NORTH SYDNEY GIRLS HIGH SCHOOL

2012
TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11–14

Total Marks – 70

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **60 Marks**

- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section.

Student Number: ____________________________ **Teacher:** ___________________________

Student Name: ______________________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

Section I **Objective-response questions**

Total marks – 10

Attempt Questions 1 – 10

Answer each question on the multiple choice answer sheet provided.

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1 The point P divides the interval joining A (4, 1) and B (−1, 11) externally in the ratio 2 : 3. Which of these are coordinates of P?

(A) (2, 5) (B) (14, −19) (C) (1, 7) (D) (−11, 31)

2 What is the sum of the zeroes of $P(x) = 2x^4 + 8x^2 - x + 4$?

(A) 4 (B) − 4 (C) 0 (D) − 8

3 The inverse function of $g(x)$, where $g(x) = \sqrt{2x-4}$ is

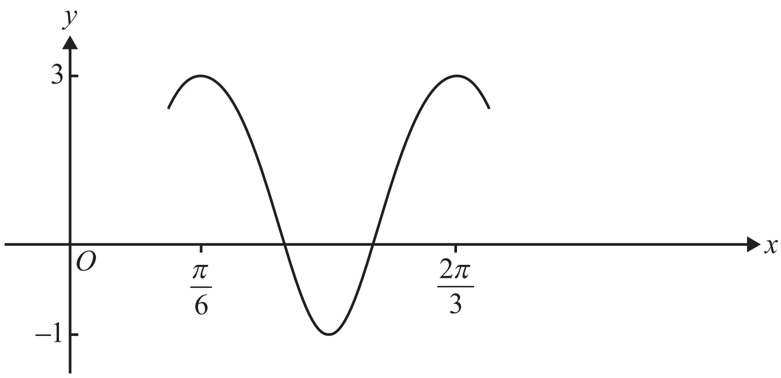
(A) $g^{-1}(x) = \dfrac{x^2+4}{2}$

(B) $g^{-1}(x) = (2x-4)^2$

(C) $g^{-1}(x) = \sqrt{\dfrac{x}{2}+4}$

(D) $g^{-1}(x) = \dfrac{x^2-4}{2}$

4 The graph below could have the equation



(A) $y = 2\cos\left(x + \dfrac{\pi}{6}\right) + 1$

(B) $y = 2\cos 2\left(x + \dfrac{\pi}{6}\right) + 1$

(C) $y = 2\cos 4\left(x - \dfrac{\pi}{6}\right) + 1$

(D) $y = 2\cos 4\left(x + \dfrac{2\pi}{3}\right) + 1$

5 The domain and range of the function $f(x)$, where $f(x)=3\sin^{-1}(4x-1)$ are respectively.

(A) $0 \le x \le \frac{1}{2}$ and $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(B) $-\frac{1}{2} \le x \le 0$ and $-\frac{\pi}{2} \le y \le \frac{5\pi}{2}$.

(C) $0 \le x \le \frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{5\pi}{2}$

(D) $-\frac{1}{2} \le x \le 0$ and $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$.

6 A particle is moving in simple harmonic motion according to the equation

$$x = 2 - 3\cos\left(2t + \frac{\pi}{3}\right)$$

In which interval does the particle oscillate?

(A) $-3 \le x \le 3$

(B) $\frac{1}{2} \le x \le 3\frac{1}{2}$

(C) $-1 \le x \le 5$

(D) $1 \le x \le 5$

7 $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = ?$

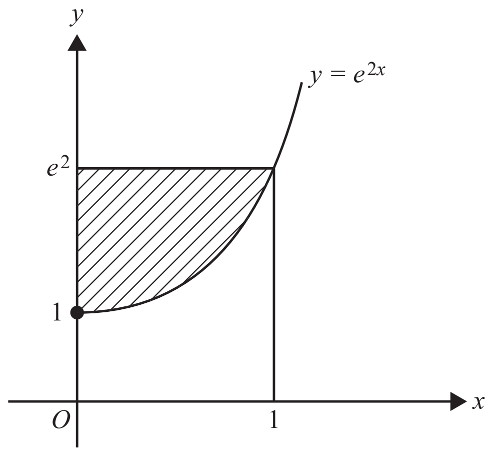
(A) $-\frac{1}{1+x^2}$

(B) $\frac{1}{1+x^2}$

(C) $-\frac{x^2}{1+x^2}$

(D) $\frac{x^2}{1+x^2}$

8 To find the area of the shaded region in the diagram below, four different students proposed the following calculations.



Student 1: $\int_0^1 e^{2x}\,dx$

Student 2: $e^2 - \int_0^1 e^{2x}\,dx$

Student 3: $\int_1^{e^2} e^{2y}\,dy$

Student 4: $\int_1^{e^2} \frac{\log_e y}{2}\,dy$

Which of the following is correct?

(A) Student 2 only.

(B) Students 2 and 3 only.

(C) Students 2 and 4 only.

(D) Students 1 and 4 only.

9 If the substitution $u = x^2 - 1$ is used then the definite integral $\int_0^2 \frac{x}{\sqrt{x^2-1}}dx$ can be simplified to

(A) $\frac{1}{2}\int_{-1}^{3} u^{-\frac{1}{2}}\,du$

(B) $2\int_{-1}^{3} u^{-\frac{1}{2}}\,du$

(C) $\frac{1}{2}\int_{0}^{2} u^{-\frac{1}{2}}\,du$

(D) $2\int_{0}^{2} u^{-\frac{1}{2}}\,du$

10. The polynomial $P(x)$ is monic and of degree 5. It has a single zero at $x=-2$ and double zero at $x=1$. Its other two zeroes are not real.
Which of the following equations best represents $P(x)$?

(A) $x^5 + bx^4 + cx^3 + dx^2 + ex + f$

(B) $(x-1)(x+2)^2(x+a)(x+b)$

(C) $(x+2)(x-1)^2(x^2+bx+c)$, where $b^2-4c \geq 0$.

(D) $(x+2)(x-1)^2(x^2+bx+c)$, where $b^2-4c < 0$.

End of Section I

Section II **Free response questions**
Total marks – 60
Attempt Questions 11 – 14

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Give that $f(x)=\sin^{-1}(2x)$, find $f''\left(\dfrac{\sqrt{2}}{4}\right)$ **3**

(b) Find $\displaystyle\int \cos^2 2x \sin 2x \, dx$ **2**

(c) Find the exact value of $\displaystyle\int_0^1 \frac{x^2+2}{x^2+1} \, dx$ **3**

(d) Evaluate $\displaystyle\lim_{x\to\frac{\pi}{3}} \frac{2\sin\left(x-\frac{\pi}{3}\right)}{x-\frac{\pi}{3}}$ **1**

(e) Evaluate $\displaystyle\int_0^1 \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}} \, dx$ using the substitution $u=\cos^{-1}x$. **3**

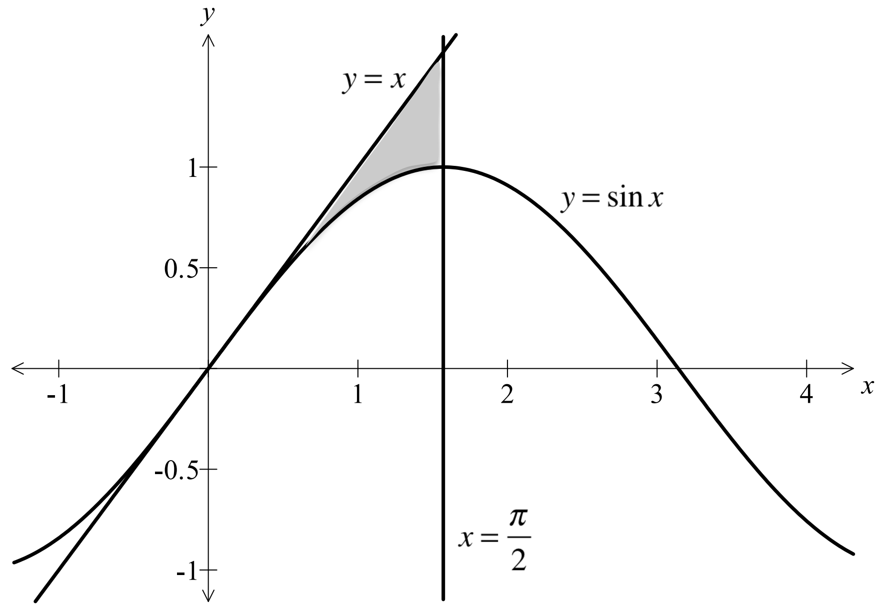
Question 11 continues on the following page

Question 11 continued

(f) In the diagram below, $y = x$ is the tangent to the curve $y = \sin x$ at $x = 0$. **3**

The shaded region is enclosed by $y = x$, $y = \sin x$ and the line $x = \frac{\pi}{2}$.

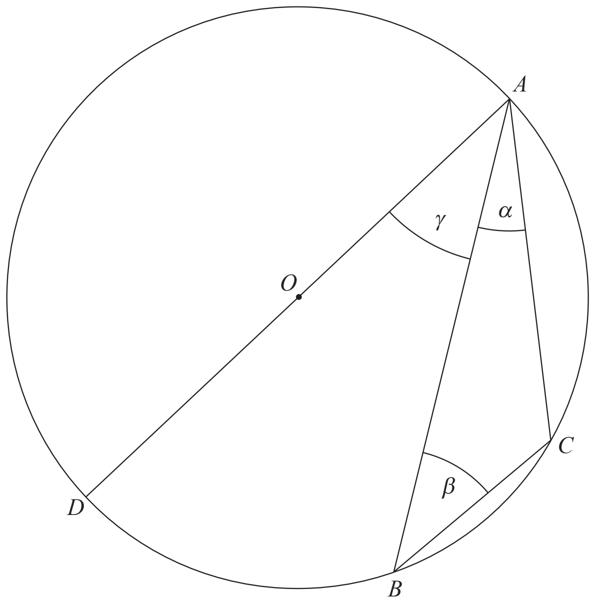
Find the exact volume when the shaded region is rotated about the x-axis.



End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows points B and C on a semi-circle with centre O and diameter AD.



Given that $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle OAB = \gamma$, find the value of $\alpha + \beta + \gamma$, giving reasons. **2**

(b) Two zeroes of the cubic $P(x) = x^3 + px^2 + qx + r$ are equal in magnitude, but opposite in sign.

(i) Show that $x = -p$ is the third zero. **1**

(ii) Show that $r = pq$ **2**

(c) The Crows Nest Twins Association has 12 members – 6 pairs of twins.

(i) In how many ways can a committee of 3 members be elected if no two members of this committee are each other's twin? **2**

(ii) If there are 4 pairs of identical twins and 2 pairs of non-identical twins, in how many ways can a committee of 4 members be elected if there is equal representation of non-identical and identical twins on the committee, but no 2 members of this committee are each other's twin? **2**

Question 12 continues on the following page

Question 12 continued

(d) Falling into cold water is particularly dangerous because the body loses body heat 25 times faster in cold water than in cold air.

The normal body temperature is $37°$ C.

Shivering begins at an approximate temperature of $36°$ C, amnesia at $34 \cdot 4°$ C and unconsciousness at $30°$ C.

The body temperature of a person, $T°$ C, who has been in the water for t minutes can be modeled by the differential equation

$$\frac{dT}{dt} = k\left(T - T_w\right),$$ where T_w is the temperature of the water and k is a constant.

(i) Show that $T = T_w + Be^{-kt}$, where B is a constant, is a solution of this differential equation. **1**

(ii) If the water temperature is $2°$ C, a person is expected to start shivering after 5 minutes. **3**
Show that $T = 2 + 35e^{-0 \cdot 0058t}$.

(iii) Find how long a person can stay in the water before coming unconsciousness. **2**
Give your answer in minutes, correct to 1 decimal place.

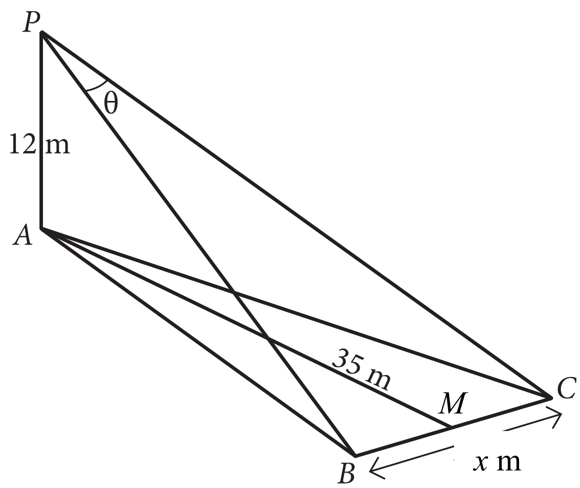
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) By sketching two appropriate graphs, or otherwise, solve $x + 4 > \dfrac{2}{x+3}$ **3**

(ii) Hence, or otherwise, deduce the values of x for which $x + 4 > \dfrac{2}{|x+3|}$ **2**

(b) A hot balloon, P, is rising vertically above A, the vertex of the isosceles triangle ABC.
M is the midpoint of BC.
$AM = 35$ m and $\angle BPC = \theta$.
At a certain time the balloon reaches a height of 12 m above A.



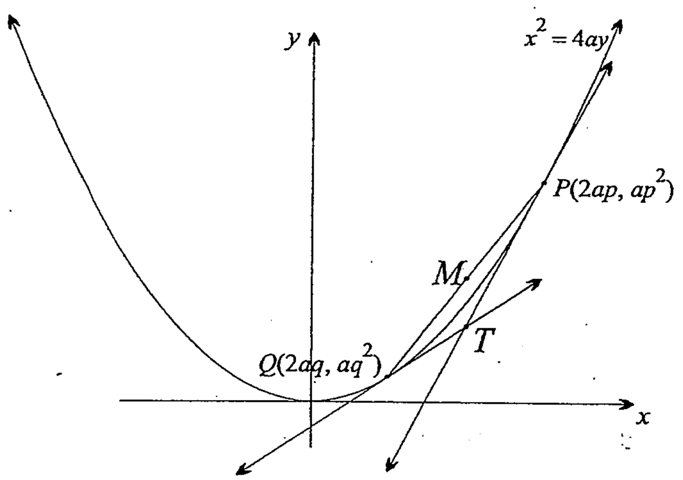
(i) Find the value of x in terms of θ. **3**

(ii) When $PA = 12$, $\dfrac{d\theta}{dt} = 0 \cdot 01$ radians/s, find $\dfrac{dx}{dt}$ when $\theta = 0 \cdot 04$ radians. **2**
Leave your answer in correct units, correct to 2 decimal places.

Question 13 continues on the following page

Question 13 continued

(c) The diagram below shows the tangents drawn at the points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ on the parabola $x^2 = 4ay$.
The tangents at P and Q intersect at T.



You may assume that the equation of the tangent at P is $y = px - ap^2$
and that point T has coordinates $T\left[a\left(p+q\right), apq\right]$.

(i) Suppose that point T lies on the line $y = a$, show that $pq = 1$. **1**

(ii) Find the Cartesian equation of the locus of the midpoint, M, of the chord PQ. **3**

(iii) State any restrictions on the x-coordinate of the locus of M. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Given that $f(k)=12^k+2\times 5^{k-1}$, **1**

show that $f(k+1)-5f(k)=a\times 12^k$, where a is an integer.

(ii) Hence, or otherwise, prove by induction that $12^n+2\times 5^{n-1}$ is divisible by 7, for all integers $n\geq 1$. **3**

(b) (i) Use the definition of the derivative i.e. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ **2**

to find $f'(0)$ where $f(x)=e^x$.

Hence find $\lim\limits_{h\to 0}\dfrac{e^h-1}{h}$.

(ii) By considering the sum of a geometric series, find $\lim\limits_{n\to\infty}\dfrac{e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n}{n}}}{n}$ **2**

(c) A particle is moving in simple harmonic motion of period T about a centre O. Its displacement at any time t is given by $x=A\sin nt$, where A is the amplitude.

(i) Draw a neat sketch of one period of this displacement-time equation, showing all intercepts. **2**

(ii) Show that $\dot{x}=\dfrac{2\pi A}{T}\cos\dfrac{2\pi t}{T}$ **1**

(iii) The point P lies D units on the positive side of O. **2**
Let V be the velocity of the particle when it first passes through P.

Show that the first time the particle is at P after passing through O is

$$\frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

(iv) Show that the time between the first two occasions when the particle passes through P is $\dfrac{T}{\pi}\tan^{-1}\dfrac{VT}{2\pi D}$. **2**

[You may assume that $\tan^{-1}x+\tan^{-1}\left(\dfrac{1}{x}\right)=\dfrac{\pi}{2}$ for $x>0$]

End of paper

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2012

TRIAL HSC EXAMINATION

Mathematics Extension 1

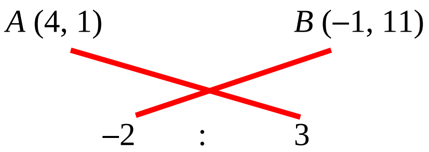
Sample Solutions

Section I **Objective-response Questions**

1	B
2	C
3	A
4	C
5	D
6	D
7	A
8	B
9	A
10	D

1 The point P divides the interval joining A (4, 1) and B (–1, 11) externally in the ratio 2 : 3. Which of these are coordinates of P?

(B) (14, –19)



$$x_P = \frac{4\times 3+(-1)\times(-2)}{3+(-2)} = 14;\quad y_P = \frac{1\times 3+11\times(-2)}{3+(-2)} = -19$$

$P(14, -19)$

2 What is the sum of the zeroes of $P(x) = 2x^4 + 8x^2 - x + 4$?

(C) 0

The coefficient of x^3 is zero, so $\sum \alpha = 0$

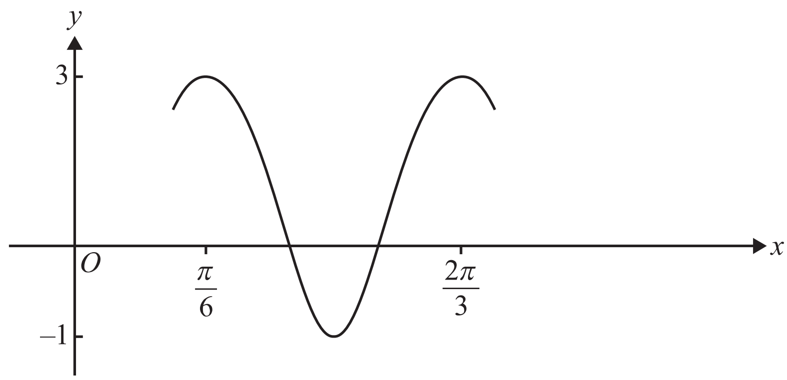
3 The inverse function of $g(x)$, where $g(x) = \sqrt{2x-4}$ is

(A) $g^{-1}(x) = \dfrac{x^2+4}{2}$

$y = \sqrt{2x-4} \Rightarrow x = \sqrt{2y-4}$

$\therefore 2y - 4 = x^2 \Rightarrow y = \frac{1}{2}\left(x^2+4\right)$

4 The graph below could have the equation



(C) $y = 1 + 2\cos 4\left(x - \dfrac{\pi}{6}\right)$

Let the period be T

$T = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$

$T = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{T} = 4$

Also, if the right shift is not obvious then when $x = \frac{\pi}{6}$, $y = 3$.

5 The domain and range of the function $f(x)$, where $f(x) = 3\sin^{-1}(4x-1)$ are respectively

(D) $0 \le x \le \frac{1}{2}$ and $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

D: $-1 \le 4x-1 \le 1$ R: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

$\therefore 0 \le 4x \le 2$ $\therefore -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

$\therefore 0 \le x \le \frac{1}{2}$

6 A particle is moving in simple harmonic motion according to the equation

$$x = 2 - 3\cos\left(2t + \frac{\pi}{3}\right)$$

What is the initial velocity?

(D) $3\sqrt{3}$

$\dot{x} = 6\sin\left(2t + \frac{\pi}{3}\right)$

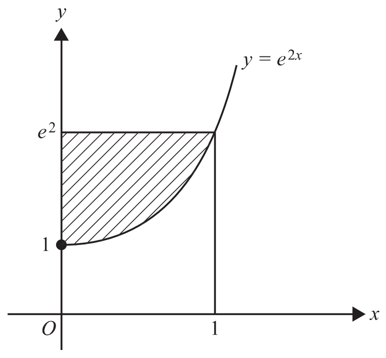
$t = 0 \Rightarrow \dot{x} = 6\sin\left(\frac{\pi}{3}\right) = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

7 $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = ?$

(A) $-\frac{1}{1+x^2}$

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x^2}} \times \left(-\frac{1}{x^2}\right) = \frac{x^2}{x^2+1} \times \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

8 To find the area of the shaded region in the diagram below, four different students proposed the following calculations.



Student 1: $\int_0^1 e^{2x}\,dx$

Student 2: $e^2 - \int_0^1 e^{2x}\,dx$

Student 3: $\int_1^{e^2} e^{2y}\,dy$

Student 4: $\int_1^{e^2} \frac{\log_e y}{2}\,dy$

Which of the following is correct?

(B) Students 2 and 4 only.

Student 2 is finding the area next to the *x*-axis first and then subtracting this from the appropriate rectangle.

Student 4 is finding the area next to the *y*-axis.

9 If the substitution $u = x^2 - 1$ is used then the definite integral $\int_0^2 \frac{x}{\sqrt{x^2-1}}dx$ can be simplified to

(A) $\frac{1}{2}\int_{-1}^{3} u^{-\frac{1}{2}}\,du$

$u = x^2 - 1 \Rightarrow du = 2x$

$x = 0 \Rightarrow u = 0 - 1 = -1$

$x = 2 \Rightarrow u = 4 - 1 = 3$

$$\int_0^2 \frac{xdx}{\sqrt{x^2-1}} = \frac{1}{2}\int_0^2 \frac{2xdx}{\sqrt{x^2-1}} = \frac{1}{2}\int_{-1}^{3} u^{-\frac{1}{2}}\,du$$

10 The polynomial $P(x)$ is monic and of degree 5. It has a single zero at $x = -2$ and double zero at $x = 1$. Its other two zeroes are not real.
Which of the following expressions best represents $P(x)$?

(D) $(x+2)(x-1)^2(x^2+bx+c)$, where $b^2 - 4c < 0$.

It has a single zero at $x = -2$ $\Rightarrow (x+2) \mid P(x)$

Double zero at $x = 1$ $\Rightarrow (x-1)^2 \mid P(x)$

Its other two zeroes are not real $\Rightarrow \Delta < 0$

Question 11

(a) Differentiate $\sin^{-1} 2x$ **1**

$$\frac{d}{dx}\left(\sin^{-1} 2x\right) = \frac{1}{\sqrt{1-(2x)^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

(b) Find the acute angle between the lines $y = 3x + 5$ and $2x - 4y + 7 = 0$. **2**

Leave your answer correct to the nearest minute.

Let the angle be θ

$m_1 = 3, m_2 = \frac{1}{2}$

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}} \right|$$

$$= 1$$

$\therefore \theta = 45°$

(c) Find $\int \cos^2 2x \sin 2x \, dx$ **2**

$$\int \cos^2 2x \sin 2x \, dx = -\frac{1}{2}\int \cos^2 2x \left(-2\sin 2x\right) dx$$

$$= -\frac{1}{2}\int \cos^2 2x \, d\cos 2x$$

$$= -\frac{1}{2} \times \frac{1}{3}\cos^3 2x + C$$

$$= -\frac{1}{6}\cos^3 2x + C$$

(d) Evaluate $\lim_{x \to 0} \frac{2\sin x}{3x}$ **1**

$$\lim_{x \to 0} \frac{2\sin x}{3x} = \frac{2}{3} \times \lim_{x \to 0} \frac{\sin x}{x} = \frac{2}{3} \times 1 = \frac{2}{3}$$

(e) By splitting the numerator, find the exact value of $\int_0^1 \frac{x^2+2}{x^2+1} dx$ **2**

$$\int_0^1 \frac{x^2+2}{x^2+1} dx = \int_0^1 \frac{\left(x^2+1\right)+1}{x^2+1} dx$$

$$= \int_0^1 \left(1 + \frac{1}{x^2+1}\right) dx$$

$$= \left[x + \tan^{-1} x\right]_0^1$$

$$= 1 + \frac{\pi}{4}$$

Question 11 continued

(f) Find $\int_0^1 \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$ using the substitution $u=\cos^{-1}x$. **3**

$$u=\cos^{-1}x \Rightarrow du=-\frac{1}{\sqrt{1-x^2}}dx$$

$$x=0\Rightarrow u=\cos^{-1}0=\frac{\pi}{2}$$

$$x=1\Rightarrow u=\cos^{-1}1=0$$

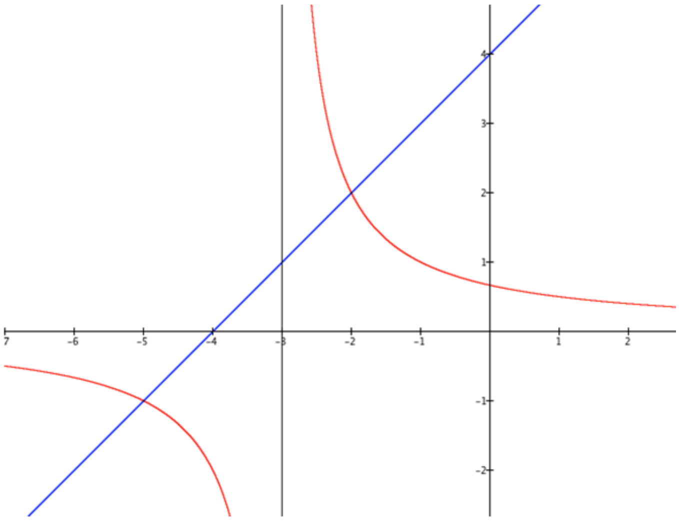
$$\int_0^1 \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx=-\int_0^1 e^{\cos^{-1}x}\times\frac{-1}{\sqrt{1-x^2}}\,dx$$

$$=-\int_{\frac{\pi}{2}}^{0} e^u\,du=\int_0^{\frac{\pi}{2}} e^u\,du$$

$$=\left[e^u\right]_0^{\frac{\pi}{2}}$$

$$=e^{\frac{\pi}{2}}-1$$

(g) (i) By sketching two appropriate graphs on the same number plane, or otherwise, solve $x+4>\frac{2}{x+3}$. **3**



Finding the points of intersection: $x+4=\frac{2}{x+3}$

$\therefore x^2+7x+12=2$

$\therefore x^2+7x+10=0$

$\therefore (x+2)(x+5)=0$

$\therefore x=-2,-5$

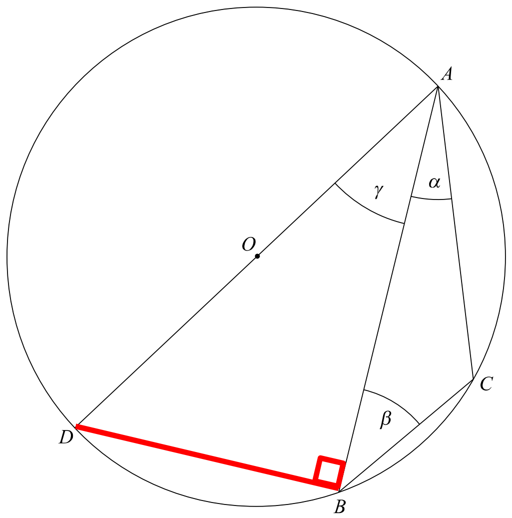
$\therefore -5<x<-3,\ x>-2$

(ii) Hence, or otherwise, deduce the values of x for which $x+4>\frac{2}{|x+3|}$. **1**

From the diagram, $x>-2$.

Question 12

(a) The diagram shows points B and C on a semi-circle with centre O and diameter AD.



Given that $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle OAB = \gamma$, find the value of $\alpha + \beta + \gamma$, giving reasons. **2**

Join D to B

$\angle ADB = 90°$ (angle in a semi-circle)

$\angle DAC + \angle DBC = 180°$ (opp. ang. of cyclic quad)

$\therefore \alpha + \gamma + 90° + \beta = 180°$

$\therefore \alpha + \beta + \gamma = 90°$

(b) Two zeroes of the cubic polynomial $P(x) = x^3 + px^2 + qx + r$ are equal in magnitude, but opposite in sign.

(i) Show that $x = -p$ is the third zero. **1**

Let the three roots be $\alpha, -\alpha$ and β.

$\sum \alpha = -p$

$\therefore \alpha + (-\alpha) + \beta = -p$

$\therefore \beta = -p$

(ii) Show that $r = pq$ **2**

$-\alpha^2 \beta = -r$ $\qquad$ $\alpha\beta + (-\alpha)\beta + (-\alpha)\alpha = q$

$\therefore \alpha^2 \beta = r$ $\qquad$ $\therefore -\alpha^2 = q \Rightarrow \alpha^2 = -q$

$\therefore r = \alpha^2 \beta = -q \times (-p) = pq$

ALTERNATIVE:

$P(-p) = (-p)^3 + p(-p)^2 + q(-p) + r$

$= -pq + r$

$P(-p) = 0$ [From (i)]

$\therefore r = pq$

Question 12 continued

(c) The Crows Nest Twins Association has 12 members i.e. 6 pairs of twins.

(i) *In how many ways can a committee of 3 members be elected if no two members of this committee are each other's twin?* **2**

Let the twins be Aa, Bb, Kk, Dd, Ee, Ff
First choose 3 sets of twins in ${}^6C_3 = 20$ ways.
Then from each set, there is a choice of choosing either twin.
So total number of ways is ${}^6C_3 \times 2 \times 2 \times 2 = 160$

ALTERNATIVE:
Pick 1 person from the 12 possible, then to avoid choosing the person's twin there are only 10 possibilities to pick the next person and then 8 for the final choice i.e. $12 \times 10 \times 8 = 960$.
But this means that if we picked **A**, **B** and **d** then this will also end up giving us **d**, **A** and **B** and all the other permutations of **A**, **B** and **d**.

So to avoid this over counting, the number of possible committees is $\dfrac{12 \times 10 \times 8}{3!} = 160$.

NB we wouldn't divide by 3! if we were picking President, VP and Secretary.

(ii) *If there are 4 pairs of identical twins and 2 pairs of non-identical twins, in how many ways can a committee of 4 members be elected if there is equal representation of non-identical and identical twins on the committee, but no 2 members of this committee are each other's twin?* **2**

Let the identical twins be Aa, Bb, Kk, Dd and the non-identical twins be Ee, Ff.
There has to be 2 identical twins and 2 non-identical twins.
First choose 2 sets of identical twins in ${}^4C_2 = 6$ ways.
Then from each set, there is a choice of choosing either twin.
So total number of ways of choosing the identical twins is ${}^4C_2 \times 2 \times 2 = 24$

Now choosing 2 sets of non-identical twins can be done in ${}^2C_2 = 1$ ways.
Then from each set, there is a choice of choosing either twin
So total number of ways of choosing the non-identical twins is ${}^2C_2 \times 2 \times 2 = 4$.

So choosing the committee can be done in $\left({}^4C_2 \times 2^2\right) \times \left({}^2C_2 \times 2^2\right) = 96$ ways.

ALTERNATIVE:
Using a similar logic to above and starting with the identical twins:
Pick 1 person from the 8 possible, then to avoid choosing the person's twin there are only 6 possibilities to pick the next person i.e. $8 \times 6 \times 4 = 48$.
But this means that if we picked **A** and **B** then this will also end up giving us **B** and **A**.

So to avoid this over counting the number of possible committees is $\dfrac{8 \times 6}{2!} = 24$.

<u>Now with the non-identical twins:</u>
Pick 1 person from the 4 possible, then to avoid choosing the person's twin there are only 2 possibilities to pick the next person i.e. $4 \times 2 = 8$.

But this means that if we picked **E** and **F** then this will also end up giving us **F** and **E** $\dfrac{4 \times 2}{2!} = 4$.

As we want both of these groups on the committee then there are $24 \times 4 = 96$ ways of this.

(d) Falling into cold water is particularly dangerous because the body loses body heat 25 times faster in cold water than in cold air.

The normal body temperature is $37°$ C.

Shivering begins at an approximate temperature of $36°$ C, amnesia at $34\cdot4°$ C and unconsciousness at $30°$ C.

The body temperature of a person, $T°$C, who has been in the water for t minutes can be modeled by the differential equation

$\frac{dT}{dt} = -k\left(T - W\right)$, where W is the temperature of the water and k is a constant.

(i) Show that $T = W + Be^{-kt}$, where B is a constant, is a solution of this differential equation. **1**

$T = W + Be^{-kt} \Rightarrow T - W = Be^{-kt}$

$\text{LHS} = \frac{dT}{dt} = -kBe^{-kt} = -k\left(T - W\right) = \text{RHS}$

(ii) If the water temperature is $2°$ C, a person is expected to start shivering after 5 minutes. **3**

Show that $T = 2 + 35e^{-0\cdot0058t}$.

$W = 2 \Rightarrow T = 2 + Be^{-kt}$

$t = 5$, $T = 37° \Rightarrow T = 2 + 35e^{-kt}$

$t = 5$, $T = 36°$:

$\therefore 36 = 2 + 35e^{-5k} \Rightarrow 35e^{-5k} = 34$

$\therefore e^{-5k} = \frac{34}{35} \Rightarrow -5k = \ln\frac{34}{35}$

$\therefore k = \frac{1}{5}\ln\frac{35}{34} \doteqdot 0\cdot0058$

(iii) Find how long a person can stay in the water before becoming unconsciousness. Give your answer in minutes, correct to 1 decimal place. **2**

Unconsciousness at $30°$ C.

$T = 2 + 35e^{-0\cdot0058t}$

$\therefore 30 = 2 + 35e^{-0\cdot0058t}$

$\therefore 35e^{-0\cdot0058t} = 28$

$\therefore e^{-0\cdot0058t} = \frac{28}{35} = \frac{4}{5}$

$\therefore -0\cdot0058t = \ln\frac{4}{5}$

$\therefore t = \frac{\ln\frac{5}{4}}{0\cdot0058} \doteqdot 38\cdot5$

So unconsciousness would occur after approximately $38\cdot5$ mins.

Question 13

(a) When the region is rotated about the x-axis, the volume of the solid formed is given the by the integral $\pi\int_0^{\frac{\pi}{2}}\left(x^2-\sin^2 x\right)dx$.

Find the exact volume, leaving your answer in exact, simplified form. **3**

$$\cos 2x = 1-2\sin^2 x \Rightarrow \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$

$$\pi\int_0^{\frac{\pi}{2}}\left(x^2-\sin^2 x\right)dx = \pi\int_0^{\frac{\pi}{2}}\left[x^2-\tfrac{1}{2}(1-\cos 2x)\right]dx$$

$$= \pi\left[\frac{x^3}{3}-\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{2}}$$

$$= \pi\left[\frac{\left(\frac{\pi}{2}\right)^3}{3}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)\right]$$

$$= \pi\left(\frac{\pi^3}{24}-\frac{\pi}{4}\right)$$

$$= \frac{\pi^2\left(\pi^2-6\right)}{24}$$

(b) Show that there is a stationary point on the curve $y = e^{-x}-e^{-2x}$ at $x=\ln 2$ and determine its nature. **3**

$$e^{\ln x} = x \text{ and } e^{-\ln x} = \frac{1}{x}$$

$$y = e^{-x}-e^{-2x}$$

$$\therefore y' = -e^{-x}+2e^{-2x}$$

$$\therefore y'' = e^{-x}-4e^{-2x}$$

$$x=\ln 2 \Rightarrow y' = -e^{-\ln 2}+2e^{-2\ln 2} = -\frac{1}{2}+2\times\frac{1}{4} = 0$$

So there is a stationary point at $x=\ln 2$

$$x=\ln 2 \Rightarrow y'' = e^{-\ln 2}-4e^{-2\ln 2} = \frac{1}{2}-4\times\frac{1}{4} = -\frac{1}{2}$$

As $y''<0$, the stationary point is a (rel) maximum turning point.

OR

$x = \ln 2 \doteqdot 0\cdot 693$

x	$0\cdot 6$	$\ln 2$	$0\cdot 7$
y'	$0\cdot 054$	0	$-0\cdot 0034$
	/	–	\

So there is a (rel) maximum turning point at $x=\ln 2$.

Question 13 continued

(c) The diagram below shows the tangents drawn at the points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ on the parabola $x^2 = 4ay$.
The tangents at P and Q intersect at T.
You may assume that the equation of the tangent at P is $y = px - ap^2$
and that point T has coordinates $T\left[a\left(p+q\right), apq\right]$.

(i) Supposing that point T lies on the line $y = a$, show that $pq = 1$. **1**
If point T lies on the line $y = a$, then $apq = a \Rightarrow pq = 1$

(ii) M is the midpoint of the chord PQ.
Find the Cartesian equation of the locus of M. **3**

$$M\left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right) = \left[a\left(p+q\right), \tfrac{a}{2}\left(p^2+q^2\right)\right]$$

Let $x = a\left(p+q\right)$ and $y = \tfrac{a}{2}\left(p^2+q^2\right)$

$x = a\left(p+q\right) \Rightarrow p+q = \tfrac{x}{a}$

$y = \tfrac{a}{2}\left(p^2+q^2\right) \Rightarrow p^2+q^2 = \tfrac{2y}{a}$

$\tfrac{2y}{a} = p^2+q^2$

$\quad = \left(p+q\right)^2 - 2pq$

$\quad = \left(\tfrac{x}{a}\right)^2 - 2$

$\therefore \tfrac{2y}{a} = \left(\tfrac{x}{a}\right)^2 - 2$

$\therefore x^2 = 2a\left(y+a\right)$ or $y = \tfrac{1}{2a}x^2 - a$

(iii) State any restrictions on the domain of the locus of M. **1**

$x^2 = 4ay \Rightarrow y = \tfrac{1}{4a}x^2$

M is constrained in its domain since T has to lie on the line $y = a$ and *external* to the curve as tangents to a parabola can only intersect externally.
So the constraints for T will be the restraints for M.

Comparing y-coordinates, M must be above the parabola and T must be below it
i.e. $\quad y = \tfrac{1}{4a}x^2 > a$.

$\therefore x^2 > 4a^2$

$\therefore x < -2a, x > 2a$ or $\left| x \right| > 2a$

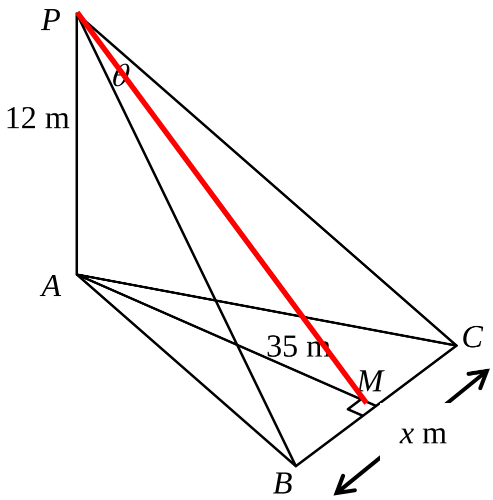
Alternative 1:
Comparing y-coordinates, M must be above the parabola.
i.e. $\quad y = \tfrac{1}{2a}x^2 - a > a$

$\therefore \tfrac{1}{2a}x^2 > 2a$.

$\therefore x^2 > 4a \Rightarrow x < -2a, x > 2a$

Alternative 2:
As M and T share the same x-coordinate, the extreme case will be when P and Q are NOT distinct points i.e. $p = q$.
When this happens T (and M) will be at $(2a, a)$.
Since T must lie outside the parabola then $\left| x \right| > 2a$.

(d) A hot air balloon P is rising vertically above A and P is equidistant from observers at B and C. M is the midpoint of BC, AM = 35 m and $\angle BPC = \theta$. At a certain time the balloon reaches a height of 12 m above A.



(i) Show that $x = 74\tan\left(\frac{\theta}{2}\right)$ **2**

$PM = 37$ (Pythagorean triad)

As ΔPBC is isosceles then $\angle MPB = \frac{1}{2}\theta$ and $BM = \frac{1}{2}x$

$$\therefore \tan\left(\tfrac{1}{2}\theta\right) = \frac{\frac{1}{2}x}{37}$$

$$\therefore x = 74\tan\left(\tfrac{1}{2}\theta\right)$$

(ii) When PA = 12, $\dfrac{d\theta}{dt} = 0\cdot 01$ radians/s. Find $\dfrac{dx}{dt}$ when $\theta = 0\cdot 04$ radians. **2**

Leave your answer correct to 2 decimal places.

$$\frac{dx}{d\theta} = \tfrac{1}{2}\times 74\times \sec^2\left(\tfrac{1}{2}\theta\right) = 37\sec^2\left(\tfrac{1}{2}\theta\right)$$

$$\frac{dx}{dt} = \frac{dx}{d\theta}\times\frac{d\theta}{dt}$$

$$= 37\sec^2\left(0\cdot 02\right)\times 0\cdot 01$$

$$= 0\cdot 37 \text{ m/s}$$

Question 14

(a) Prove by induction that $12^n + 2\times 5^{n-1}$ is divisible by 7, for all integers $n \ge 1$. **3**

Test n = 1: $12^n + 2\times 5^{n-1} = 12 + 2\times 14 = 2\times 7$.
So true for n = 1.

Assume true for n = k i.e. $12^k + 2\times 5^{k-1} = 7M$, $M \in \mathbb{Z}$
$\therefore 12^k = 7M - 2\times 5^{k-1}$

Need to prove true for $n = k+1$ i.e. $12^{k+1} + 2\times 5^k = 7N$, $N \in \mathbb{Z}$

$$
\begin{aligned}
12^{k+1} + 2\times 5^k &= 12\times 12^k + 2\times 5^k \\
&= 12\left(7M - 2\times 5^{k-1}\right) + 2\times 5\times 5^{k-1} \\
&= 7\times 12M - 24\times 5^{k-1} + 10\times 5^{k-1} \\
&= 7\left(12M - 2\times 5^{k-1}\right) \\
&= 7N \qquad \left(N = 12M - 2\times 5^{k-1} \in \mathbb{Z}\right)
\end{aligned}
$$

So the statement is true for $n = k+1$ if it is true for n = k.

So by the principle of mathematical induction the statement is true for all integers $n \ge 1$.

Question 14 continued

(b) (i) $f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{e^h-1}{h}$.

Now $f'(x)=e^x$

$\therefore f'(0)=1$

$\therefore f'(0)=\lim_{h\to 0}\frac{e^h-1}{h}=1$

(ii) $e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n}{n}}=\frac{e^{\frac{1}{n}}\left[\left(e^{\frac{1}{n}}\right)^n-1\right]}{e^{\frac{1}{n}}-1}$

$=\frac{e^{\frac{1}{n}}(e-1)}{e^{\frac{1}{n}}-1}$

(iii) Using (i), if $h=e^{\frac{1}{n}}$, then as $n\to\infty, h=\frac{1}{n}\to 0$.

NB $\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{h\to 0}\frac{h}{e^h-1}=1$

From (ii) and replacing n with $\frac{1}{h}$.

$\frac{e^{\frac{1}{n}}}{n\left(e^{\frac{1}{n}}-1\right)}(e-1)=\frac{e^h}{\frac{1}{h}\left(e^h-1\right)}(e-1)=\frac{he^h}{\left(e^h-1\right)}(e-1)$

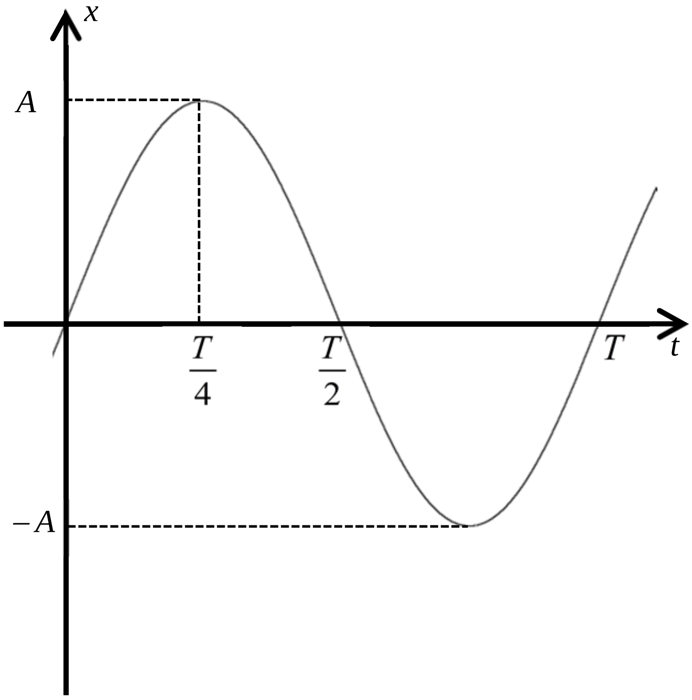
$\therefore \lim_{n\to\infty}\frac{e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n}{n}}}{n}=(e-1)\times\lim_{h\to 0}\left[\frac{h}{\left(e^h-1\right)}\right]\times\lim_{h\to 0}e^h$

$=(e-1)\times 1\times 1$

$=e-1$

Question 14 continued

(c) (i)



(ii) $T=\frac{2\pi}{n}\Rightarrow n=\frac{2\pi}{T}$

$x=A\sin nt=A\sin\left(\frac{2\pi}{T}\right)t$

$\therefore \dot{x}=A\times\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}\right)t=\frac{2\pi A}{T}\cos\left(\frac{2\pi}{T}\right)t$

(iii) $A, D > 0$

At P, $x = D$, and $\dot{x}=V$

$D=A\sin\left(\frac{2\pi}{T}\right)t \qquad -(1)$

$V=\frac{2\pi A}{T}\cos\left(\frac{2\pi}{T}\right)t \qquad -(2)$

$(1)\div(2)\Rightarrow \frac{T}{2\pi}\tan\left(\frac{2\pi}{T}\right)t=\frac{D}{V}$

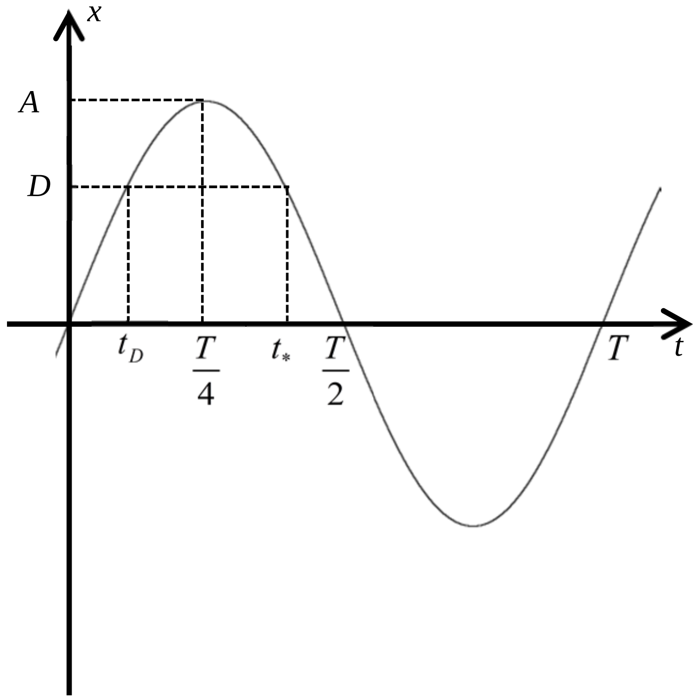
$\therefore \tan\left(\frac{2\pi}{T}\right)t=\frac{2\pi D}{VT}$

As D, V and T are positive, then the first positive solution is $t=\frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$

Question 14 continued

(iv) **Method 1:** Using (i)

Let $t_D = \dfrac{T}{2\pi}\tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$ and let t_* be the second time that the particle is at P



Using the symmetry of the sine curve about $t = \dfrac{T}{4}$, for $0 \le t \le \dfrac{T}{2}$ then

$$\begin{aligned}
t_* - t_D &= \frac{T}{2} - 2t_D \\
&= \frac{T}{2} - \frac{T}{\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right) \\
&= \frac{T}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{2\pi D}{VT}\right)\right] \\
&= \frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi D}\right) \qquad \left[\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}x\right]
\end{aligned}$$

Question 14 continued

Method 2: Using SHM

Given that the particle is travelling in SHM, then the velocity of the particle when it comes back to P is $-V$

$$D = A\sin\left(\frac{2\pi}{T}\right)t \qquad -(1)$$

$$-V = \frac{2\pi A}{T}\cos\left(\frac{2\pi}{T}\right)t \qquad -(2)$$

$$(1) \div (2) \Rightarrow \frac{T}{2\pi}\tan\left(\frac{2\pi}{T}\right)t = -\frac{D}{V}$$

$$\therefore \tan\left(\frac{2\pi}{T}\right)t = -\frac{2\pi D}{VT}$$

As D, V and T are positive then $\dfrac{2\pi t}{T} = \pi - \tan^{-1}\left(\dfrac{2\pi D}{VT}\right)$

$$\therefore t = \frac{T}{2} - \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

$$\therefore t = \frac{T}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{2\pi D}{VT}\right)\right]$$

$$\therefore t = \frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi D}\right) \qquad \left[\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1} x\right]$$