Locus and Complex Numbers

$\omega = f(z)$, find the locus of ω or z

given some condition for ω or z

(Make the condition the subject)

ω is purely real $\Rightarrow \text{Im}(\omega) = 0, \arg\omega = 0$ or π

ω is purely imaginary $\Rightarrow \text{Re}(\omega) = 0, \arg\omega = \pm\frac{\pi}{2}$

$$\arg\left(\frac{\text{linear function}}{\text{linear function}}\right) = \theta \Rightarrow \text{locus is an arc of a circle}$$

* minor arc if $\theta > \frac{\pi}{2}$

* major arc if $\theta < \frac{\pi}{2}$

* semicircle if $\theta = \frac{\pi}{2}$

$e.g.(i)$ Find the locus of w if $w=\dfrac{z+2}{2}, |z|=4$

$$w=\frac{z+2}{z}$$

$$zw=z+2$$

$$z(w-1)=2$$

$$z=\frac{2}{(w-1)}$$

$$\therefore \left|\frac{2}{(w-1)}\right|=4$$

$$\frac{2}{|w-1|}=4$$

$$|w-1|=\frac{1}{2}$$

$\therefore$ locus is a circle, centre $(1,0)$ and radius $\dfrac{1}{2}$

i.e. $(x-1)^2+y^2=\dfrac{1}{4}$

(ii) Find the locus of z if $w=\dfrac{z+1}{z-1}$ and w is purely real

$$w=\frac{(x+1)+iy}{(x-1)+iy}\times\frac{(x-1)-iy}{(x-1)-iy}$$

$$=\frac{(x^2-1)-i(x+1)y+i(x-1)y+y^2}{(x-1)^2+y^2}$$

If w is purely real then $\text{Im}(w)=0$

$$i.e. -(x+1)y+(x-1)y=0$$

$$-xy-y+xy-y=0$$

$$-2y=0$$

$$y=0$$

$\therefore$ locus is $y=0$, excluding $(1,0)$

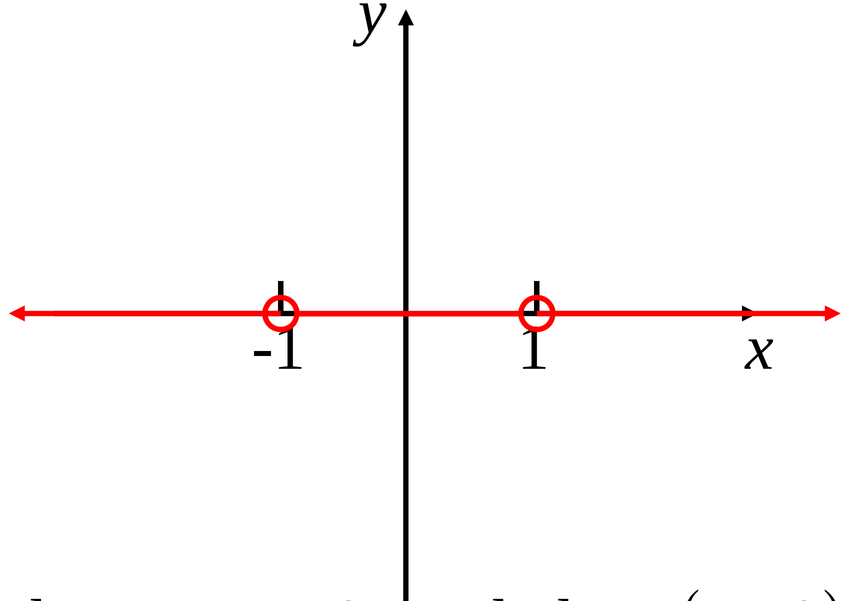
$(z-1\neq 0,$ bottom of fraction $\neq 0$ $)$

OR

If w is purely real then $\arg w = 0$ or π

$$i.e. \arg\left(\frac{z+1}{z-1}\right)=0 \text{ or } \pi$$

$$\arg(z+1)-\arg(z-1)=0 \text{ or } \pi$$



locus is $y=0$, excluding $(\pm 1,0)$

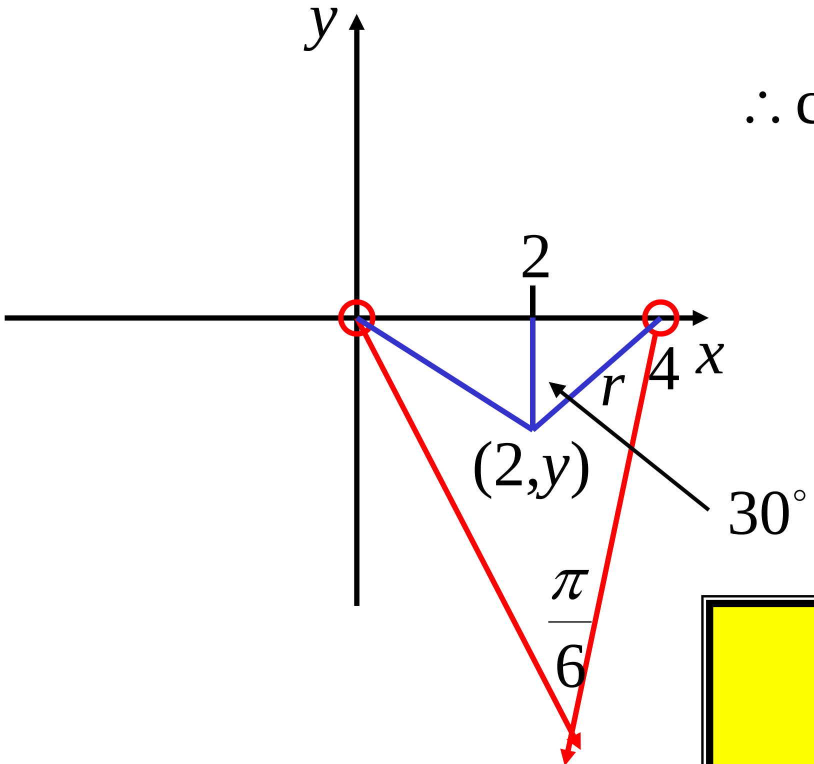
Note: locus is $y=0$, excluding $(1,0)$ only

i.e. answer the original question

(iii) Find the locus of z if $\arg\left(\dfrac{z}{z-4}\right)=\dfrac{\pi}{6}$

$$\arg\left(\frac{z}{z-4}\right)=\frac{\pi}{6}$$

$$\arg z-\arg(z-4)=\frac{\pi}{6}$$



NOTE: $\arg z > \arg(z\text{-}4)$

$\therefore$ below axis

$$\frac{y}{2}=\tan 60$$

$$y=2\tan 60$$

$$=2\sqrt{3}$$

$$r^2=2^2+\left(2\sqrt{3}\right)^2$$

$$r^2=16$$

$$r=4$$

$\therefore$ centre is $\left(2,-2\sqrt{3}\right)$

$\therefore$ locus is the major arc of the circle $(x-2)^2+\left(y+2\sqrt{3}\right)^2=16$ formed by the chord joining $(0,0)$ and $(4,0)$ but not including these points.

Exercise 4N; 5, 6

Exercise 4O; 3 to 10, 12, 13a, 14, 17, 20b, 21a, 22, 25, 26

HSC Geometrical Complex Numbers Questions