + (g-kV_0^2)e^{-2kx}$$

$$v^2 = \frac{g}{k} + \left(V_0^2 - \frac{g}{k}\right)e^{-2kx}$$

$$v^2 = V^2 + (V_0^2 - V^2)e^{-2kx}$$

(c) (i) Let $z = x+iy$

$$z = x+iy + \frac{1}{x+iy} \times \frac{x-iy}{x^2-iy}$$

$\text{Im}(z) = 0$ so $\quad y + -\frac{y}{x^2+y^2} = 0$

$$y\left(1 - \frac{1}{x^2+y^2}\right) = 0$$

$$y = 0 \quad \text{or} \quad x^2+y^2 = 1$$

(ii) $x + \frac{1}{x} = k$

$x^2 - kx + 1 = 0$ has real roots. $\quad \therefore \Delta \geq 0$

$$k^2 - 4 \geq 0$$

$$|k| \geq 2$$

Question Seven

(a) (i) $$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

(ii) $$\cos 4\theta = \frac{1-\tan^2 2\theta}{1+\tan^2 2\theta}$$

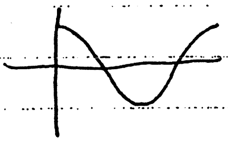
$$= \frac{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2}{1+\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta}{1-2\tan^2\theta+\tan^4\theta+4\tan^2\theta}$$

$$= \frac{1-6\tan^2\theta+\tan^4\theta}{1+2\tan^2\theta+\tan^4\theta}$$

Consider $\cos 4\theta = 0$

$$4\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$



$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}$$

Consider the quadratic

$$1-6x+x^2=0 \quad \text{—} \quad *$$

where $x = \tan^2\theta$

The solutions to $*$ are

$\tan^2\frac{\pi}{8}$ and $\tan^2\frac{3\pi}{8}$.

$$\Sigma \text{roots} = \tan^2\frac{\pi}{8} + \tan^2\frac{3\pi}{8} = \frac{-b}{a}$$

$$\tan^2\frac{\pi}{8} + \tan^2\frac{3\pi}{8} = 6.$$

Question Seven (b)

(i) $\rho^9 = \rho^7 \times \rho^2$
$= \rho^2$ ✓

(ii) $\alpha + \theta = \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6$ ✓

In the equation $z^7 - 1 = 0$
coeff of $x^6 = 0$

$\therefore \quad 1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = 0$

$\rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = -1$ ✓

(iii) $\alpha\theta = (\rho + \rho^2 + \rho^4)(\rho^3 + \rho^5 + \rho^6)$

$= \rho^4 + \rho^6 + \rho^7 + \rho^5 + \rho^7 + \rho^8 + \rho^7 + \rho^9 + \rho^{10}$

$= \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 + \rho^7 + 2$ ✓

$= 0 + 2$ ✓

$= 2$

(iv) The quadratic is
$x^2 - (\alpha + \theta)x + \alpha\theta = 0$ ✓

$x^2 + x + 2 = 0$ — $*$

(v) The roots of $*$ are

$x = \dfrac{-1 \pm \sqrt{7}i}{2}$ ✓

$\text{Re}(\rho + \rho^2 + \rho^4) = -\frac{1}{2}$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$ ✓

8 (a) $a_1 = \dfrac{2!}{2^1 \times 1!} = 1$, which is odd. ✓

Suppose that $a_k = \dfrac{(2k)!}{2^k k!}$ is odd.

Prove that a_{k+1} is odd: ✓

$$a_{k+1} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

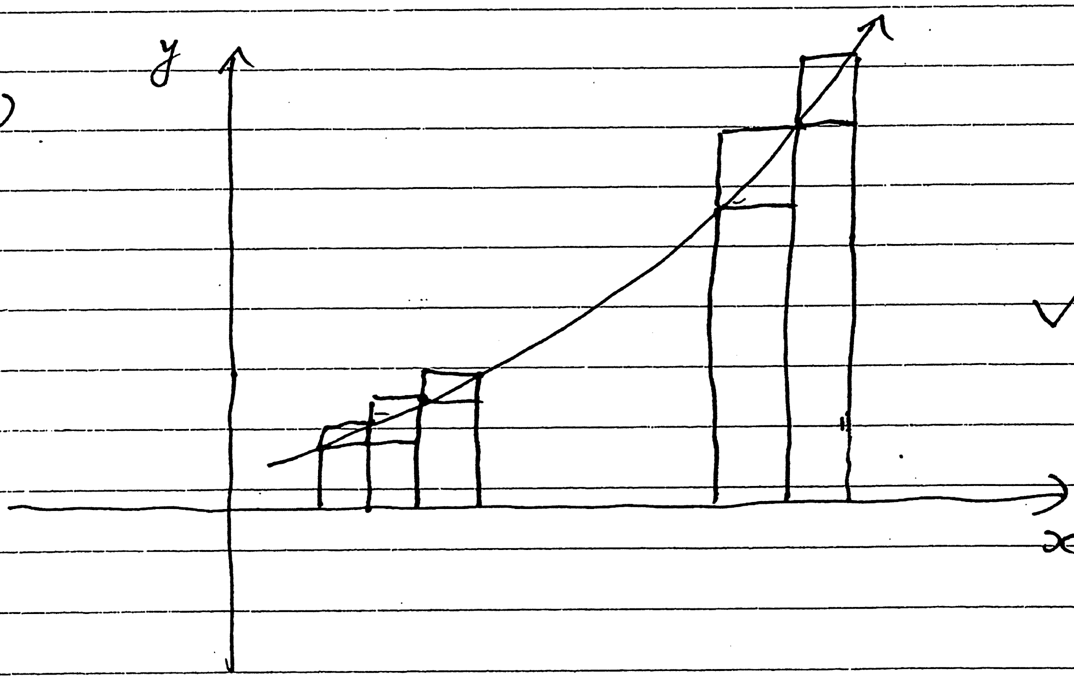
$$= \frac{(2k+2)(2k+1)(2k)!}{2 \times 2^k \times (k+1) \times k!}$$ ✓

$$= \frac{2(k+1)(2k+1)}{2(k+1)} \cdot \frac{(2k)!}{2^k k!}$$

$$= \underbrace{(2k+1)}_{\text{odd}} \cdot \underbrace{a_k}_{\text{odd}}$$ ✓

$\therefore a_{k+1}$ is odd

(b) (i)



✓

sum of area of lower rectangles < exact area under curve from $x=1$ to $x=n$ < sum of areas of upper rectangles ✓

$$f(1) + f(2) + \ldots + f(n-1) < \int_1^n f(x)\,dx < f(2) + f(3) + \ldots + f(n)$$

(ii) $\int_1^n \ln x \frac{d}{dx}(x)\,dx$

$= [x\ln x]_1^n - \int_1^n 1\,dx$ ✓

$= n\ln n - n + 1$ ✓

(iii) (α) Let $f(x) = \ln x$

From (i): $\int_1^n \ln x\,dx < \ln 2 + \ln 3 + \cdots + \ln n$

Using (ii) $n\ln n - n + 1 < \ln n!$ ✓

$n! > e^{\ln n^n - n + 1}$ ✓

$n! > n^n e^{1-n}$

$n! > \frac{n^n}{e^{n-1}}$ ✓

(β) From (i) and (ii)

$\ln 1 + \ln 2 + \cdots + \ln(n-1) < n\ln n - n + 1$ ✓

$\ln(n-1)! < \ln n^n - n + 1$

$(n-1)! < n^n \cdot e^{1-n}$ ✓

$n! < \frac{n^{n+1}}{e^{n-1}}$

(iv) From (iii), $\frac{n^n}{e^{n-1}} < n! < \frac{n^{n+1}}{e^{n-1}}$

Taking nth roots : $\frac{n}{e^{1-\frac{1}{n}}} < (n!)^{\frac{1}{n}} < \frac{n^{1+\frac{1}{n}}}{e^{1-\frac{1}{n}}}$ ✓

$\therefore\ e^{\frac{1}{n}-1} < \frac{\sqrt[n]{n!}}{n} < \sqrt[n]{n}\, e^{\frac{1}{n}-1}$

As $n\to\infty$ $e^{\frac{1}{n}-1} \to e^{-1}$ and $\sqrt[n]{n} \to 1$ given

$\therefore \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ ✓