NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 2)

2012 HSC Course Assessment Task 3 (Trial Examination)
June 21, 2012

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 10)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Mr Weiss ◯ 12M4B – Mr Ireland ◯ 12M4C – Mr Fletcher

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. The region in the first quadrant between the x axis and $y = 6x - x^2$ is rotated about the y axis. The volume of this solid of revolution is **1**

(A) $\displaystyle\int_0^6 \pi\left(6x - x^2\right)\,dx$

(B) $\displaystyle\int_0^6 2\pi x\left(6x - x^2\right)\,dx$

(C) $\displaystyle\int_0^6 \pi x\left(6x - x^2\right)^2\,dx$

(D) $\displaystyle\int_0^6 \pi\left(3 + \sqrt{9-y}\right)^2\,dx$

2. What are all the values of k for which the graph of $y = x^3 - 3x^2 + k$ will have three distinct x intercepts? **1**

(A) all $k > 0$

(B) all $k < 4$

(C) $k = 0, 4$

(D) $0 < k < 4$

3. Which of the following is the triple root of the equation **1**

$$8x^4 + 12x^3 - 30x^2 + 17x - 3 = 0$$

(A) $\frac{1}{2}$ (B) $-\frac{5}{4}$ (C) -3 (D) 0

4. If n is a non-negative integer, then for what values of n is $\displaystyle\int_0^1 x^n\,dx = \int_0^1 (1-x)^n\,dx$ true? **1**

(A) no solution

(B) n even, only

(C) non zero n, only

(D) all values of n

5. What are the coordinates of the foci of $xy = 18$? **1**

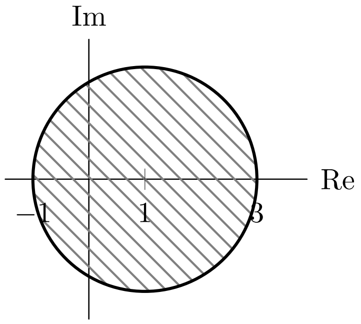
(A) $(0, 6)$, $(0, -6)$

(B) $\left(0, 3\sqrt{2}\right)$, $\left(0, -3\sqrt{2}\right)$

(C) $\left(3\sqrt{2}, 3\sqrt{2}\right)$, $\left(-3\sqrt{2}, -3\sqrt{2}\right)$

(D) $(6, 6)$, $(-6, 6)$

6. Which of the following inequalities is represented by the Argand diagram? **1**



(A) $|z-1| \leq 2$

(B) $|z-i| \leq 2$

(C) $|z+1| \leq 2$

(D) $|z+i| \leq 2$

7. What does $\displaystyle\int \frac{dx}{(x-1)(x+2)}$ evaluate to? **1**

(A) $\frac{1}{3}\log_e\left|\frac{x-1}{x+2}\right| + C$

(B) $\frac{1}{3}\log_e\left|\frac{x+2}{x-1}\right| + C$

(C) $\frac{1}{3}\log_e|(x-1)(x+2)| + C$

(D) $(\log_e|x-1|)(\log_e|x+2|)$

8. What is the value of $\displaystyle\int_0^1 xe^{-x}\,dx$? **1**

(A) $1-2e$

(B) -1

(C) $1-2e^{-1}$

(D) $2e-1$

9. What is the value of the eccentricity of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$? **1**

(A) $\dfrac{3}{\sqrt{13}}$

(B) $\dfrac{\sqrt{13}}{3}$

(C) $\dfrac{11}{\sqrt{5}}$

(D) $\sqrt{2}$

10. What is the value of $\dfrac{dy}{dx}$ at the point $(1,2)$ if $xy^2 + 2xy = 8$? **1**

(A) $-\dfrac{5}{2}$

(B) $-\dfrac{4}{3}$

(C) -1

(D) $-\dfrac{1}{2}$

End of Section I
Examination continues overleaf...

Section II: Short answer

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Evaluate:

i. $\displaystyle\int \frac{dx}{\sqrt{7-9x-x^2}}$ **2**

ii. $\displaystyle\int \frac{dx}{x\log_e x}$ **2**

(b) Evaluate $\displaystyle\int_1^2 \frac{dx}{x\,(1+x^2)}$. **4**

(c) Evaluate $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$. **3**

(d) Find $\displaystyle\int e^{-2x}\cos x\,dx$. **4**

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Show that $3i$ is a root of $P(x) = x^4 - 3x^3 + 5x^2 - 27x - 36$, and hence solve $P(x) = 0$ completely. **3**

(b) If α, β and γ are roots of $3x^3 + 4x^2 + 5x + 1 = 0$, find the value of **3**

$$\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\alpha^2\gamma^2}$$

(c) Given $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a root of multiplicity 2, solve $Q(x) = 0$ over $\mathbb{C}$. **3**

(d) The roots of the polynomial equation $x^3 - 2x^2 + 3x + 1 = 0$ are α, β and γ. **3**

Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(e) The polynomial $x^5 - ax^2 + b = 0$ has a multiple root. **3**

Show that $108a^5 = 3\,125b^3$.

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) Sketch the region in the Argand diagram which simultaneously satisfies the following inequalities: **2**

$$\begin{cases} |z - 2i| \le 2 \\ \mathrm{Im}(z) \ge 2 \end{cases}$$

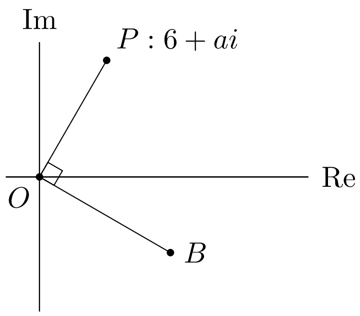
(b) What is the locus in the Argand diagram of the point z such that **3**

$$z\overline{z} - 2\left(z + \overline{z}\right) = 5$$

(c) Find the value of z^{10} in Cartesian form, given that **3**

$$z = \sqrt{2} - \sqrt{2}i$$

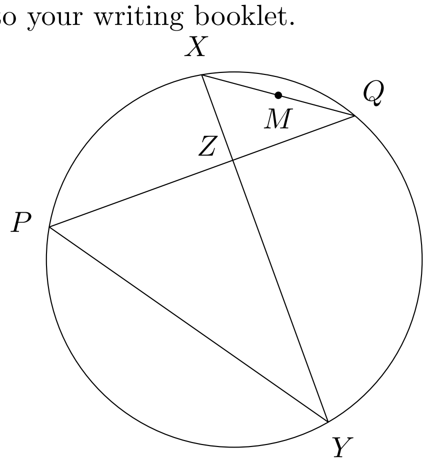
(d) In the following Argand diagram, P represents the point $6 + ai$, and O is the origin. **3**



Find the complex number represented by the point B, given $\angle POB = 90^\circ$ and

$$2\left|OB\right| = 3\left|OP\right|$$

(e) Two perpendicular chords PQ and XY of a circle intersect at Z. **4**
Copy the diagram into your writing booklet.



If M is the midpoint of the chord QX, prove that MZ produced is perpendicular to the chord PY.

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) Sketch the following graphs:

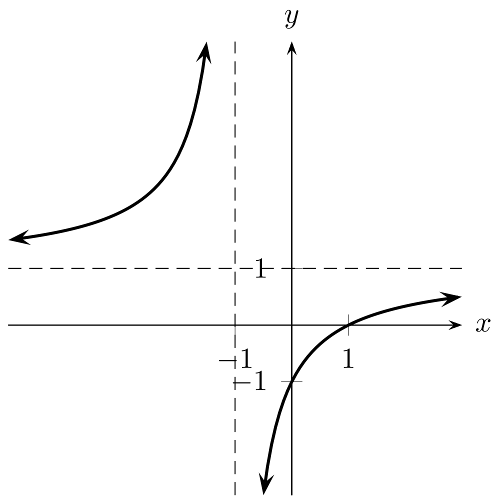
i. $y = |\sin x|$ for $-2\pi \leq x \leq 2\pi$. **1**

ii. $y = \sqrt{x^2 - 4}$ **2**

iii. $y^2 = x^2 - 9x$ **2**

(b) Sketch $y = \dfrac{1}{(x-1)^2(x+2)}$. **2**

(c) The diagram shows the graph of $f(x)$.



Sketch the following curves on separate diagrams, clearly indicating any turning points and asymptotes.

i. $y = \dfrac{1}{f(x)}$ **2**

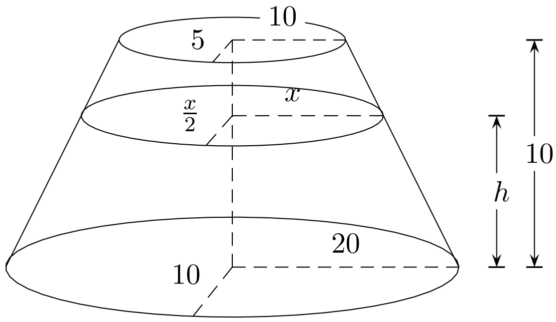
ii. $y = f(|x|)$ **2**

iii. $y = \log_e (f(x))$ **2**

iv. $y = e^{f(x)}$ **2**

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) A solid of height 10 m stands on horizontal ground. The base of the solid is an ellipse with semi-axes of 20 m and 10 m. Horizontal cross-sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres so that the centres of these elliptical cross-sections lie on a vertical straight line, and the extremitites of their semi-axes line on sloping straight lines as shown in the diagram. The top of the solid is an ellipse with semi-axes 10 m and 5 m. **5**

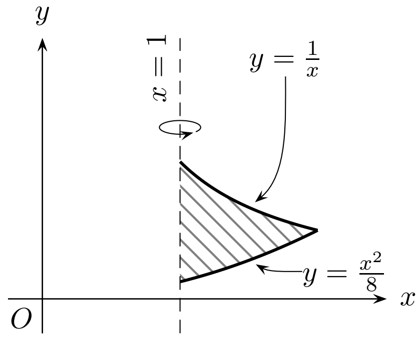


Show that the volume V m^3 of the solid is given by

$$V = \frac{\pi}{2}\int_0^{10} (20-h)^2\, dh$$

and hence find the volume correct to the nearest cubic metre.

(b) The shaded region shown in the diagram below is bounded by $y = \dfrac{1}{x}$, $y = \dfrac{x^2}{8}$ and $x = 1$. This region is rotated about the line $x = 1$.



i. Find an integral which gives the volume of the resulting solid of revolution using the method of cylindrical shells. **4**

ii. Find the volume of the solid of revolution. **2**

(c) On the number plane, shade the region **4**

$$(x-a)^2 + (y-b)^2 \leq R^2$$

where $R < b < a$.

Find the volume when this shape is rotated about the y axis using the method of slices.

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) i. Determine the real values of p for which the equation

$$\frac{x^2}{3+p} + \frac{y^2}{8+p} = 1$$

defines

(α) an ellipse **1**

(β) a hyperbola **2**

ii. For the value $p = -4$ in the above equation, find the **2**

- eccentricity
- coordinates of the foci, and
- the equations of the directrices

of the conic.

(b) P is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with centre at the origin O.

A line drawn from the origin O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

i. Derive the equation of the tangent at $P\,(a\cos\theta, b\sin\theta)$. **2**

ii. Hence or otherwise, prove that the area of $\triangle OPQ$ is independent of the position of P. **3**

(c) i. Find the equation of the normal at $P(a\sec\theta, b\tan\theta)$ to the hyperbola **2**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

ii. This normal intersects the x and y axes at Q and R respectively. $M(x, y)$ is the midpoint of QR. Find the equation of the locus of M as P varies on the hyperbola. **3**

End of paper.

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1} + C,$	$n \neq -1; \quad x \neq 0$ if $n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x + C,$	$x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax} + C,$	$a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax + C,$	$a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax + C,$	$a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax + C,$	$a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax + C,$	$a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$	$a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a} + C,$	$a > 0, -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 - a^2} \right) + C,$	$x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln \left(x + \sqrt{x^2 + a^2} \right) + C$	

NOTE: $\ln x = \log_e x, \; x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 12M4A – Mr Weiss ○ 12M4B – Mr Ireland ○ 12M4C – Mr Fletcher

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
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8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

2012 Ext. 2 – Test 3

?11) a) i/ $\int \frac{dx}{\sqrt{7-9x-x^2}}$

$= \int \frac{dx}{\sqrt{\frac{109}{4} - \left(\frac{9}{2}-x\right)^2}}$

$= \sin^{-1}\left(\frac{2x+9}{\sqrt{109}}\right) + c$

ii/ $\int \frac{\frac{1}{x}\,dx}{\ln x}$

$= \ln(\ln|x|) + c$

b) $\int_1^2 \frac{dx}{x(1+x^2)}$

Partial fractions

$\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}$

$1 = a(1+x^2) + bx^2 + c$

$\therefore a+b=0$

$c=0$

$a=1 \quad b=-1$

$\int_1^2 \frac{dx}{x(1+x^2)} = \int_1^2 \frac{1}{x} - \frac{x}{1+x^2}$

$= \left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^2$

$= \ln 2 - \ln 1 - \frac{1}{2}\ln 5 + \frac{1}{2}\ln 2$

$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$

$= \ln 2\sqrt{\frac{2}{5}}$

$\doteqdot 0.235$ to 3D.P.

c) $\int \frac{x\,dx}{\sqrt{1-x}}$

$u^2 = 1-x$

$2u\,du = -dx$

$\therefore \int \left(\frac{1-u^2}{u}\right) \cdot -2u\,du$

$= \int -\frac{2u}{u} + \frac{2u^{3}}{u}\,du$

$= \frac{2u^3}{3} - 2u + c$

$= \frac{2}{3}(1-x)^{3/2} - 2(1-x)^{1/2} + c$

d) $\int e^{-2x}\cos x\,dx$ (u = e^{-2x}, dV = $\cos x$)

$I = e^{-2x}\sin x + 2\int \sin x\, e^{-2x}\,dx$

$= e^{-2x}\sin x - 2\cos x\, e^{-2x} - 4\int \cos x\, e^{-2x}\,dx$

$\therefore 5I = e^{-2x}\sin x - 2\cos x\, e^{-2x}$

$= \frac{1}{5}e^{-2x}(\sin x - 2\cos x) + c$

Q1 – Q10

1. B	6. A
2. D	7. A
3. A	8. C
4. D	9. B
5. D	10 B

Q12

(a) $P(x) = x^4 - 3x^3 + 5x^2 - 27x - 36$

$P(3i) = (3i)^4 - 3(3i)^3 + 5(3i)^2 - 27(3i) - 36$

$= 81 + 81i - 45 - 81i - 36$

$= 0$

$\therefore$ $3i$ is a root ✓

Real coeffs $\therefore$ $-3i$ also a root.

$P(x) = (x+3i)(x-3i)\,Q(x)$

$= (x^2+9)(x^2-3x-4)$

$= (x^2+9)(x-4)(x+1)$

$\therefore$ roots are $\pm 3i,\ 4,\ -1$ ✓✓

(ALT. use sum & product of roots).

(b) $3x^3 + 4x^2 + 5x + 1 = 0$

$$\frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\beta^2\gamma^2} = \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)}{(\alpha\beta\gamma)^2}$$ ✓

$$= \frac{\left(-\frac{4}{3}\right)^2 - 2\left(\frac{5}{3}\right)}{\left(-\frac{1}{3}\right)^2}$$ ✓

$= -14$ ✓

(ALT: Create the equation with roots equal to α^2, β^2 and γ^2, and proceed from there.)

Q12

(c) $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

$Q'(x) = 4x^3 - 15x^2 + 8x + 3$

First, find roots of $Q'(x) = 0$.

$Q'(1) = 0$ but $Q(1) \neq 0$

$Q'(3) = 0$, and $Q(3) = 0$ $\therefore$ $x = 3$ is the double root. ✓

$\therefore$ $Q(x) = (x-3)^2 \cdot S(x)$

$= (x-3)^2(x^2+x+1)$ (by inspection or otherwise) ✓

$\therefore$ $(x-3)^2 = 0$ or $x^2+x+1 = 0$

$\therefore$ $x = 3, 3, \dfrac{-1 \pm i\sqrt{3}}{2}$ ✓

(d) $P(x) = x^3 - 2x^2 + 3x + 1 = 0$

α, β, γ are roots, $\therefore$

$\alpha^3 - 2\alpha^2 + 3\alpha + 1 = 0$

$\beta^3 - 2\beta^2 + 3\beta + 1 = 0$

$\gamma^3 - 2\gamma^2 + 3\gamma + 1 = 0$

adding:

$\alpha^3+\beta^3+\gamma^3 - 2(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 3 = 0$

✓ (method).

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (2)^2 - 2(3) = -2$ ✓

$\therefore$ $\alpha^3+\beta^3+\gamma^3 - 2(-2) + 3(2) + 3 = 0$

$\therefore$ $\alpha^3+\beta^3+\gamma^3 = -13$. ✓ (final answer)

Q12 (e)

$P(x) = x^5 - ax^2 + b = 0$ has a multiple root.

call it α.

Now $P'(x) = 5x^4 - 2ax$.

We have $P(\alpha) = P'(\alpha) = 0$

$\therefore \quad \alpha^5 - a\alpha^2 + b = 0 \quad \ldots\ldots (1)$

$5\alpha^4 - 2a\alpha = 0 \quad \ldots\ldots (2)$

From (2), $\alpha(5\alpha^3 - 2a) = 0$

$\therefore \alpha = 0$ or $\alpha^3 = \frac{2a}{5}$

But $P(0) \neq 0 \quad \therefore \quad \alpha^3 = \frac{2a}{5} \quad \therefore \quad \alpha = \left(\frac{2a}{5}\right)^{\frac{1}{3}}$ ✓

Sub. into (1):

$$\left(\frac{2a}{5}\right)^{\frac{5}{3}} - a \cdot \left(\frac{2a}{5}\right)^{\frac{2}{3}} = -b$$ ✓

Factorise:–

$$a^{\frac{5}{3}} \cdot \left(\frac{2}{5}\right)^{\frac{2}{3}} \left(\frac{2}{5} - 1\right) = -b$$

$$a^{\frac{5}{3}} \cdot \left(\frac{2}{5}\right)^{\frac{2}{3}} \cdot \left(-\frac{3}{5}\right) = -b$$

Cube both sides:

$$a^5 \left(\frac{2}{5}\right)^2 \left(-\frac{3}{5}\right)^3 = -b^3$$

$$-\frac{108a^5}{3125} = -b^3$$

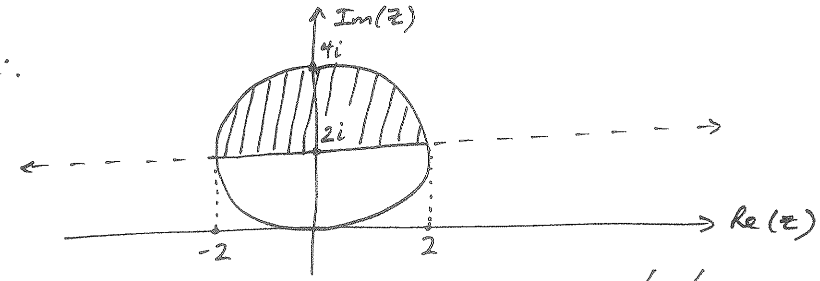
$$\therefore \quad 108a^5 = 3125b^3$$ ✓

[Note: Failure to eliminate $\alpha = 0$ as a possibility costs a mark.]

Q13

(a) $|z - 2i| \leq 2$ and $\text{Im}(z) \geq 2$

$\therefore$



✓✓

(b) $z\bar{z} - 2(z + \bar{z}) = 5$.

Let $z = x + iy \quad \therefore \quad \bar{z} = x - iy$

$\therefore \quad z\bar{z} = x^2 + y^2, \quad z + \bar{z} = 2x$ ✓

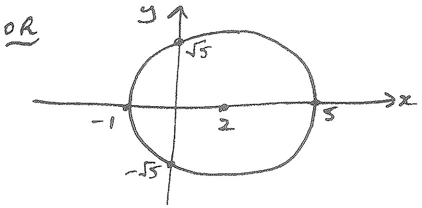
$\therefore \quad x^2 + y^2 - 4x = 5$

$x^2 - 4x + 4 + y^2 = 9$

$(x-2)^2 + y^2 = 3^2$ ✓

This is a circle, centre $(2,0)$, radius 3 ✓

OR



[NOTE: full marks requires a description of the locus or a sketch depicting it.]

Q13

(c)

$z = \sqrt{2} - \sqrt{2} i$

$= 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$= 2 \text{ cis}\left(-\frac{\pi}{4}\right)$ ✓

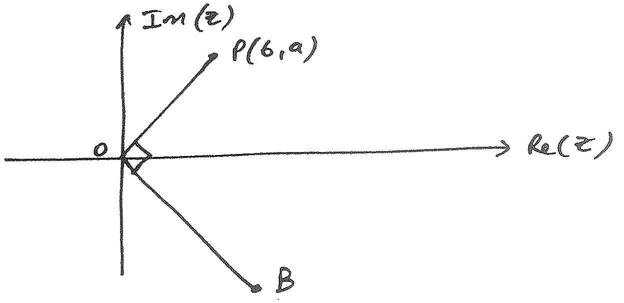
$\therefore z^{10} = 2^{10} \cdot \text{cis}\left(-\frac{5\pi}{2}\right)$ ✓

(by De Moivre)

$= 2^{10} \text{ cis}\left(-\frac{\pi}{2}\right) = 1024 \cdot -i$

$= -1024 i$ ✓

(d)



We're told $|\overrightarrow{OB}| = \frac{3}{2} \cdot |\overrightarrow{OP}|$

Thus $\overrightarrow{OB}$ is obtained from $\overrightarrow{OP}$ by a clockwise rotation of $\frac{\pi}{2}$ and a stretching by a factor of $\frac{3}{2}$.

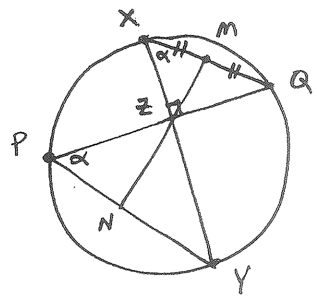
$\therefore \overrightarrow{OB}$ represents $\frac{3}{2} \cdot -i \cdot (6+ai)$ ✓✓

$= -9i + \frac{3a}{2}$

$\therefore$ B represents $\frac{3a}{2} - 9i$. ✓ (final answer)

Q13

(e)



Given: $PQ \perp XY$

$MX = MQ$

Let MZ produced meet PY at N.

To prove: $MN \perp PY$

Since $\angle XZQ = 90°$ and $MX = MQ$,

$\therefore$ XQ is the diameter of a circle, centre M, that passes through Z. [converse to angle in a semicircle is 90°] ✓

Thus $MZ = MX = MQ$

Let $\angle YPQ = \alpha$

$\therefore \angle ZXQ = \alpha$ (angles standing on same arc QY)

$\therefore \angle XQZ = 90 - \alpha$ (angle sum of $\triangle XZQ$, given $\angle XZQ = 90°$) ✓

Now $\angle MZQ = \angle MQZ$ (base angles of isosceles $\triangle MZQ$)

$= 90 - \alpha$.

Thus $\angle PZN = \angle MZQ$ (vertically opposite $\angle$'s)

$= 90 - \alpha$.

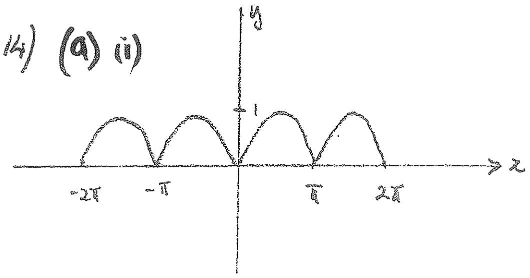
Thus $\angle PNZ = 180 - \alpha - (90 - \alpha)$ [angle sum of $\triangle PZN$]

$= 90°$ ✓✓

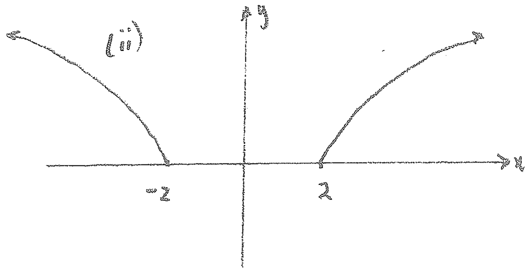
That is, $MN \perp PY$. #

[Note: no marks awarded without a viable strategy towards solution being supplied. eg. 'angles on same arc' by itself gets no marks.].

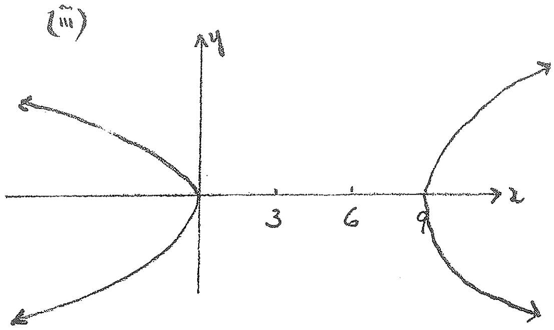
Q 14) (a) (i)



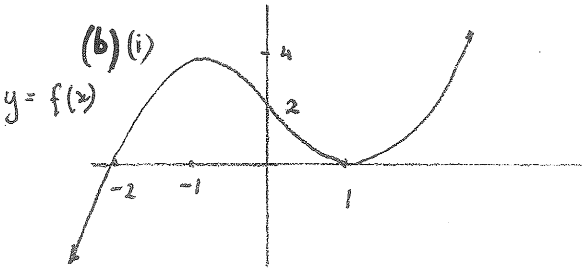
(ii)



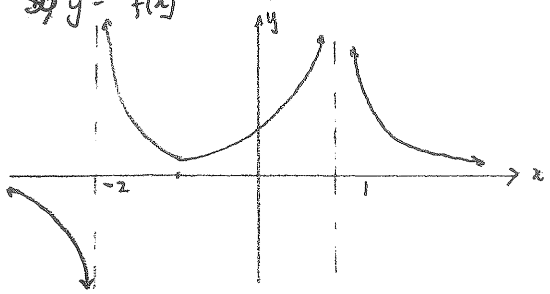
(iii)



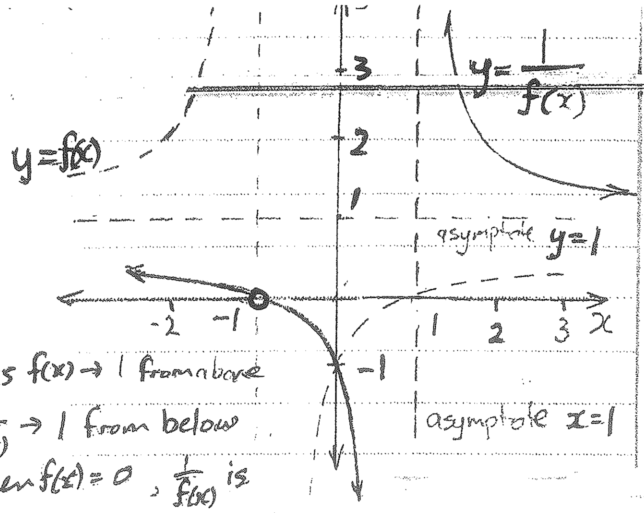
(b) (i)



(ii) $y = \frac{1}{f(x)}$



(c) (i)

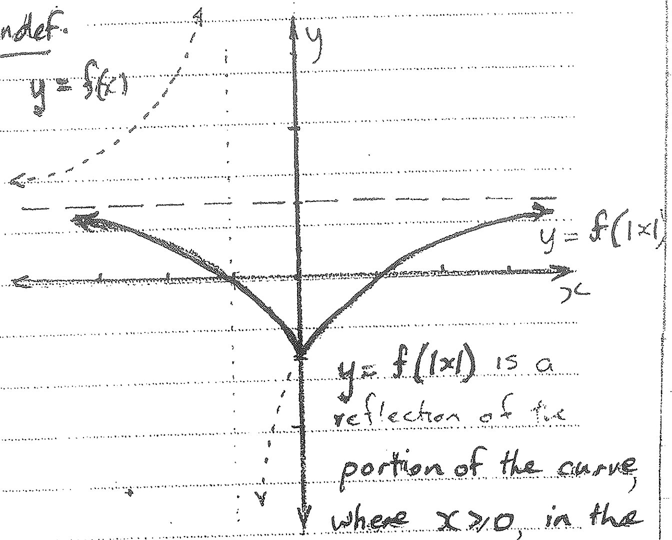


as $f(x) \to 1$ from above

$\frac{1}{f(x)} \to 1$ from below

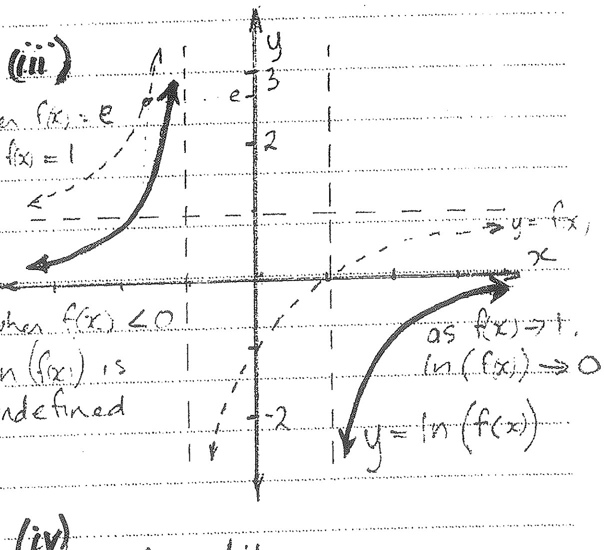
when $f(x) = 0$, $\frac{1}{f(x)}$ is undef.

(ii)



$y = f(|x|)$ is a reflection of the portion of the curve, where $x \geq 0$, in the y axis.

(iii)

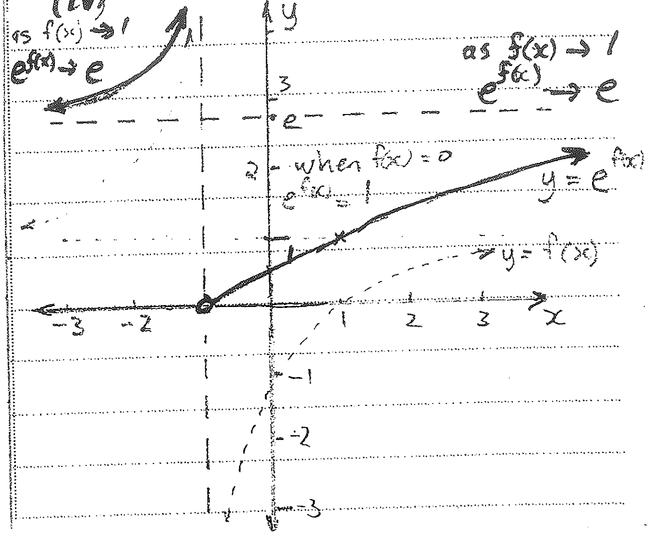


when $f(x) = e$
$\ln f(x) = 1$

when $f(x) < 0$
$\ln(f(x))$ is undefined

as $f(x) \to 1$,
$\ln(f(x)) \to 0$

(iv)

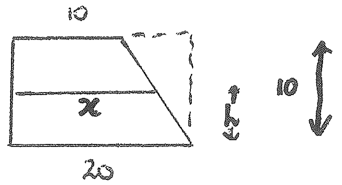


as $f(x) \to 1$
$e^{f(x)} \to e$

as $f(x) \to 1$
$e^{f(x)} \to e$

when $f(x) = 0$
$e^{f(x)} = 1$

(Q15) a)



Using similar triangles

$$\frac{20-x}{10} = \frac{h}{10}$$

$$x = 20 - h$$

Area of slice $A = \pi \times \frac{x}{2}$

$$= \frac{\pi}{2}(20-h)^2$$

Volume of solid

$$V = \lim_{\delta h \to 0} \sum_{h=0}^{10} \frac{\pi}{2}(20-h)^2 \, \delta h$$

$$V = \frac{\pi}{2}\int_0^{10} (20-h)^2 \, dh$$

$$= -\frac{\pi}{6}\Big[(20-h)^3\Big]_0^{10}$$

$$= -\frac{\pi}{6}(10^3 - 20^3)$$

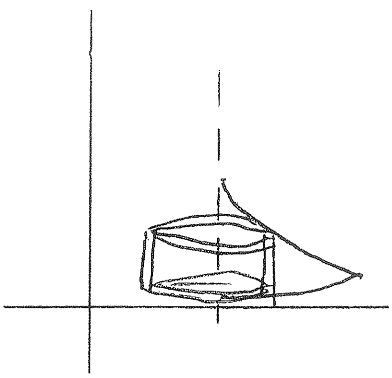
$$= \frac{7000\pi}{6} \text{ m}^3$$

$$= 3665 \text{ m}^3 \text{ (nearest m}^3\text{)}$$

b) $y = \frac{1}{x}$ and $y = \frac{x^2}{8}$

$\therefore x^3 = 8$

$x = 2$



$$V = \lim_{\delta x \to 0} \sum_{x=1}^{2} 2\pi(x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right)\delta x$$

$$= 2\pi \int_1^2 (x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right) dx$$

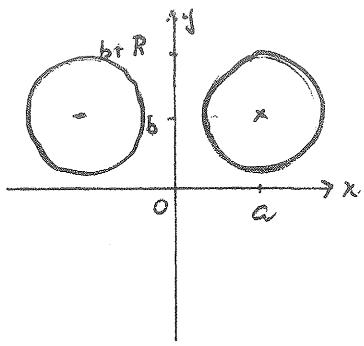
$$= 2\pi \int_1^2 \left(1 - \frac{x^3}{8} - \frac{1}{x} + \frac{x^2}{8}\right) dx$$

$$= 2\pi \left[x - \frac{x^4}{32} - \ln x + \frac{x^3}{24}\right]_1^2$$

$$= 2\pi\left(\frac{79}{96} - \ln 2\right)$$

$$= \frac{79\pi}{48} - 2\pi \ln 2$$

c)



Rotation about y-axis

Area $= \pi(R^2 - r^2)$

Now $R = a + \sqrt{R^2 - (y-b)^2}$

$r = a - \sqrt{R^2 - (y-b)^2}$

$$V_{slice} = \pi(R+r)(R-r).\,\delta y$$

$$V_{slice} = \pi(2a)\left(2\sqrt{R^2-(y-b)^2}\right)\delta y$$

$$V = 4\pi a \int_{b-R}^{b+R} \sqrt{R^2 - (y-b)^2} \, dy$$

↑ semi-circle rad R, centre b.

$$= 4\pi a \cdot \frac{1}{2}\pi R^2$$

$$= 2\pi^2 R^2 a \quad u^3$$

Q16) (a) $\frac{x^2}{3+p} + \frac{y^2}{8+p} = 1$

(i) α) ellipse if $3+p>0$
and $8+p>0$
$\therefore p>-3$

β) hyperbola if $3+p>0$
and $8+p<0$
∴ Not possible

or if $3+p<0$ and $8+p>0$

$\therefore -8<p<-3$

(ii) if $p=-4$

$\frac{x^2}{-1} + \frac{y^2}{4} = 1$

$\frac{y^2}{4} - \frac{x^2}{1} = 1$

Now $b^2 = a^2(e^2-1)$

$1 = 4(e^2-1)$

$e = \pm\frac{\sqrt5}{2}$

foci $(0, \pm\sqrt5)$

directrices $y = \pm\frac{4}{\sqrt5}$

vertices $(0, \pm 2)$

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

eqn of tangent

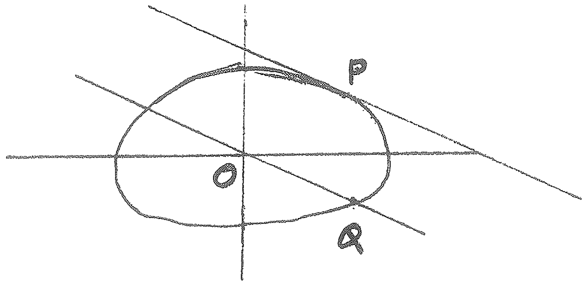
$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

(ii)



eqn of OQ $y = \frac{-b\cos\theta\, x}{a\sin\theta}$ ①

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ②

$\frac{x^2}{a^2} + \frac{1}{b^2}\frac{b^2\cos^2\theta\, x^2}{a^2\sin^2\theta} = 1$

$\sin^2\theta\, x^2 + \cos^2\theta\, x^2 = a^2\sin^2\theta$

$\therefore x^2 = a^2\sin^2\theta$

$x = \pm a\sin\theta$

from diagram $x = a\sin\theta$

$\therefore y = -b\cos\theta$

Dist from P to OQ

OQ $b\cos\theta\, x + a\sin\theta\, y = 0$

P $(a\cos\theta, b\sin\theta)$

⊥ Dist $= \frac{ab\cos^2\theta + ab\sin^2\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

dist OQ $= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

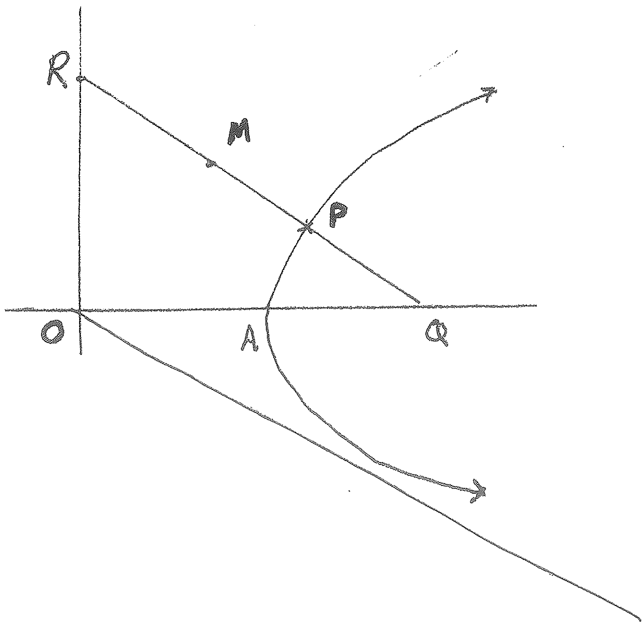
Δ Area $= \frac12 \cdot OQ \cdot$ ⊥ dist from P

$= \frac12 \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \times \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$= \frac{ab}{2}$ indep of P.

(Q16 - continued)

c) eqn. of normal at P



(i) $ax \sin\theta + by = (a^2+b^2)\tan\theta$

ii) at Q $y=0$

$\therefore x = \dfrac{(a^2+b^2)\tan\theta}{a\sin\theta}$

$= \dfrac{(a^2+b^2)}{a}\sec\theta$

at R $x=0$

$y = \dfrac{a^2+b^2}{b}\tan\theta$

at M midpoint of RQ

$X = \dfrac{1}{2a}(a^2+b^2)\sec\theta \quad Y = \dfrac{1}{2b}(a^2+b^2)\tan\theta$

locus of M

$(2aX)^2 - (2bY)^2 = (a^2+b^2)^2(\sec^2\theta - \tan^2\theta)$

$\therefore 4a^2X^2 - 4b^2Y^2 = (a^2+b^2)^2$

which is another hyperbola.

Derivation of c) i/

$\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{xb^2}{ya^2}$

$= \dfrac{b^2 x \sec\theta}{a^2 b \tan\theta}$

eqn of normal $\quad y - b\tan\theta = -\dfrac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$

$(\div\sec\theta)\quad ax\sin\theta + by = (a^2+b^2)\tan\theta$

OR $\quad \dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2+b^2$

$(\div \sec\theta\tan\theta)$