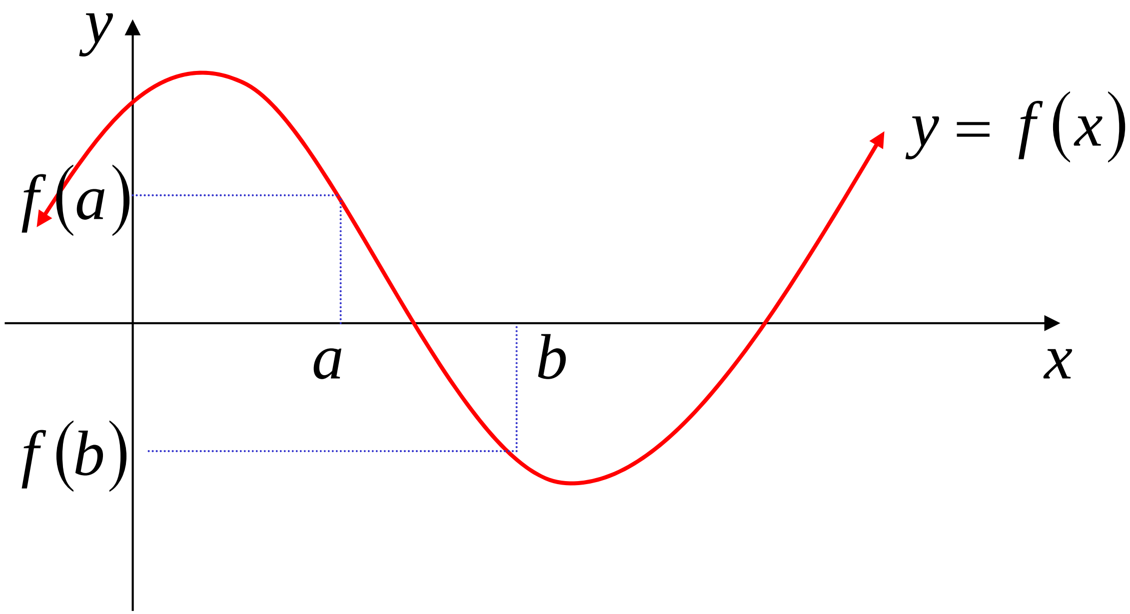
Approximations To Roots

(1) Halving The Interval



If $y = f(x)$ is a continuous function over the interval $a \le x \le b$, and $f(a)$ and $f(b)$ are opposite in sign, then at least one root of the equation $f(x) = 0$ lies in the interval $a \le x \le b$

e.g Find an approximation to two decimal places for a root of

$x^4 + 2x - 19 = 0$ in the interval $1 \le x \le 3$

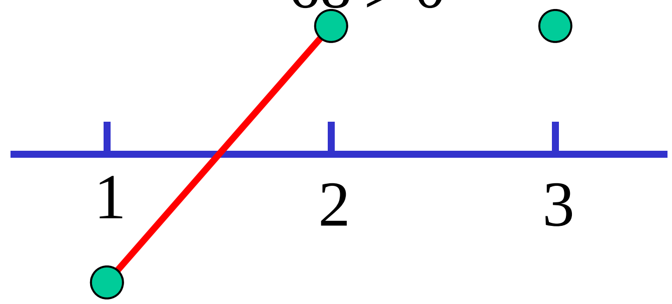
$f(x) = x^4 + 2x - 19$

$f(1) = 1^4 + 2 - 19 = -16 < 0$

$f(3) = 3^4 + 2(3) - 19 = 68 > 0$

$x_1 = \frac{1+3}{2} = 2$

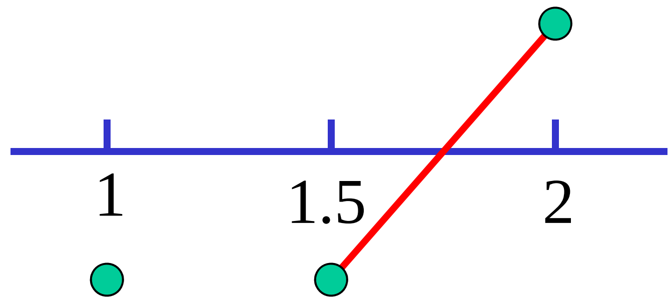
$f(2) = 2^4 + 2(2) - 19 = 1 > 0$



$\therefore$ solution lies in interval $1 \le x \le 2$

$x_2 = \frac{1+2}{2} = 1.5$

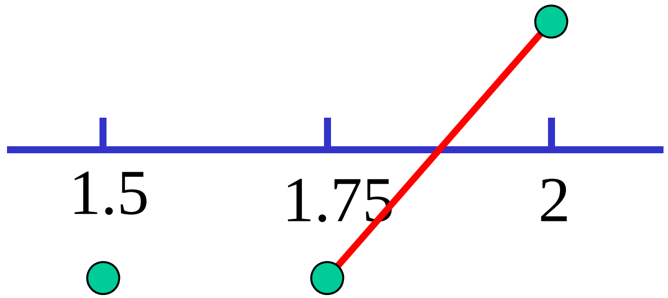
$f(1.5) = 1.5^4 + 2(1.5) - 19 = -10.9 < 0$



$\therefore$ solution lies in interval $1.5 \le x \le 2$

$$x_3 = \frac{1.5+2}{2} = 1.75$$

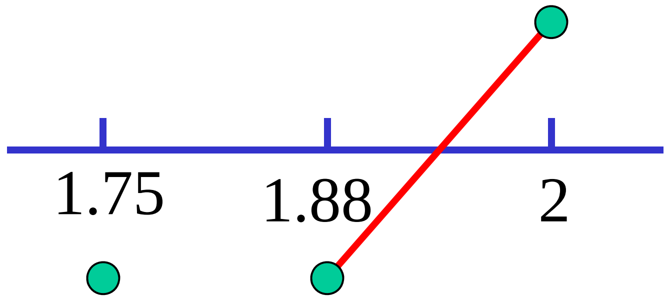
$$f(1.75) = 1.75^4 + 2(1.75) - 19 = -6.12 < 0$$



$\therefore$ solution lies in interval $1.75 \le x \le 2$

$$x_4 = \frac{1.75+2}{2} = 1.88$$

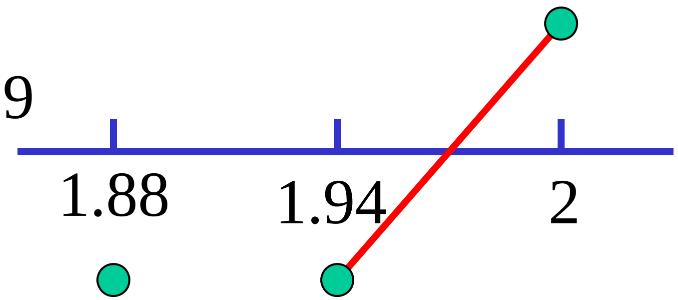
$$f(1.88) = 1.88^4 + 2(1.88) - 19 = -2.75 < 0$$



$\therefore$ solution lies in interval $1.88 \le x \le 2$

$$x_5 = \frac{1.88+2}{2} = 1.94$$

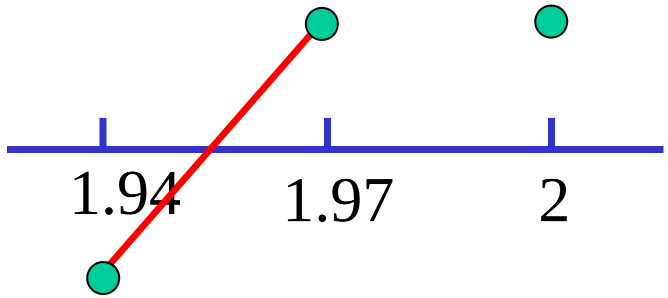
$$f(1.94) = 1.94^4 + 2(1.94) - 19 = -0.96 < 0$$



$\therefore$ solution lies in interval $1.94 \le x \le 2$

$$x_6 = \frac{1.94+2}{2}$$
$$= 1.97$$

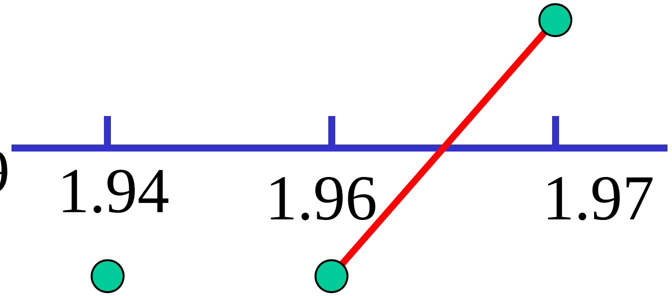
$$f(1.97) = 1.97^4 + 2(1.97) - 19$$
$$= 0.001 > 0$$



$\therefore$ solution lies in interval $1.94 \le x \le 1.97$

$$x_7 = \frac{1.94+1.97}{2}$$
$$= 1.96$$

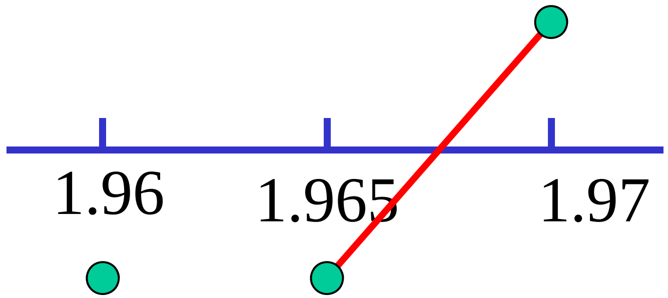
$$f(1.96) = 1.96^4 + 2(1.96) - 19$$
$$= -0.32 < 0$$



$\therefore$ solution lies in interval $1.96 \le x \le 1.97$

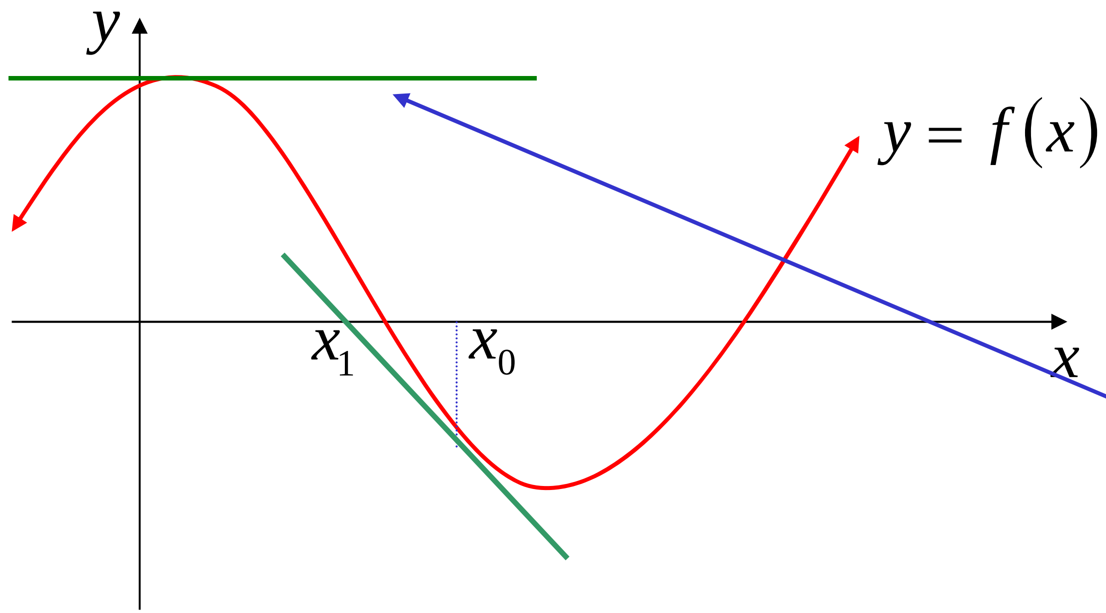
so is the solution closer to 1.96 or 1.97?

$$f(1.965) = 1.965^4 + 2(1.965) - 19$$
$$= -0.16 < 0$$



$\therefore$ an approximation for the root is $x = 1.97$

(2) Newton's Method of Approximation



NOTE:

x_0 must be a good first approximation

Newton's method finds where the tangent at x_0 cuts the x axis

If $f'(x_0) = 0$

i.e. tangent $\parallel$ x axis

the method will fail

Using the tangent at x_0 to find x_1

$$\text{slope of tangent} = \frac{f(x_0) - 0}{x_0 - x_1}$$

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

$$\left(x_0 - x_1\right) f'\left(x_0\right) = f\left(x_0\right)$$

$$x_0 - x_1 = \frac{f\left(x_0\right)}{f'\left(x_0\right)}$$

If x_0 is a good first approximation to a root of the equation $f(x) = 0$, then a closer approximation is given by;

$$x_1 = x_0 - \frac{f\left(x_0\right)}{f'\left(x_0\right)}$$

Successive approximations $x_2, x_3, \ldots, x_n, x_{n+1}$ are given by;

$$\boxed{x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}}$$

e.g Find an approximation to two decimal places for a root of

$$x^4 + 2x - 19 = 0$$

$$f(x) = x^4 + 2x - 19$$

$$f'(x) = 4x^3 + 2$$

$$x_0 = 1.5$$

$$f(1.5) = 1.5^4 + 2(1.5) - 19 = -10.9375$$

$$f'(1.5) = 4(1.5)^3 + 2 = 15.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{-10.9375}{15.5} = 2.21$$

$$f(2.21) = 2.21^4 + 2(2.21) - 19 = 9.2744$$

$$f'(2.21) = 4(2.21)^3 + 2 = 45.1754$$

$$x_2 = 2.21 - \frac{9.2744}{45.1754}$$
$$= 2.00$$

$$f(2) = 2^4 + 2(2) - 19$$
$$= 1$$
$$f'(2) = 4(2)^3 + 2$$
$$= 35$$

$$x_3 = 2 - \frac{1}{35}$$
$$= 1.97$$

$$f(1.97) = 1.97^4 + 2(1.97) - 19$$
$$= 0.001$$
$$f'(1.97) = 4(1.97)^3 + 2$$
$$= 32.58$$

$$x_4 = 1.97 - \frac{0.001}{32.58}$$
$$= 1.97$$

$\therefore x = 1.97$ is a better approximation for the root

(ii) Use Newton's Method to obtain an approximation to $\sqrt{23}$ correct to two decimal places

$$f(x) = x^2 - 23$$

$$f'(x) = 2x$$

$$x_n = x_{n-1} - \frac{{x_{n-1}}^2 - 23}{2x_{n-1}}$$

$$= \frac{{x_{n-1}}^2 + 23}{2x_{n-1}}$$

$$x_0 = 5$$

$$x_1 = \frac{5^2 + 23}{2(5)} \qquad x_2 = \frac{4.8^2 + 23}{2(4.8)}$$

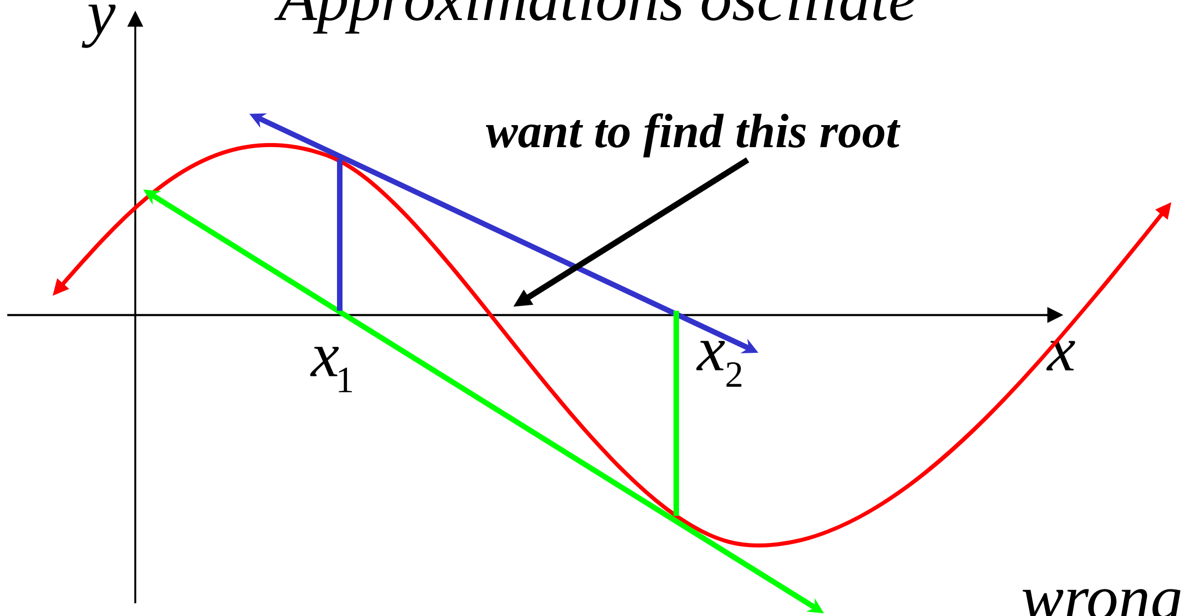
$$x_1 = 4.8 \qquad x_2 = 4.795833333$$

$$x_2 = 4.80 \text{ (to 2 dp)}$$

$$\therefore \sqrt{23} = 4.80 \text{ (to 2 dp)}$$

Other Possible Problems with Newton's Method

Approximations oscillate



Exercise 6E; 1, 3ac, 6adf, 8a, 10, 12

wrong side of stationary point converges to wrong root