Sydney Grammar School
Mathematics Department
Trial Examinations 2006

FORM VI

MATHEMATICS EXTENSION 2

Examination date

Tuesday 1st August 2006

Time allowed

Three hours (plus 5 minutes reading time)

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet and on the tear-off sheet.
Hand in the eight questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Bundle the tear-off sheet with the question it belongs to.
Keep the printed examination paper and bring it to your next Mathematics lesson.

6A: REP 6B: BDD 6C: GJ 6D: MLS

Checklist

SGS booklets: 8 per boy. A total of 750 booklets should be sufficient.
Candidature: 61 boys.

Examiner

GJ

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find the following integrals:

(i) $\displaystyle\int \frac{1}{x \ln x}\, dx$ **2**

(ii) $\displaystyle\int x \ln x \, dx$ **2**

(iii) $\displaystyle\int \frac{2x+1}{x^2+2x+5}\, dx$ **3**

(b) Use integration by parts to evaluate **3**

$$\int_0^{\frac{1}{2}} \cos^{-1} x \, dx.$$

(c) (i) Find the values of A, B and C so that **2**

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}.$$

(ii) Use the substitution $t = \tan\theta$, and part (i) above, to find **3**

$$\int \frac{2}{1-\tan\theta}\, d\theta.$$

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Given that $w = 3 - 4i$, find $\dfrac{|w| - \overline{w}}{w}$ in the form $a + ib$, where a and b are real. **3**

0 (b) Find the roots of the equation $(1+i)z^2 + 2z + 1 - i = 0$. **3**

(c) (i) Write $1 - i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$. **1**

(ii) Hence find $(1 - i\sqrt{3})^6$ in the form $a + ib$, where a and b are real. **2**

(d) If ω is one of the complex roots of $z^3 = 1$, simplify **3**

$$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8).$$

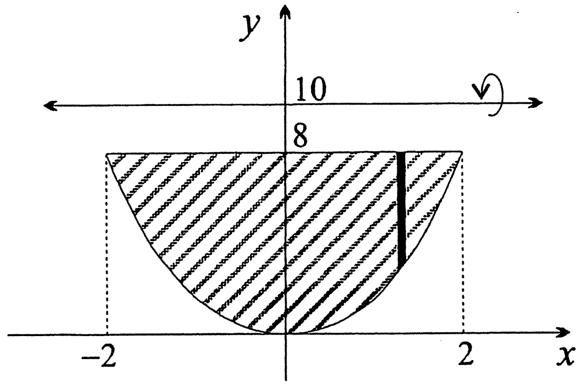
(e) Shade on an Argand diagram the region given by **3**

$$|z-1| \le 1 \qquad \text{and} \qquad \frac{\pi}{6} < \arg z < \frac{\pi}{3}.$$

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $y = 2x^2$ and the line $y = 8$. This region is rotated about the line $y = 10$ to form a solid of revolution.

(i) The solid is sliced perpendicular to the axis of rotation. Show that the area of each cross-section formed is **2**

$$4\pi(24 - 10x^2 + x^4).$$

(ii) Hence find the volume of the solid. **2**

(b) (i) Let $I_n = \int_0^1 x^n e^x \, dx$, for $n \ge 0$, where n is an integer. **4**

Show that $I_{n+1} = e - (n+1)I_n$.

(ii) Hence find $\int_0^1 t^3 e^t \, dt$. **2**

(c) The tangents to the hyperbola $xy = c^2$ at the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ intersect at the point T.

(i) Given that the equation of the tangent at P is $x + p^2 y = 2cp$, show that the coordinates of the point T are **3**

$$\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right).$$

(ii) Prove that the origin, the point T and the midpoint of PQ are collinear. **2**

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) Given that α, β and γ are the roots of the equation $2x^3 + 3x^2 - 5x + 8 = 0$, find the polynomial equation with roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. $\boxed{3}$

(b) (i) Using the sums-to-products formulae, or otherwise, prove that $\boxed{3}$

$$\frac{\sin 2x + \sin 3x + \sin 4x}{\cos 2x + \cos 3x + \cos 4x} = \tan 3x.$$

(ii) Hence find the general solution of $\boxed{1}$

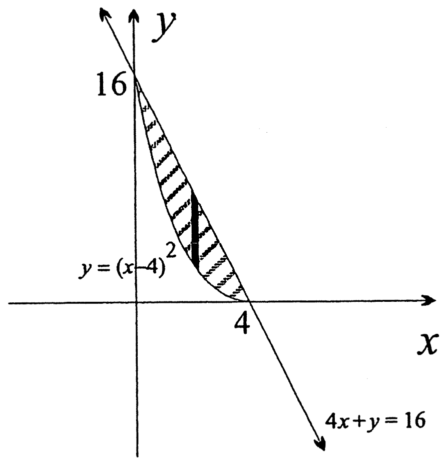
$$\frac{\sin 2x + \sin 3x + \sin 4x}{\cos 2x + \cos 3x + \cos 4x} = \frac{1}{\sqrt{3}}.$$

(c) The parametric equations of an ellipse are $x = 5\cos\theta$ and $y = 4\sin\theta$.

(i) Find the Cartesian equation of the ellipse and show that its eccentricity is $\frac{3}{5}$. $\boxed{2}$

(ii) Sketch the ellipse showing its intercepts, foci and directrices. $\boxed{2}$

(d)



The region enclosed by the curve $y = (x-4)^2$ and the line $4x + y = 16$ is shaded in the diagram above. A solid is formed with this region as its base.

When the solid is sliced perpendicular to the x-axis, each cross-section is an equilateral triangle with its base in the xy-plane.

(i) Show that the area of the cross-section x units to the right of the y-axis is $\boxed{2}$

$$\frac{\sqrt{3}}{4}x^2(4-x)^2, \text{ where } 0 \le x \le 4.$$

(ii) Hence find the volume of the solid. $\boxed{2}$

Exam continues next page ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) De Moivre's theorem with $n = 5$ states that

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta.$$

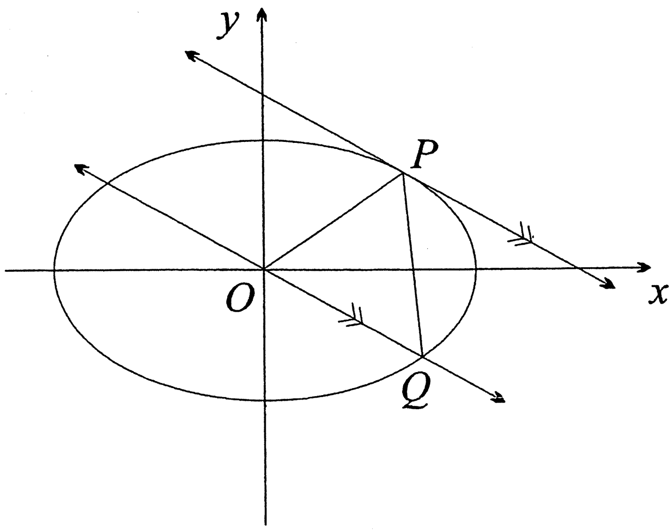
(i) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **2**

(ii) Hence find all five roots of the equation $16x^5 - 20x^3 + 5x = 0$. **2**

(iii) Show that $\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$. **1**

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, where $p > q$, lie on the parabola $x^2 = 4ay$, and the difference in their x-coordinates is $2a$. Show that the locus of the midpoint M of the chord PQ is a parabola, and find the coordinates of its focus. **3**

(c)



In the diagram above, $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P lies in the first quadrant.

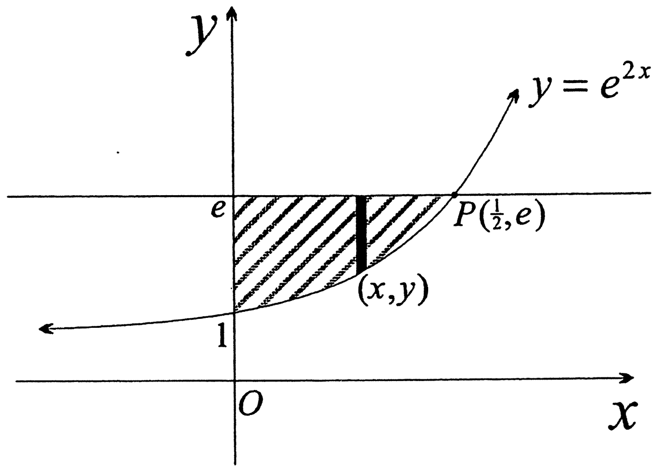
A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q, where P and Q both lie on the same side of the y-axis.

(i) Prove that the equation of the line OQ is $xb\cos\theta + ya\sin\theta = 0$, and find the coordinates of the point Q. **3**

(ii) Prove that the area of $\triangle OPQ$ is independent of the position of P. **4**

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a) **5**



In the diagram above, $P(\frac{1}{2}, e)$ is the point of intersection of the curve $y = e^{2x}$ and the line $y = e$.

Use the method of cylindrical shells to find the volume of the solid generated when the shaded region enclosed by the y-axis, the curve $y = e^{2x}$ and the line $y = e$ is rotated about the y-axis.

(b) A particle of unit mass is projected vertically upwards from a point O, with initial velocity of u m/s, in a medium whose resistance has magnitude kv^2, where k is a positive constant and v m/s is the velocity after t seconds. Let x be the vertical displacement of the object above the origin after t seconds.

After reaching its maximum height, the particle then falls back to O, experiencing the same resistance.

(i) Taking upwards as positive, show that **1**

$$\ddot{x} = -(g + kv^2).$$

(ii) Hence show that the maximum height attained by the particle is **3**

$$\frac{1}{2k} \ln\left(\frac{g + ku^2}{g}\right).$$

(iii) Show that the speed of the particle when it returns to O is **4**

$$\sqrt{\frac{gu^2}{g + ku^2}}.$$

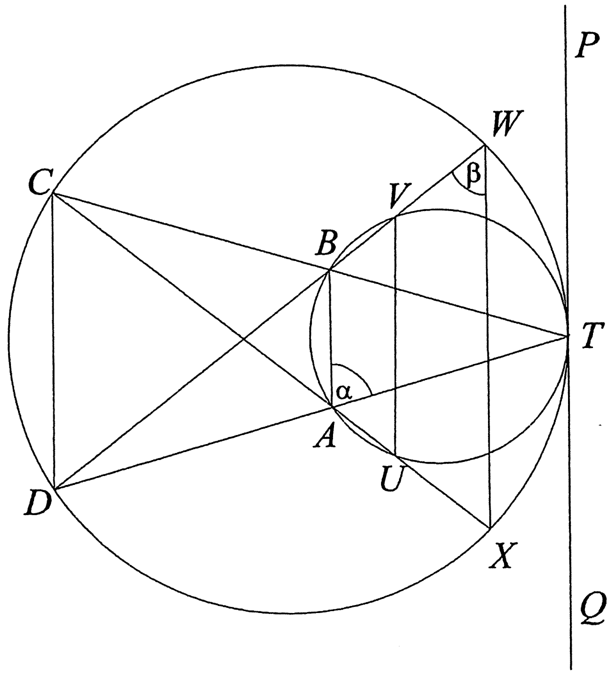
(iv) Find the terminal velocity V of any particle of unit mass falling in this medium subject to the resistance kv^2. Hence prove that if the particle in part (i) above is projected upwards with velocity V, it will return to O with speed $\frac{V}{\sqrt{2}}$. **2**

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x) = 3x^3 - 11x^2 + 24x - 12$ has one rational non-integer zero. Find its value. **2**

(b) Given that $x = -\frac{2}{5}$ is a zero of the polynomial $P(x) = 5x^3 - 3x^2 + 8x + 4$, factor $P(x)$ into its real and complex linear factors. **3**

(c) NOTE: The diagram below has been reprinted on Page 11 so that working can be done on the diagram. Tear out Page 11, write your candidate number on the top of the sheet in the space provided, and place the sheet inside your answer booklet for Question Seven.



The diagram above shows two circles touching internally at the point T. The line PQ is the common tangent at T. The points A and B lie on the small circle so that $TB = TA$, and TA and TB produced meet the larger circle at the points D and C respectively.

The line DB produced meets the smaller circle at the point V and the larger circle at the point W, while the line CA produced meets the smaller circle at the point U and the larger circle at the point X.

Let $\angle BAT = \alpha$ and $\angle VWX = \beta$.

(i) Show that $CD \parallel AB$. **2**

(ii) Show that $ABCD$ is a cyclic quadrilateral. **2**

(iii) Show that $UVWX$ is a cyclic quadrilateral. **3**

(iv) Given that $TU = TV$, prove that T is the centre of a circle passing through the points U, V, W and X. **3**

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $\alpha^k + \beta^k = (\alpha + \beta)(\alpha^{k-1} + \beta^{k-1}) - \alpha\beta(\alpha^{k-2} + \beta^{k-2})$, for $k \geq 2$. $\boxed{1}$

(ii) By substituting $\alpha = \cos\theta + i\sin\theta$ and $\beta = \cos\theta - i\sin\theta$, show that $\boxed{2}$

$$\cos k\theta = 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta, \text{ for } k \geq 2.$$

(iii) Using part (ii) with $k = 2$, 3 and 4, show that $\boxed{2}$

$$\cos 2\theta = 2\cos^2\theta - 1,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

(b) (i) The *Tschebyshev polynomials* are defined by the recurrence formula $\boxed{1}$

$$t_0(x) = 1,$$
$$t_1(x) = x,$$
$$t_k(x) = 2x\,t_{k-1}(x) - t_{k-2}(x), \text{ for } k \geq 2.$$

Show that the Tchebyshev polynomials $t_2(x)$, $t_3(x)$ and $t_4(x)$ are

$$t_2(x) = 2x^2 - 1,$$
$$t_3(x) = 4x^3 - 3x,$$
$$t_4(x) = 8x^4 - 8x^2 + 1.$$

(ii) To find a formula for $t_k(x)$ let $F(z)$ be the power series in z with the coefficient of z^k being $t_k(x)$. That is, let

$$F(z) = 1 + xz + (2x^2 - 1)z^2 + (4x^3 - 3x)z^3 + (8x^4 - 8x^2 + 1)z^4 + \cdots + t_k(x)z^k + \cdots$$

(α) Show that $\boxed{3}$

$$(1 - 2xz)F(z) = 1 - xz - z^2F(z),$$

and hence show that

$$F(z) = \frac{1 - xz}{1 - 2xz + z^2}.$$

(β) Given that α and β are the zeroes of $1 - 2xz + z^2$ show that $\boxed{1}$

$$1 - 2xz + z^2 = \left(1 - \frac{z}{\alpha}\right)\left(1 - \frac{z}{\beta}\right).$$

Exam continues next page ...

(γ) Using the partial fraction decomposition of $F(z)$, **2**

$$F(z) = \frac{1 - xz}{1 - 2xz + z^2} = \frac{A}{1 - \dfrac{z}{\alpha}} + \frac{B}{1 - \dfrac{z}{\beta}}$$

where A and B are independent of z, show that the coefficient $t_k(x)$ is

$$A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k, \text{ for } |z| \text{ sufficiently small.}$$

(δ) Deduce that the formula for $t_k(x)$ is **3**

$$t_k(x) = \frac{1}{2}\left(\frac{1}{x + \sqrt{x^2 - 1}}\right)^k + \frac{1}{2}\left(\frac{1}{x - \sqrt{x^2 - 1}}\right)^k.$$

END OF EXAMINATION

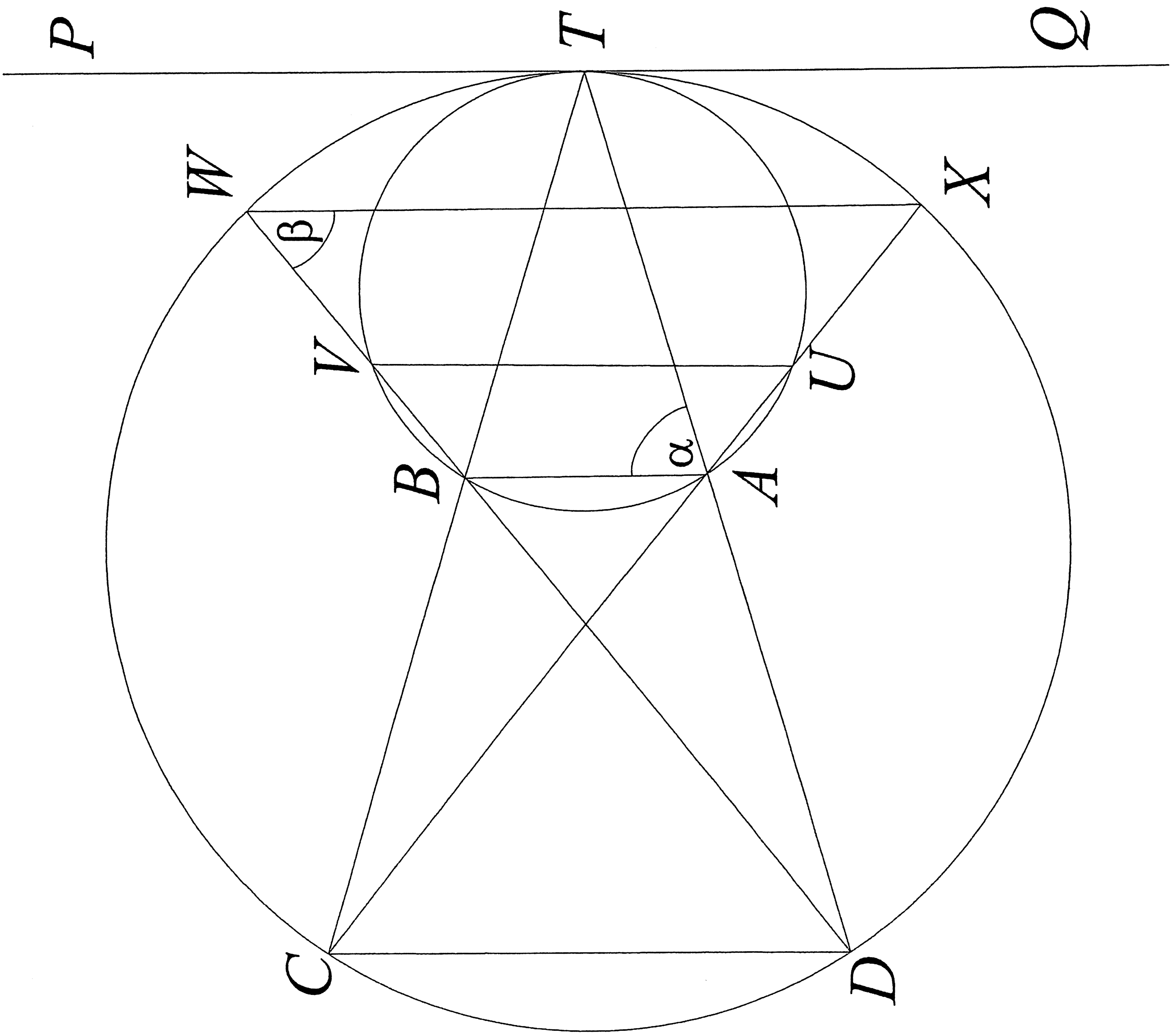
Tear-off pages follow ...

CANDIDATE NUMBER: ....

DETACH THIS SHEET AND BUNDLE IT WITH THE REST OF QUESTION SEVEN.

QUESTION SEVEN

(b)



QUESTION ONE

a (i) $\int \frac{1}{x \ln x} dx$ Let $u = \ln x$, $du = \frac{1}{x} dx$

$= \int \frac{1}{u} du$ ✓

$= \ln|u| + c$

$= \ln|\ln x| + c$ ✓

(ii) $\int x \ln x \, dx$ $u = \ln x$, $du = \frac{1}{x} dx$ $dv = x dx$, $v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln|x| - \int \frac{1}{2}x^2 \times \frac{1}{x} dx$ ✓

$= \frac{1}{2}x^2 \ln|x| - \int \frac{1}{2} x \, dx$

$= \frac{1}{2}x^2 \ln|x| - \frac{1}{4}x^2 + c$ ✓

(iii) $\int \frac{2x+1}{x^2+2x+5} dx$

$= \int \frac{2x+2}{x^2+2x+5} dx - \int \frac{1}{x^2+2x+5} dx$ ✓

$= \ln(x^2+2x+5) - \int \frac{1}{(x+1)^2+4} dx$ ✓

$= \ln(x^2+2x+5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$ ✓

(b) $\int_0^{\frac{1}{2}} \cos^{-1}x \, dx = \int_0^{\frac{1}{2}} \cos^{-1}x \frac{d}{dx}(x) dx$ ✓ SETUP

$= \left[x\cos^{-1}x\right]_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$ ✓

$= \frac{\pi}{6} + \int_0^{\frac{1}{2}} x(1-x^2)^{-\frac{1}{2}} dx$

$= \frac{\pi}{6} + \left[2 \times -\frac{1}{2}(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

$= \frac{\pi}{6} - \left[(1-\frac{1}{4})^{\frac{1}{2}} - 1\right]$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$ ✓

(c)(i) $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$= \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$

Consider $2 = A(1+x^2) + (Bx+C)(1-x)$

Let $x=1$: $2 = 2A$.

$A = 1$

Coefficient of term in x^0

$2 = 1 + C$

$C = 1$

Coefficient of term in x^1

$0 = B - C$

$B = 1$

So $A = 1$, $B = 1$ and $C = 1$ ✓✓

(ii) $t = \tan\theta$,

$dt = \sec^2\theta \, d\theta$

$= 1 + \tan^2\theta \, d\theta$

$d\theta = \frac{dt}{1+t^2}$

$\int \frac{2}{1-\tan\theta} d\theta = \int \frac{2}{1-t} \times \frac{dt}{1+t^2}$

$= \int \frac{2\,dt}{(1-t)(1+t^2)}$ ✓

$= \int \left(\frac{1}{1-t} + \frac{t+1}{1+t^2}\right) dt$ from (i),

$= \int \left(\frac{1}{1-t} + \frac{t}{1+t^2} + \frac{1}{1+t^2}\right) dt$ ✓

$= -\ln|1-t| + \frac{1}{2}\ln(1+t^2) + \tan^{-1}t + c$

$= -\ln|1-\tan\theta| + \frac{1}{2}\ln(1+\tan^2\theta) + \tan^{-1}(\tan\theta) + c$

$= -\ln|1-\tan\theta| + \ln(\sec\theta) + \theta + c$ ✓

QUESTION TWO

(a) $w = 3-4i$

$|w| = \sqrt{9+16}$

$= 5$

$\bar{w} = 3+4i$

$\dfrac{|w| - \bar{w}}{w} = \dfrac{5-3-4i}{3-4i}$ ✓✓

$= \dfrac{2-4i}{3-4i} \times \dfrac{3+4i}{3+4i}$

$= \dfrac{6+16+8i-12i}{9+16}$

$= \dfrac{22}{25} - \dfrac{4}{25}i$ ✓

(b) $(1+i)z^2 + 2z + 1-i = 0$

$\Delta = 4 - 4(1+i)(1-i)$

$= 4-8$

$= -4$

?

$z = \dfrac{-2+\sqrt{-4}}{2(1+i)}$ or $\dfrac{-2-\sqrt{-4}}{2(1+i)}$

$= \dfrac{-2-2i}{2+2i}$ or $\dfrac{-2+2i}{2(1+i)}$ ✓✓

$= -1$ or $\dfrac{-1+i}{1+i} \times \dfrac{1-i}{1-i}$

$= -1$ or $\dfrac{2i}{2}$

$= -1$ or i ✓

c(i) OVER

c(i) Let $z = 1+\sqrt{3}i$

$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ ✓

(ii) $z^6 = 2^6 \cos\left(-\frac{\pi}{3} \times 6\right)$

$= 2^6 \cos(-2\pi)$ ✓

$= 2^6\left(\cos(-2\pi) + i\sin(-2\pi)\right)$

$= 64$ ✓

(d) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

So $w^3 = 1$ and $1+w+w^2 = 0$ — * ✓

?

$(1-w)(1-w^2)(1-w^4)(1-w^8)$

$= (1-w)(1-w^2)(1-w)(1-w^2)$?

$= (1-w-w^2+w^3)^2$

$= (1-w-w^2+1)^2$

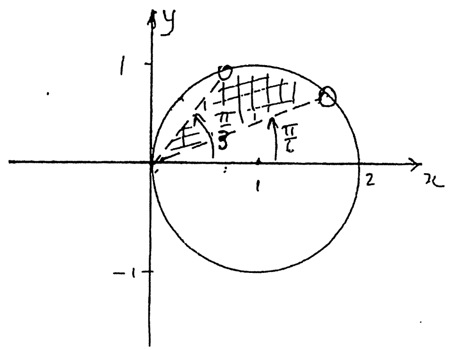
$= (2-w-w^2)^2$

$= (3-1-w-w^2)^2$

$= 3^2$ from *

$= 9$ ✓✓

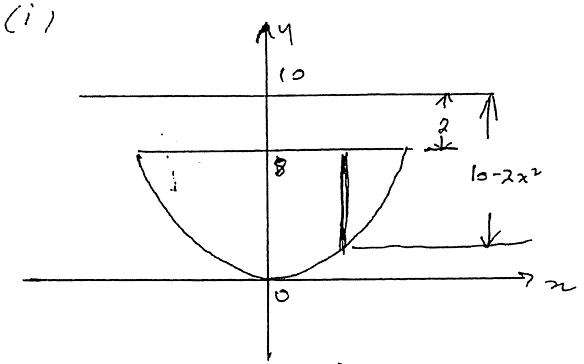
(e)



✓✓✓

QUESTION THREE

(a) (i)



$$\text{Area} = \pi(R^2 - r^2)$$
$$= \pi(R-r)(R+r)$$
$$= \pi(10-2x^2-2)(10-2x^2+2)$$
$$= \pi(8-2x^2)(12-2x^2)$$
$$= 4\pi(4-x^2)(6-x^2)$$
$$= 4\pi(24-10x^2+x^4)$$ ✓2

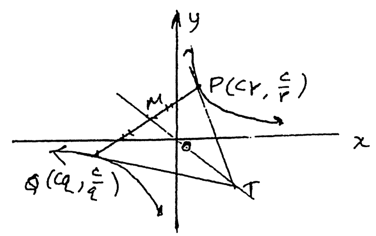
(ii) $V = 2\int_0^2 4\pi(24-10x^2+x^4)\,dx$ ✓
$$= 8\pi\int_0^2(24-10x^2+x^4)\,dx$$
$$= 8\pi\left[24x - \frac{10x^3}{3} + \frac{x^5}{5}\right]_0^2$$
$$= 8\pi\left(48 - \frac{80}{3} + \frac{32}{5}\right)$$
$$= \frac{3328\pi}{15} \text{ units}^3$$ ✓

(c)(i) $I_n = \int_0^1 x^n e^x\,dx$
$$= \int_0^1 x^n \frac{d}{dx}(e^x)\,dx$$ ✓
$$= \left[x^n e^x\right]_0^1 - \int n x^{n-1} e^x\,dx$$ ✓
$$= e - n\int x^{n-1}e^x\,dx$$ ✓
$$= e - n I_{n-1}$$ ✓

It follows that
$$I_{n+1} = e - (n+1)I_n$$

(ii) $I_0 = \int_0^1 e^x\,dx$
$$= \left[e^x\right]_0^1$$ ✓
$$= e - 1$$
$I_1 = e - I_0$
$$= e - e + 1$$
$$= 1$$
$I_2 = e - 2I_1$
$$= e - 2$$
$I_3 = e - 3I_2$
$$= e - 3e + 6$$
$$= 6 - 2e$$
So $\int_0^1 t^3 e^t\,dt = 6 - 2e$. ✓

(d)



(i) $x + p^2 y = 2cp$ —(1)
So $x + q^2 y = 2cq$ —(2) is the tangent at Q ✓
(1)−(2) $y(p^2-q^2) = 2c(p-q)$
$$y = \frac{2c(p-q)}{(p-q)(p+q)}, \quad p \neq q$$
$$= \frac{2c}{p+q}$$ ✓
$$x = 2cp - \frac{2cp^2}{p+q}$$
$$= \frac{2cpq}{p+q}$$ ✓

T is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

(iii) Midpoint $M = \left(\frac{c(p+q)}{2}, \frac{c(\frac{1}{p}+\frac{1}{q})}{2} \right)$

$= \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq} \right)$

gradient of $OT = \frac{2c}{p+q} \div \frac{2cpq}{p+q}$

$= \frac{1}{pq}$ ✓

Gradient of $OM = \frac{c(p+q)}{2pq} \div \frac{c(p+q)}{2}$

$= \frac{1}{pq}$ ✓

Since the gradients are equal and O is a common point the points T, O and M are collinear.

QUESTION FOUR

(a) $2x^3+3x^2-5x+8=0$ ✓

Let $x = \frac{1}{y}$

$\frac{2}{y^3} + \frac{3}{y^2} - \frac{5}{y} + 8 = 0$ ✓

So $8y^3 - 5y^2 + 3y + 2 = 0$ is the required equation ✓

(b)(i) $\sin 2x + \sin 4x = 2\sin\frac{2x+4x}{2}\cos\frac{4x-2x}{2}$

$= 2\sin 3x \cos x$

$\cos 2x + \cos 4x = 2\cos\frac{2x+4x}{2}\cos\frac{4x-2x}{2}$

$= 2\cos 3x \cos x$

So LHS $= \frac{\sin 2x + \sin 3x + \sin 4x}{\cos 2x + \cos 3x + \cos 4x}$

$= \frac{2\sin 3x \cos x + \sin 3x}{2\cos 3x \cos x + \cos 3x}$ ✓ ✓

$= \frac{\sin 3x(2\cos x + 1)}{\cos 3x(2\cos x + 1)}$

$= \tan 3x$ ✓

$=$ RHS

(ii) $\frac{\sin 2x + \sin 3x + \sin 4x}{\cos 2x + \cos 3x + \cos 4x} = \frac{1}{\sqrt{3}}$

So $\tan 3x = \frac{1}{\sqrt{3}}$

So $3x = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$

So $x = \frac{n\pi}{3} + \frac{\pi}{18}, n \in \mathbb{Z}$ ✓

(b)(i) $x = 5\cos\theta$, $y = 4\sin\theta$

$\cos^2\theta + \sin^2\theta = \frac{x^2}{25} + \frac{y^2}{16}$

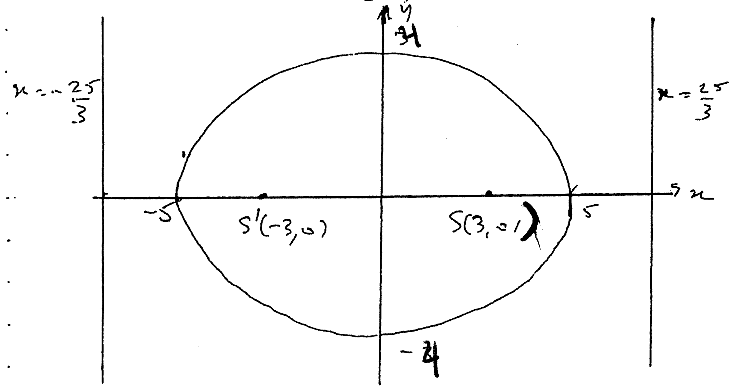
So $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ✓

Now $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

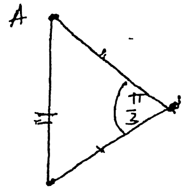
$e^2 = \frac{9}{25}$

$e = \frac{3}{5}$, $e > 0$ ✓



✓✓

(d) (i)



a typical slice x units from the y-axis has

$AB = 16 - 4x - (x-4)^2$

$= 16 - 4x - x^2 + 8x - 16$

$= 4x - x^2$

$= x(4-x)$, where $0 \le x \le 4$

Area of each slice

$= \frac{1}{2}x^2(4-x)^2 \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{4} x^2(4-x^2)$ ✓✓

(ii) Volume $V = \int_0^4 \frac{\sqrt{3}}{4}(x^2)(x-4)^2\,dx$

$= \int_0^4 \frac{\sqrt{3}}{4}(x^4 - 8x^3 + 16x^2)\,dx$

$= \frac{\sqrt{3}}{4}\left[\frac{x^5}{5} - 2x^4 + \frac{16x^3}{3}\right]$

$= \frac{\sqrt{3}}{4}\left(\frac{1024}{5} - 192 + \frac{1024}{3}\right)$

$= \frac{128\sqrt{3}}{15}$ units^3 ✓✓

QUESTION FIVE

(a)(i)

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

So $\cos 5\theta + i\sin 5\theta$

$= (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta . i\sin\theta + 10\cos^3\theta . i^2\sin^2\theta + 10\cos^2\theta . i^3\sin^3\theta + 5\cos\theta . i^4\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating real parts.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(ii) To solve $16x^5 - 20x^3 + 5x = 0$ let $x = \cos\theta$.

so $\cos 5\theta = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

So $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$ are the five unique solutions

i.e. $x = \cos\frac{k\pi}{10}$, for $k = 1, 3, 5, 7, 9$.

(iii) Now $16x^5 - 20x^3 + 5x = x(16x^4 - 20x^2 + 5)$

Roots are given by

$x(16x^4 - 20x^2 + 5) = 0$

So $x = 0$ or $16x^4 - 20x^2 + 5 = 0$

The roots of $16x^4 - 20x^2 + 5 = 0$ are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}$ or $\cos\frac{9\pi}{10}$

Product of roots

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10} = \frac{5}{16}$

$\cos\frac{\pi}{10} \times -\cos\frac{\pi}{10} \times \cos\frac{3\pi}{10} \times -\cos\frac{3\pi}{10} = \frac{5}{16}$

$\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10} = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$

since $\cos\frac{\pi}{10} > 0$ and $\cos\frac{3\pi}{10} > 0$.

(b) Midpoint of PQ

$M = \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

The locus is defined by

$x = a(p+q)$ — (1)

$y = \frac{a(p^2+q^2)}{2}$ — (2)

Also $2ap - 2aq = 2a$

So $p - q = 1$ — (3)

From (3) $p^2 + q^2 - 2pq = 1$

From (1) $x^2 = a^2(p^2 + q^2 + 2pq)$

$= a^2(p^2 + q^2 + p^2 + q^2 - 1)$

$= a^2\left(\frac{4y}{a} - 1\right)$

$x^2 = 4a\left(y - \frac{a}{4}\right)$

which is a parabola with vertex $(0, \frac{a}{4})$ and focal length a

The focus is $(0, \frac{a}{4} + a)$

$= (0, \frac{5a}{4})$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

gradient of tangent at P

$m = -\frac{b^2 a\cos\theta}{a^2 b\sin\theta}$

$= -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$

Equation of OQ.

$y = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}x$

$x b\cos\theta + y a\sin\theta = 0$.

For point Q

$$\frac{x^2}{a^2} + \frac{b^2\cos^2\theta}{a^2b^2\sin^2\theta}\,x^2 = 1$$

$$\frac{x^2}{a^2}\left(1+\cot^2\theta\right) = 1$$

$$\frac{x^2}{a^2}\,\mathrm{cosec}^2\theta = 1$$

$$\frac{x^2}{a^2} = \frac{1}{\mathrm{cosec}^2\theta}$$

$$x^2 = a^2\sin^2\theta$$

$$x = a\sin\theta$$

$$y = -\frac{b}{a}\frac{\cos\theta}{\sin\theta} \times a\sin\theta$$

$$= -b\cos\theta$$

So Q is the point $(a\sin\theta, -b\cos\theta)$

(ii) Equation of OP

$$y = \frac{b\sin\theta}{a\cos\theta}\,x$$

$$xb\sin\theta - ay\cos\theta = 0$$

Perpendicular distance from Q to OP

$$p = \left|\frac{ab\sin^2\theta + ab\cos^2\theta}{\sqrt{b^2\sin^2\theta + a^2\cos^2\theta}}\right|$$

$$= \frac{ab}{\sqrt{b^2\sin^2\theta + a^2\cos^2\theta}}$$

Distance OP

$$d = \sqrt{b^2\sin^2\theta + a^2\cos^2\theta}$$

So area of $\Delta OPQ = \frac{1}{2}pd$

$$= \frac{1}{2}ab$$

which is independent of the position of point P

QUESTION SIX

(a) Surface area of shell $= 2\pi x(e-y)$

$$= 2\pi x(e-e^{2x})$$

Volume of shell of width $\delta x = 2\pi x(e-e^{2x})\,\delta x$

So $V = \int_0^{\frac{1}{2}} 2\pi x(e-e^{2x})\,dx$

Now $\int_0^{\frac{1}{2}} 2\pi x e\,dx = 2\pi e\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}$

$$= \frac{\pi e}{4}$$

$$\int_0^{\frac{1}{2}} 2\pi x e^{2x}\,dx = 2\pi\int_0^{\frac{1}{2}} x\frac{d}{dx}\left(\tfrac{1}{2}e^{2x}\right)dx$$

$$= 2\pi\left\{\left[\tfrac{1}{2}xe^{2x}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\tfrac{1}{2}e^{2x}\,dx\right\}$$

$$= 2\pi\left\{\frac{e}{4} - \left[\tfrac{1}{4}e^{2x}\right]_0^{\frac{1}{2}}\right\}$$

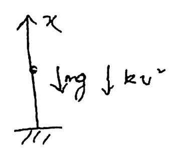
$$= 2\pi\left\{\frac{e}{4} - \left(\frac{e}{4} - \frac{1}{4}\right)\right\}$$

$$= -\frac{\pi}{2}$$

Volume $= \frac{\pi e}{4} - \frac{\pi}{2}$

$$= \frac{\pi}{4}(e-2)\ \text{units}^3$$

(b) Motion upwards



$mg<0$, $kv^2<0$

Force on particle

$$m\ddot{x} = -mg - kv^2$$

$$\ddot{x} = -g - kv^2$$

$$\ddot{x} = -(g+kv^2)$$

Some explanation is required.

$$v\frac{dv}{dx} = -(g+kv^2)$$

$$\frac{dx}{dv} = -\frac{v}{g+kv^2}$$

$$x = -\int_u^0 \frac{v}{g+kv^2}\,dv$$

$x = \frac{1}{2k}\int_0^u \frac{2kv}{g+kv^2}\,dv$

$= \frac{1}{2k}\left[\ln|g+kv^2|\right]_0^u$ ✓ or $x = \frac{-1}{2k}\ln|g+kv^2| + c$

when $x=0$, $v=u$

$= \frac{1}{2k}\ln\left|\frac{g+ku^2}{g}\right|$ $c = \frac{1}{2k}\ln|g+ku^2|$

$x = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kv^2}\right)$

$= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$ ✓ when $v=0$, $x = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$

Maximum height $= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$

(iii) Downwards motion

Take x downwards as positive

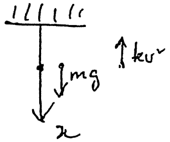
$mg > 0$, $kv^2 < 0$

$m\ddot{x} = mg - kv^2$ ✓

So $\ddot{x} = g - kv^2$

$v\frac{dv}{dx} = g - kv^2$

$\frac{dx}{dv} = \frac{v}{g-kv^2}$



$x = \int_0^v \frac{v}{g-kv^2}\,dv$

$= -\frac{1}{2k}\left[\ln|g-kv^2|\right]_0^v$

$= -\frac{1}{2k}\{\ln|g-kv^2| - \ln|g|\}$

$= \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$

$= \frac{1}{2k}\ln\frac{g}{g-kv^2}$, $g - kv^2 > 0$ ✓

When the particle returns to O

$\frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$

So $\frac{g+ku^2}{g} = \frac{g}{g-kv^2}$ ✓

$g - kv^2 = \frac{g^2}{g+ku^2}$

$kv^2 = g - \frac{g^2}{g+ku^2}$

$kv^2 = \frac{g^2 + kgu^2 - g^2}{g+ku^2}$

$v^2 = \frac{gu^2}{g+ku^2}$ ✓

So speed $|v| = \sqrt{\frac{gu^2}{g+ku^2}}$ when particle returns to O.

(iii) Now $\ddot{x} \to 0$

If $g - kv^2 \to 0$

$v^2 \to \frac{g}{k}$

So the terminal velocity is $V = \sqrt{\frac{g}{k}}$ ✓

If $u = V$ the return speed is

$|V| = \sqrt{\frac{gV^2}{g+kV^2}}$

$= \sqrt{\frac{V^2}{1+\frac{k}{g}V^2}}$

$= \sqrt{\frac{V^2}{1+1}}$

$= \frac{V}{\sqrt{2}}$. ✓

QUESTION SEVEN

(a) $P(x) = 3x^3 - 11x^2 + 24x - 12$

Given that $\frac{p}{q}$ is a zero, $q/3$ and $p/12$

So q may be 1 or 3

and p may be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

So $\frac{p}{q}$ may be $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$.

$$P\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3 - 11\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) - 12$$

$$= 0$$

So $x = \frac{2}{3}$ is the non-integer zero. ✓✓

(b) $x + \frac{2}{5}$ is a factor of $P(x)$

By long division

$5x^3 - 3x^2 + 8x + 4 = (x + \frac{2}{5})(5x^2 - 5x + 10)$ ✓

So $P(x) = (x + \frac{2}{5})(5x^2 - 5x + 10)$

$= (5x+2)(x^2 - x + 2)$

$= (5x+2)\left((x - \frac{1}{2})^2 + \frac{7}{4}\right)$ ✓

$= (5x+2)(x - \frac{1}{2} + i\frac{\sqrt{7}}{2})(x - \frac{1}{2} - i\frac{\sqrt{7}}{2})$

$= \frac{1}{4}(5x+2)(2x - 1 + i\sqrt{7})(2x - 1 - i\sqrt{7})$ ✓

(b)(i)

$\angle BAT = \angle BTP = \alpha$ (alternate segment theorem)

$\angle CDA = \angle BTP = \alpha$ (alternate segment theorem) ✓

So $\angle BAT = \angle CDA = \alpha$

So $CD \parallel AB$ (equal corresponding angles) ✓

(ii) $\angle ABT = \angle BAT = \alpha$ (base angles of isosceles ΔABT) ✓

So $\angle ABT = \angle CDA = \alpha$

So $ABCD$ is a cyclic quadrilateral (exterior angles is equal to the interior opposite angle). ✓

(iii) $\angle VWX = \angle DCX = \beta$ (angles at circumference on chord DX) ✓

$\angle DCW = \angle DBA = \beta$ (angles at circumference on chord DA) ✓

$\angle DBA = \angle AUV = \beta$ (exterior angle of cyclic quadrilateral $ABVU$)

So $\angle VWX = \angle AUV = \beta$

So $UVWX$ is a cyclic quadrilateral (exterior angle is equal to the interior opposite angle) ✓

(iv) $\angle DCB = \alpha$ (corresponding angles, $AB \parallel CD$)

Now $\angle DCT = \angle DWT = \alpha$ (angles at circumference on chord DX)

$\angle DWT = \angle TVW = \alpha$ (exterior angle of cyclic quad $ABVT$)

So $TV = TW$ (base angles of isosceles ΔTVW) ✓✓

Similarly $TU = TX$

But $TU = TV$ (given)

So $TU = TX = TV = TW$

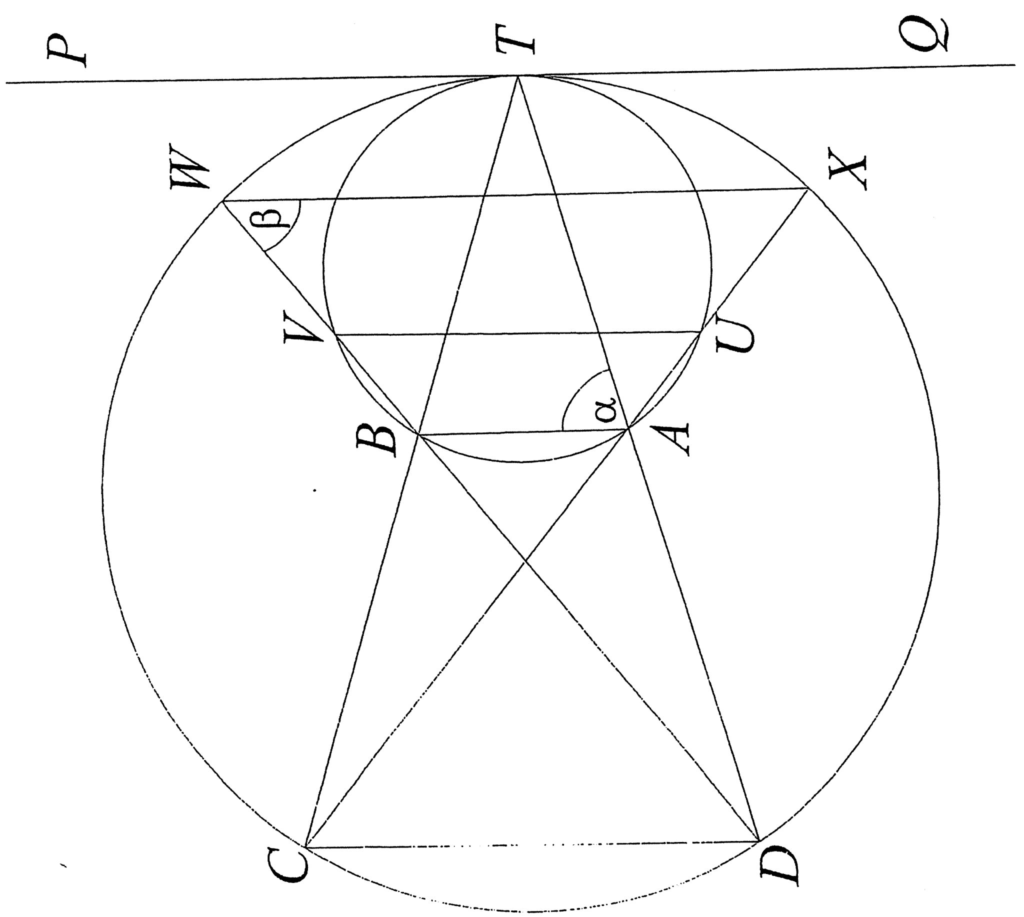
So T is the centre of a circle passing through the points U, V, W and X (equal radii) ✓

CANDIDATE NUMBER: ....

DETACH THIS SHEET AND BUNDLE IT WITH THE REST OF QUESTION SEVEN.

QUESTION SEVEN

(b)



QUESTION EIGHT

a(i)

$RHS = (\alpha+\beta)(\alpha^{k-1}+\beta^{k-1}) - \alpha\beta(\alpha^{k-2}+\beta^{k-2})$

$= \alpha^{k} + \alpha\beta^{k-1} + \beta\alpha^{k-1} + \beta^{k} - \beta\alpha^{k-1} - \alpha\beta^{k-1}$

$= \alpha^{k} + \beta^{k}$

$= LHS \quad , k \geq 2$

(ii) $\alpha = \cos\theta + i\sin\theta$

$\beta = \cos\theta - i\sin\theta$

$\alpha^k + \beta^k = \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta$

$= 2\cos k\theta$

But $\alpha^k+\beta^k = (\alpha+\beta)(\alpha^{k-1}+\beta^{k-1}) - \alpha\beta(\alpha^{k-2}+\beta^{k-2})$

$\alpha^k+\beta^k = 2\cos\theta \times 2\cos(k-1)\theta - 1 \times 2\cos(k-2)\theta$

So $\cos k\theta = 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta, k \geq 2$

(iii) $\cos k\theta = 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta$

When $k=2$,

$\cos 2\theta = 2\cos^2\theta - 1$

When $k=3$,

$\cos 3\theta = 2\cos\theta\cos 2\theta - \cos\theta$

$= \cos\theta(2\cos 2\theta - 1)$

$= \cos\theta(4\cos^2\theta - 2 - 1)$

$= 4\cos^3\theta - 3\cos\theta$

When $k=4$,

$\cos 4\theta = 2\cos\theta\cos 3\theta - \cos 2\theta$

$= 2\cos\theta(4\cos^3\theta - 3\cos\theta) - 2\cos^2\theta + 1$

$= 8\cos^4\theta - 6\cos^2\theta - 2\cos^2\theta + 1$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

(c)(i) $t_2(x) = 2x\,t_1(x) - t_0(x)$

$= 2x^2 - 1$

$t_3(x) = 2x\,t_2(x) - t_1(x)$

$= 4x^3 - 2x - x$

$= 4x^3 - 3x$

$t_4(x) = 2x\,t_3(x) - t_2(x)$

$= 2x(4x^3 - 3x) - 2x^2 + 1$

$= 8x^4 - 8x^2 + 1$

(ii) (α) $F(z) = 1 + xz + (2x^2-1)z^2 + (4x^3-3x)z^3 + (8x^4-8x^2+1)z^4 + \cdots$

$2xz\,F(z) = 2xz + 2x^2z^2 + (4x^3-2x)z^3 + (8x^4-6x^2)z^4 + (16x^5-16x^3+2x)z^5 + \cdots$

So $F(z) - 2xz\,F(z)$

$= 1 + (x-2x)z + (2x^2-1-2x^2)z^2 + (4x^3-3x-4x^3+2x)z^3 + (8x^4-8x^2+1-8x^4+6x^2)z^4 + \cdots$

$= 1 - xz + z^2(-1 - xz - (2x^2-1)z^2 - \cdots)$

$= 1 - xz - z^2(1 + xz + (2x^2-1)z^2 + \cdots)$

$= 1 - xz - z^2 F(z)$

Now $(1-2xz)F(z) = 1 - xz - z^2 F(z)$

$(1-2xz+z^2)F(z) = 1 - xz$

$F(z) = \dfrac{1-xz}{1-2xz+z^2}$

(β) $1 - 2xz + z^2 = (\alpha - z)(\beta - z)$

$= \alpha\left(1-\frac{z}{\alpha}\right)\beta\left(1-\frac{z}{\beta}\right)$

$= \alpha\beta\left(1-\frac{z}{\alpha}\right)\left(1-\frac{z}{\beta}\right)$

$= \left(1-\frac{z}{\alpha}\right)\left(1-\frac{z}{\beta}\right)$, since $\alpha\beta = 1$.

(γ) $F(z) = \dfrac{1-xz}{1-2xz+z^2}$

$= \dfrac{1-xz}{\left(1-\frac{z}{\alpha}\right)\left(1-\frac{z}{\beta}\right)}$

$= \dfrac{A}{1-\frac{z}{\alpha}} + \dfrac{B}{1-\frac{z}{\beta}}$

Now $\dfrac{1}{1-\frac{z}{\alpha}} = 1 + \frac{z}{\alpha} + \left(\frac{z}{\alpha}\right)^2 + \cdots$ for $|z|$ sufficiently small,

and $\dfrac{1}{1-\frac{z}{\beta}} = 1 + \frac{z}{\beta} + \left(\frac{z}{\beta}\right)^2 + \cdots$ for $|z|$ sufficiently small.

So $F(z) = A\left(1 + \frac{z}{\alpha} + \left(\frac{z}{\alpha}\right)^2 + \cdots\right) + B\left(1 + \frac{z}{\beta} + \left(\frac{z}{\beta}\right)^2 + \cdots\right)$

$t_k(x) = A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k$.

(δ) Now $t_0(x) = 1$

So $A + B = 1$

$t_1(x) = x$

$\frac{A}{\alpha} + \frac{B}{\beta} \equiv x$

Also the zeroes of $1 - 2xz + z^2$ are:

$z = \dfrac{2x \pm \sqrt{4x^2-4}}{2}$

$= x \pm \sqrt{x^2-1}$

So $\alpha = x + \sqrt{x^2-1}$ and $\beta = x - \sqrt{x^2-1}$

So $\dfrac{A}{x+\sqrt{x^2-1}} + \dfrac{B}{x-\sqrt{x^2-1}} \equiv x$

So $\dfrac{A(x-\sqrt{x^2-1}) + B(x+\sqrt{x^2-1})}{x^2-x^2+1} \equiv x$

So $(A+B)x - (A-B)\sqrt{x^2-1} \equiv x$

So $A + B = 1$

$A - B = 0$

So $A = B = \frac{1}{2}$

But $t_k(x) = A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k$

$= \frac{1}{2}\left(\dfrac{1}{x+\sqrt{x^2-1}}\right)^k + \frac{1}{2}\left(\dfrac{1}{x-\sqrt{x^2-1}}\right)^k$