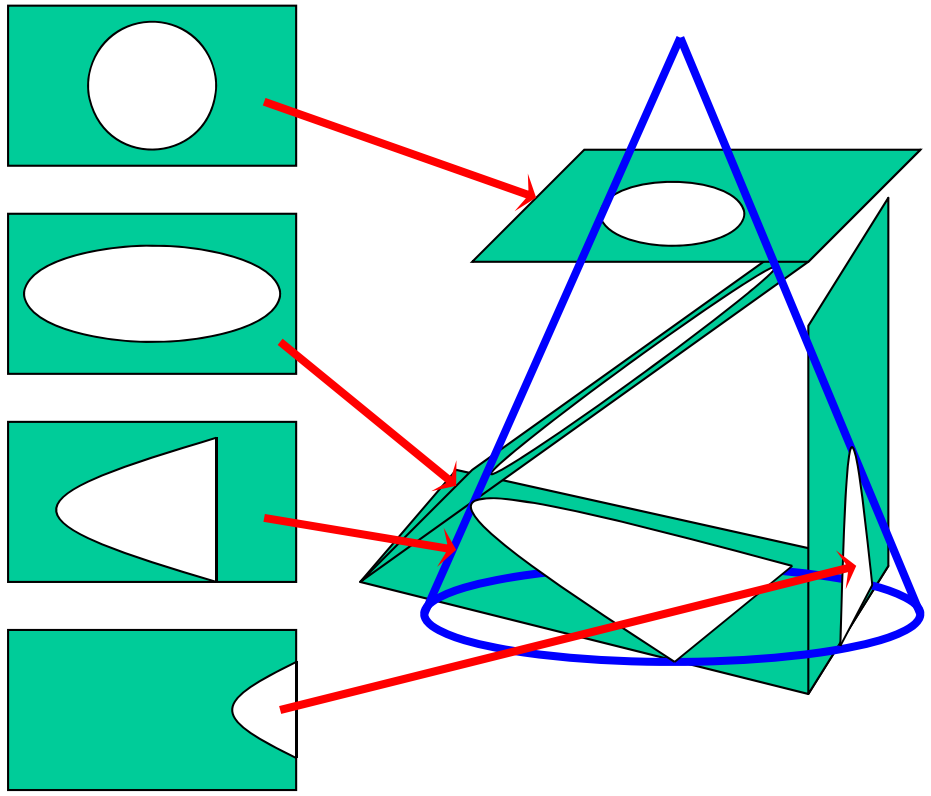
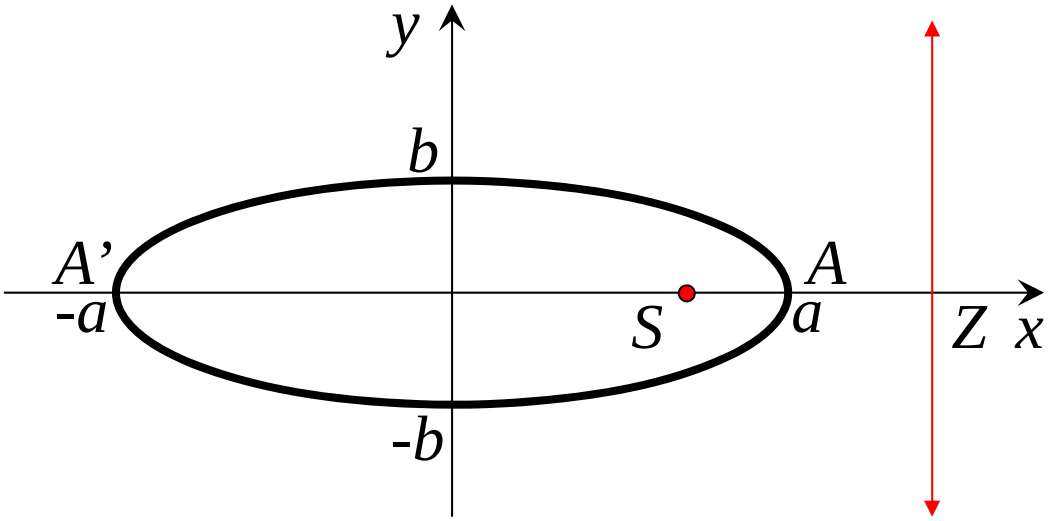
Conics

The locus of points whose distance from a fixed point (**focus**) is a multiple, *e*, (**eccentricity**) of its distance from a fixed line (**directrix**)

$e = 0$	circle
$e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola



Ellipse (e < 1)



$SA = eAZ$ and $SA' = eA'Z$

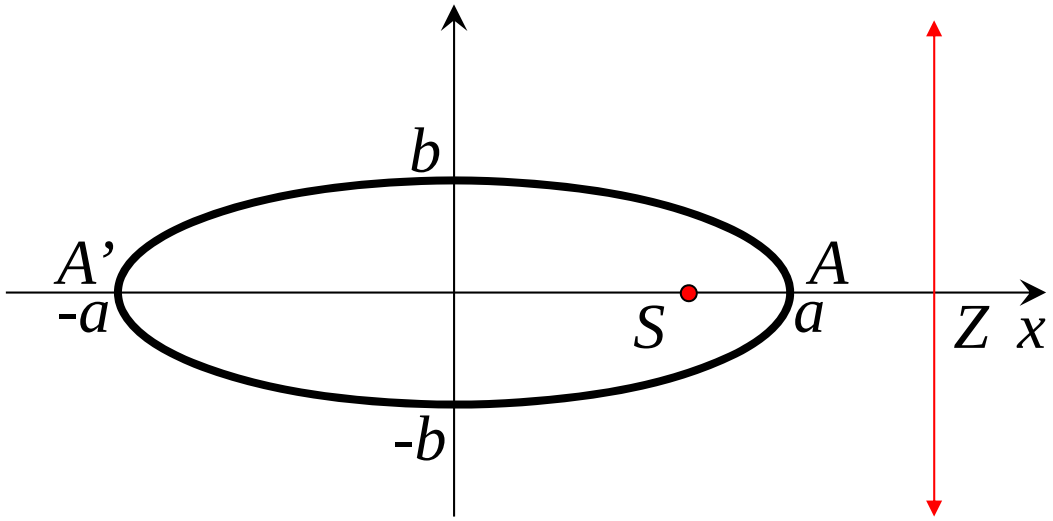
(1) $SA' + SA = 2a$

(2) $SA' - SA = e(A'Z - AZ)$

$= e(AA')$

$= e(2a)$

$= 2ae$



(1) + (2); $2SA' = 2a(1 + e)$ $SA' = a(1 + e)$

(1) - (2); $2SA = 2a(1 - e)$ $SA = a(1 - e)$

Focus

$OS = OA - SA$

$= a - a(1 - e)$

$= ae$

$\therefore S(\pm ae, 0)$

Directrix

$OZ = OA + AZ$

$= OA + \frac{SA}{e}$ $\left(\because SA = eAZ\right)$

$= \frac{ae}{e} + \frac{a(1-e)}{e}$

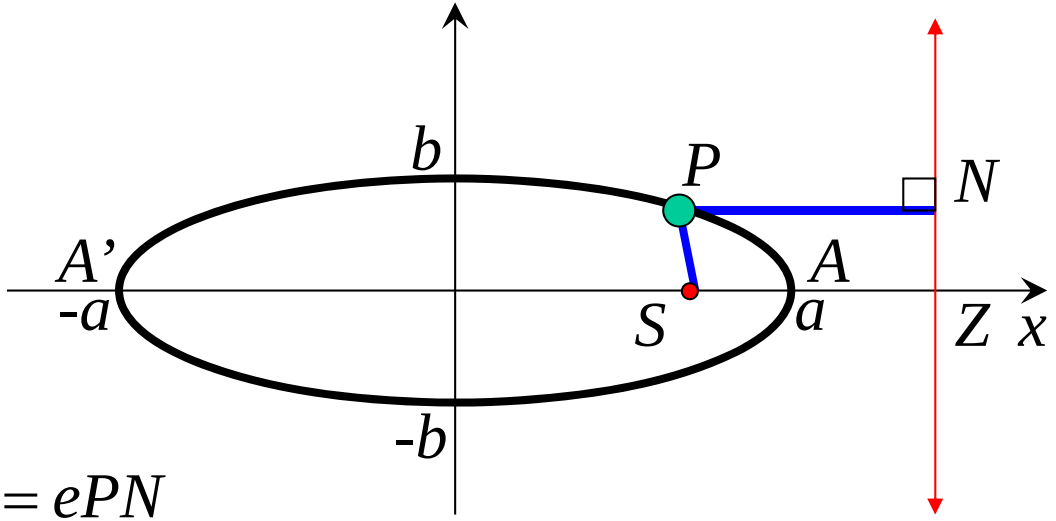
$= \frac{a}{e}$

$\therefore$ directrices $x = \pm \frac{a}{e}$

$S(ae,0)$

$P(x,y)$

$N\left(\frac{a}{e},y\right)$



$$SP = ePN$$

$$\sqrt{(x-ae)^2+(y-0)^2} = e\sqrt{\left(x-\frac{a}{e}\right)^2+(y-y)^2}$$

$$(x-ae)^2+y^2 = e^2\left(x-\frac{a}{e}\right)^2$$

$$x^2-2aex+a^2e^2+y^2 = e^2x^2-2aex+a^2$$

$$x^2(1-e^2)+y^2 = a^2(1-e^2)$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1$$

when $x = 0, y = \pm b$ i.e. $\dfrac{b^2}{a^2\left(1-e^2\right)} = 1$

$$b^2 = a^2\left(1-e^2\right)$$

Ellipse: ($a > b$)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where; $b^2 = a^2\left(1-e^2\right)$

$\text{focus} : (\pm ae, 0)$

$\text{directrices} : x = \pm\dfrac{a}{e}$

e is the eccentricity

note: $e^2 = \dfrac{a^2 - b^2}{a^2}$

major semi-axis = a units

minor semi-axis = b units

Note: If $b > a$

foci on the y axis

$a^2 = b^2\left(1-e^2\right)$

$\text{focus} : (0, \pm be)$

$\text{directrices} : y = \pm\dfrac{b}{e}$

$Area = \pi ab$

e.g. Find the eccentricity, foci and directrices of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ and sketch the ellipse showing all of the important features.

$\frac{x^2}{9}+\frac{y^2}{5}=1$

$a^2=9$

$a=3$

$e^2=\frac{a^2-b^2}{a^2}$

$e^2=\frac{9-5}{9}$

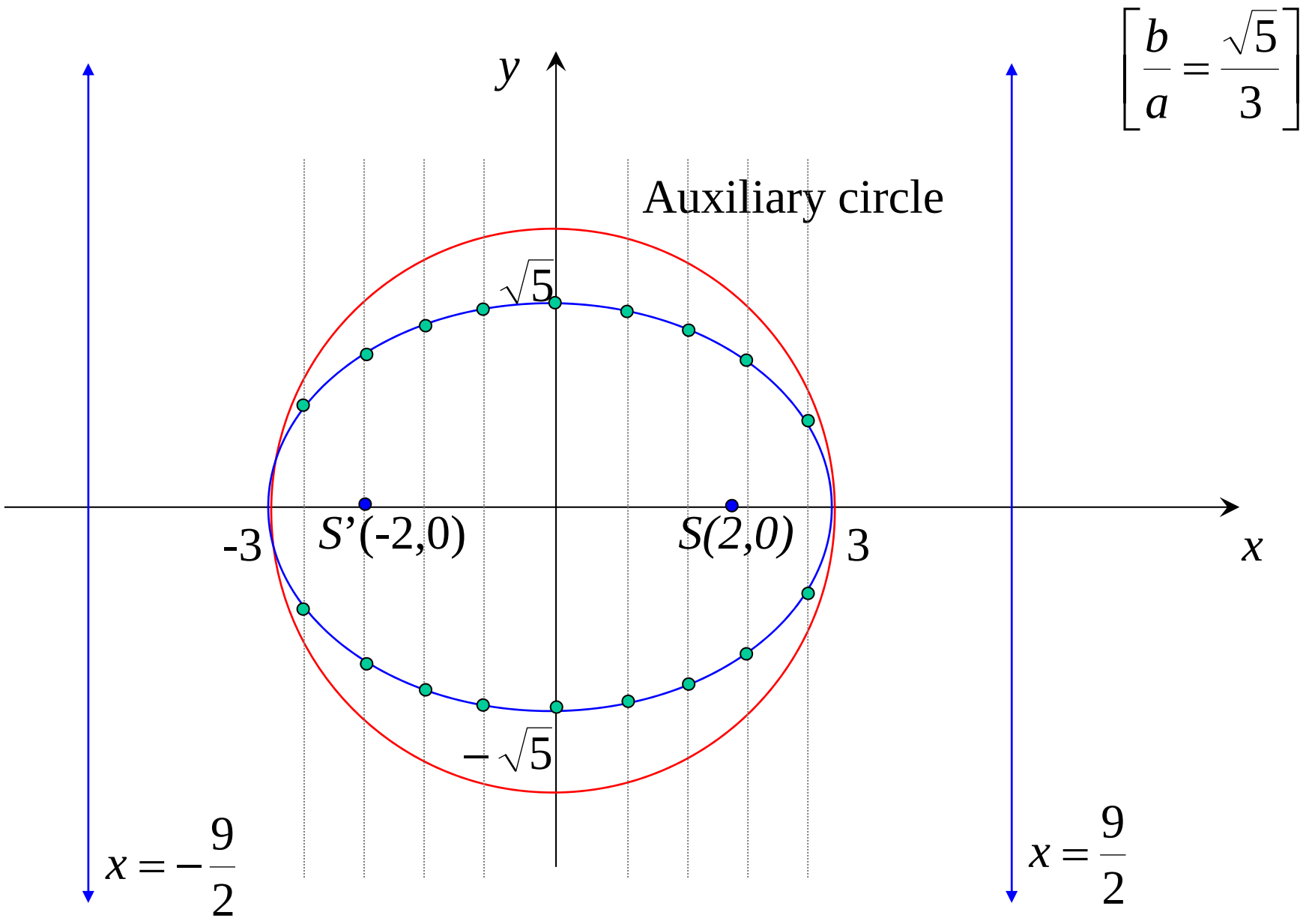
$=\frac{4}{9}$

$\therefore \text{eccentricity}=\frac{2}{3}$

$\text{foci}:(\pm 2,0)$

$\text{directrices}: x=\pm 3\times\frac{3}{2}$

$x=\pm\frac{9}{2}$


$\left[\frac{b}{a}=\frac{\sqrt{5}}{3}\right]$
y
Auxiliary circle
$\sqrt{5}$
-3
S'(-2,0)
S(2,0)
3
x
$-\sqrt{5}$
$x=-\frac{9}{2}$
$x=\frac{9}{2}$
Major axis = 6 units
Minor axis = $2\sqrt{5}$ units

(*ii*) $9x^2+4y^2+18x-16y-11=0$

$$\frac{x^2+2x}{4}+\frac{y^2-4y}{9}=\frac{11}{36}$$

$$\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}=\frac{11}{36}+\frac{1}{4}+\frac{4}{9}$$

$$\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}=1$$

centre : $(-1,2)$

$b^2=9$

$b=3$

$e^2=\frac{b^2-a^2}{b^2}$

$e^2=\frac{9-4}{9}$

$e=\frac{\sqrt{5}}{3}$

foci : $\left(-1,2\pm\sqrt{5}\right)$

directrices : $y=2\pm\frac{9}{\sqrt{5}}$

Exercise 6A; 1, 2, 3, 5, 7, 8, 9, 11, 13, 15