SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2012
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions:

- Reading Time: 5 minutes
- Working Time: 2 hours

- Write in black or blue pen
  Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of the Multiple Choice answer sheet
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total Marks – 70 Marks

Section I: Pages 2 – 5
10 marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II: Pages 6 – 11
60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section
- For Questions 11 – 14, start a new answer booklet per question

Examiner: External Examiner

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section I   Multiple Choice

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the Multiple Choice answer sheet to record your answers to Questions 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

1. Given the coordinates of the points *A* and *B* are (-1, 1) and (3, -1) respectively. Find the coordinates of the point *P* if it divides the interval *AB* externally in the ratio of 1:3.

(A) $\left(2, -\frac{1}{2}\right)$

(B) $\left(-1\frac{1}{2}, \frac{1}{2}\right)$

(C) $(-3, 2)$

(D) $(4, -2)$

2. Which of the polynomials are divisible by $x+1$?

(I) $x^{2012}-1$ (II) $x^{2011}-1$ (III) $x^{2010}+1$ (IV) $x^{2009}+1$

(A) (I) and (III) only

(B) (II) and (III) only

(C) (II) and (IV) only

(D) (I) and (IV) only

3. The number of different arrangements of the letters of the word *SERVICES* which begin and end with the letter *S* is:

(A) $\dfrac{6!}{(2!)^2}$

(B) $\dfrac{8!}{(2!)^2}$

(C) $\dfrac{6!}{2!}$

(D) $\dfrac{8!}{2!}$

4. Given $\sin x = k$ and $\frac{\pi}{2} < x < \pi$ then $\tan\left(\frac{\pi}{2} + x\right)$ equals:

(A) $\frac{\sqrt{1-k^2}}{k}$

(B) $\frac{k}{\sqrt{1-k^2}}$

(C) $-\frac{\sqrt{1-k^2}}{k}$

(D) $-\frac{k}{\sqrt{1-k^2}}$

5. Given that a is a first approximation to a root of the equation $g(x) + a - x = 0$, then a second approximation obtained by one application of Newton's method is:

(A) $a + \frac{g(a)}{1-g'(a)}$

(B) $a - \frac{g(a)}{1-g'(a)}$

(C) $a - \frac{g(a)}{1+g'(a)}$

(D) $a + \frac{g(a)}{1+g'(a)}$

6. Given that $x = t^3, y = t^2$ then $\frac{d^2y}{dx^2} =$

(A) $\frac{2}{3t}$

(B) $-\frac{2}{9t^4}$

(C) $\frac{4}{9t^2}$

(D) $-\frac{2}{3t^2}$

7. A particle moving in *Simple Harmonic Motion* oscillates about a fixed point O in a straight line with a period of 10 seconds. The maximum displacement of P from O is 5 m. Which of the following statements is/are true?

If P is at O moving to the ***right***, then 22 seconds later P will be:

(I) moving towards O.

(II) moving with a decreasing speed

(III) at a distance $5\sin\frac{22\pi}{5}$ m to the right of O

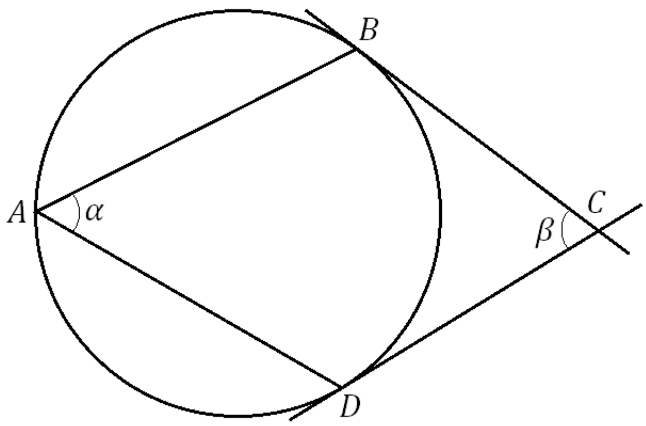
(A) (I), (II) and (III)

(B) (I) and (II) only.

(C) (II) and (III) only.

(D) none of the above.

8. In the diagram below, BC and DC are tangents, then:



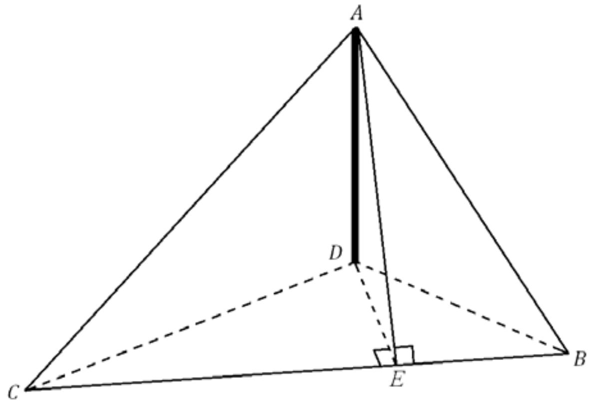
(A) $\alpha+\beta=180^\circ$

(B) $2\alpha+\beta=180^\circ$

(C) $\alpha+2\beta=180^\circ$

(D) $2\alpha-\beta=90^\circ$

9. In the figure below, AD is a vertical pole standing on the horizontal ground BCD. If E is a point lying on BC such that DE and AE are perpendicular to BC, then the angle between the plane ABC and the horizontal ground is:



(A) $\angle ABD$

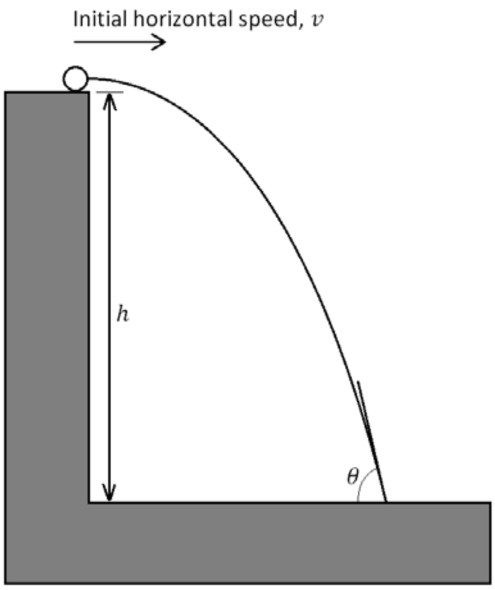
(B) $\angle ABE$

(C) $\angle ACD$

(D) $\angle AED$

10. The diagram below shows the path of a projectile fired with a horizontal velocity v from a cliff of height h.

Which pair of the following values of v and h will give the greatest value of the angle θ?



(A) $v = 10ms^{-1}$ and $h = 30m$

(B) $v = 30ms^{-1}$ and $h = 50m$

(C) $v = 50ms^{-1}$ and $h = 10m$

(D) $v = 10ms^{-1}$ and $h = 50m$

End of Section I

Section II   Short Answer Questions

60 Marks
Attempt Question 11 – 14.
Allow approximately 1 hour 45 minutes for this section.

Use a SEPARATE writing booklet to record your answers to Questions 11 – 14.

Question 11 (15 Marks) Use a SEPARATE writing booklet

(a) Solve the inequality $\frac{1}{x} < 4x$ **2**

(b) Differentiate $x\sin^{-1}2x$ **2**

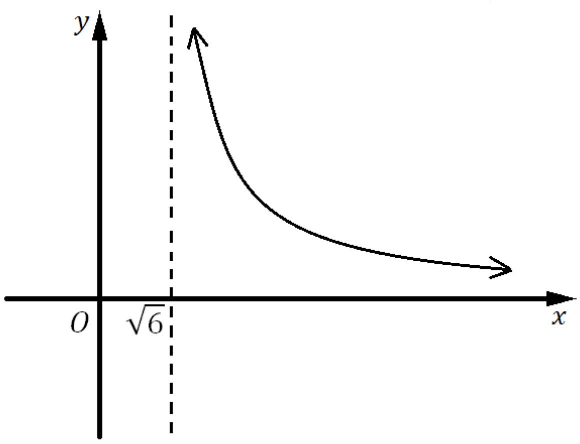
(c) Using the substitution $u = \tan x$, find the exact value of

$$\int_0^{\frac{\pi}{3}} \frac{\sec^2 x}{3+\tan^2 x}dx$$ **3**

(d) Simplify $\frac{3!}{(n+1)!} - \frac{2!}{n!}$ **1**

(e) Find the solutions for the equation $2\cos^2\theta = \sin 2\theta$, for $0 \le \theta \le 2\pi$ **3**

(f) The diagram below shows that the graph of $f(x) = \frac{1}{x^2-6}$ where $x > \sqrt{6}$



(i) Find an expression for the inverse function $y = f^{-1}(x)$ **2**

(ii) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at $x = \alpha$.
Explain why α is a root of the equation $x^3 - 6x - 1 = 0$. **2**

End of Question 11

Question 12 (15 Marks) Use a SEPARATE writing booklet

(a) If $\cos 3x - \sqrt{3}\sin 3x = R\cos(3x + \theta)$ where $R > 0$ and $0 < \theta < \dfrac{\pi}{2}$

(i) Find R and θ **2**

(ii) Hence, find the general solution of $\cos 3x - \sqrt{3}\sin 3x = 2$ **2**

(b) The acceleration of a particle moving on the x - axis is given by $\ddot{x} = x - 2$ where x is the displacement from the origin O after t seconds.
Initially, the particle is at rest at $x = 3$.

(i) Show that its velocity at any position x is $v^2 = (x-1)(x-3)$ **2**

(ii) Find its acceleration when its velocity is $2\sqrt{6}$ m s^{-1} **2**

(c) Prove by mathematical induction that $7^n - 6n - 1$ is divisible by 36 for all positive integers $n \geq 2$ **3**

(d) When the polynomial $P(x)$ is divided by $(x-4)(x+1)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Why is the most general form of $R(x)$ given by $R(x) = ax + b$? **1**

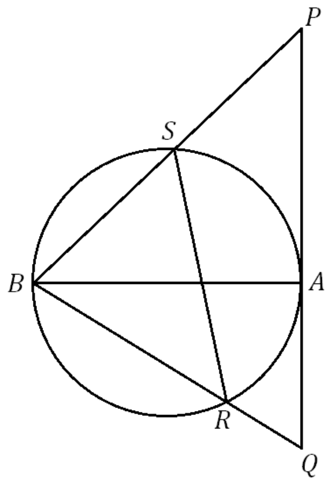
(ii) Given that $P(4) = -5$, show that $R(4) = -5$. **1**

(iii) Further, when $P(x)$ is divided by $x+1$, the remainder is 5. Find $R(x)$. **2**

End of Question 12

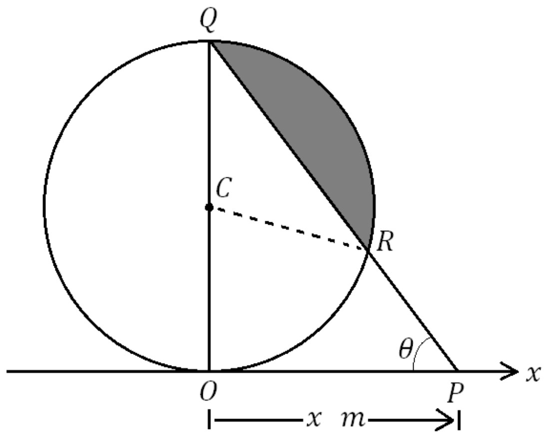
Question 13 (15 Marks) Use a SEPARATE writing booklet

(a)



PAQ is the tangent to a circle at *A*. *AB* is the diameter and *PB, QB* cut the circle at *S, R* respectively. Prove that *PQRS* is a cyclic quadrilateral. **3**

(b)



The above figure shows a circle with radius 1 metre touching the x-axis at *O*. *OQ* is a diameter and *C* is the centre of the circle. *P* is a point on the positive x-axis, which is x metres from *O*. *PQ* cuts the circle at *R*.

Let $A\,m^2$ be the area of the segment (shaded region) and $\angle OPQ = \theta$

(i) Find $\angle QCR$ in terms of θ. Hence show that $A = \theta - \frac{1}{2}\sin 2\theta$ **2**

(ii) Find $\frac{dA}{d\theta}$ in terms of θ **1**

(iii) Express x in terms of θ **1**

The point *P* is moving towards O at a constant speed 2 ms^{-1}

(iv) Find $\frac{d\theta}{dt}$ when $x = 2\sqrt{3}$ **2**

(v) Find the rate of change of A with respect to time when $x = 2\sqrt{3}$ **2**

Question 13 (Continued)

(c) Two points P and Q whose coordinates are $\left(2ap, ap^2\right)$ and $\left(2aq, aq^2\right)$ are on the parabola $x^2 = 4ay$. The equation of chord PQ is given by $y = \left(\frac{p+q}{2}\right)x - apq$.

(DO NOT prove this)

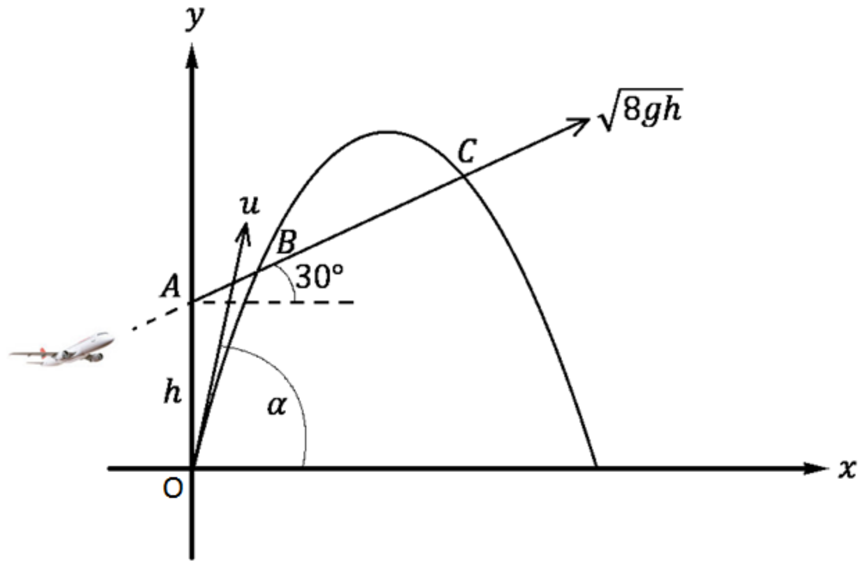
(i) Show that if chord PQ passes through the focus $S\ (0, a)$ then $pq = -1$. **1**

(ii) X is the mid-point of the focal chord PQ. T lies on the directrix such that XT is perpendicular to the directrix. W is the mid-point of XT. Find the locus of W. **3**

End of Question 13

Question 14 (15 Marks) Use a SEPARATE writing booklet

(a) The diagram below shows a plane P which is flying at a constant speed of $\sqrt{8gh}$ m/s upwards at an angle of elevation of 30°.
At the instant when the plane is at a height h metres vertically above a missile silo located at a point O on the ground, a missile from the silo is launched at an angle of elevation α to hit the plane where $0^\circ < \alpha < 90^\circ$.



With the axes shown in the diagram above, you may assume that the position of the missile is given by (DO NOT prove this)

$$x = ut\cos\alpha$$

$$y = ut\sin\alpha - \tfrac{1}{2}gt^2$$

where the launching speed of the missile is u m/s; t is the time in seconds after launch and g is the acceleration due to gravity.

(i) Show that the trajectory of the plane is given by $y = \dfrac{x}{\sqrt{3}} + h$. **1**

(ii) Assuming that the missile can hit the plane, hence, from part (i), show that the x-coordinates of the points of collision must satisfy **3**

$$\frac{x^2}{12} + \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)hx + h^2 = 0$$

(iii) Suppose that $\tan\alpha > \dfrac{2}{\sqrt{3}}$

(α) Show that there are two possible points of collision B and C between the plane and the missile. **2**

(β) Show that the time T (in seconds) elapsed between the two points of collision is given by **2**

$$T = \sqrt{\frac{8h\tan\alpha}{g}\left(3\tan\alpha - 2\sqrt{3}\right)}$$

Question 14 (Continued)

(b) A particle is oscillating between A and B, 7 m apart, in Simple Harmonic Motion. The time for the particle to travel from B to A and back is 3 seconds. Find the velocity and acceleration at M, the mid-point of OB where O is the centre of AB. **3**

(c)

(i) In how many ways can n different coloured balls be placed in 2 non-identical urns so that neither urn is empty? **1**

(ii) Hence, or otherwise, find the number of ways that 6 different coloured balls can be placed in 3 non-identical urns so that no urn is empty. **3**

End of Question 14

End of Examination

Student Number:______________________________

Mathematics Extension 1
Trial HSC 2012

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ☒ B ☒ (correct) C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. (A) (B) ● (D)
2. (A) (B) (C) ●
3. (A) (B) ● (D)
4. ● (B) (C) (D)
5. ● (B) (C) (D)
6. (A) ● (C) (D)
7. (A) (B) ● (D)
8. (A) ● (C) (D)
9. (A) (B) (C) ●
10. (A) (B) (C) ●

Question 11

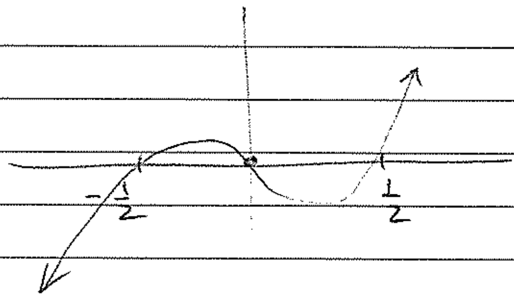
(a) $\frac{1}{x} < 4x$

$x < 4x^3$

$4x^3 - x > 0$

$x(4x^2-1) > 0$

$x(2x-1)(2x+1) > 0$



$-\frac{1}{2} < x < 0 \quad , \quad x > \frac{1}{2}$.

(b) $\frac{d(x\sin^{-1}2x)}{dx} = x . \frac{1}{\sqrt{1-(2x)^2}} . 2 + 1 . \sin^{-1}2x$

$= \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1}2x$

(c) $\int_0^{\frac{\pi}{3}} \frac{\sec^2 x}{3+\tan^2 x} dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

limit change

$x = \frac{\pi}{3}$ $\quad$ $x = 0$

$u = \sqrt{3}$ $\quad$ $u = 0$

$= \int_0^{\sqrt{3}} \frac{\sec^2 x}{3+u^2} . \frac{du}{\sec^2 x}$

$= \int_0^{\sqrt{3}} \frac{du}{3+u^2}$

$a = \sqrt{3}$

$$= \left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right]_0^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{0}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} 1$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{4} \right)$$

$$= \frac{\pi}{4\sqrt{3}} \quad \text{or} \quad \frac{\sqrt{3}\pi}{12}$$

(d) $$\frac{3!}{(n+1)!} - \frac{2!}{n!}$$

$$= \frac{3! - 2!(n+1)}{(n+1)!}$$

$$= \frac{6 - 2n - 2}{(n+1)!}$$

$$= \frac{4 - 2n}{(n+1)!}$$

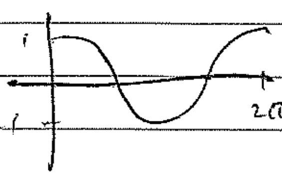
(e) $2\cos^2\theta = \sin 2\theta$ $\qquad 0 \le \theta \le 2\pi$

$2\cos^2\theta - \sin 2\theta = 0$

$2\cos^2\theta - 2\sin\theta\cos\theta = 0$

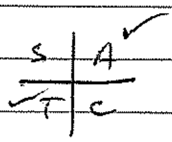
$2\cos\theta(\cos\theta - \sin\theta) = 0$

$\cos\theta = 0$ or $\cos\theta - \sin\theta = 0$



$\frac{\cos\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$1 = \tan\theta$



$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$.

f) i) $f(x) = \frac{1}{x^2-6}$ where $x > \sqrt{6}$

interchange x & y & make y the subject.

$x = \frac{1}{y^2-6}$, $y > \sqrt{6}$

$y^2 - 6 = \frac{1}{x}$

$y^2 = 6 + \frac{1}{x}$

$y = \sqrt{6 + \frac{1}{x}}$ since $y > \sqrt{6}$.

$\therefore f^{-1}(x) = \sqrt{6 + \frac{1}{x}}$, $x > 0$.

ii) $y = f(x)$ & $y = f^{-1}(x)$ meet at $x = a$ on the line $y = x$.

$\therefore \alpha$ is a root of $x = f(x)$

$x = \frac{1}{x^2-6}$

$x^3 - 6x = 1$

$x^3 - 6x - 1 = 0$.

2012 Extension 1 Mathematics THSC:
Solutions— Question 12

Marks

Question 12 (15 marks)

(a) If $\cos 3x - \sqrt{3}\sin 3x = R\cos(3x+\theta)$, where $R > 0$ and $0 < \theta < \frac{\pi}{2}$

(i) Find R and θ. [2]

Solution: $R = \sqrt{1^2 + (\sqrt{3})^2}, = 2.$ $\quad \tan\theta = \sqrt{3}/1, \quad \theta = \frac{\pi}{3}.$

(ii) Hence find the general solution of $\cos 3x - \sqrt{3}\sin 3x = 2$. [2]

Solution:
$$\begin{aligned} 2\cos(3x + \tfrac{\pi}{3}) &= 2, \\ \cos(3x + \tfrac{\pi}{3}) &= 1, \\ 3x + \tfrac{\pi}{3} &= 2n\pi,\ n \in \mathbb{Z}, \\ 3x &= 2n\pi - \tfrac{\pi}{3}, \\ x &= \frac{2n\pi}{3} - \frac{\pi}{9} \text{ or } \frac{\pi(6n-1)}{9}. \end{aligned}$$

(b) The acceleration of a particle moving on the x-axis is given by $\ddot{x} = x - 2$ where x is the displacement from the origin O after t seconds.
Initially, the particle is at rest at $x = 3$.

(i) Show that its velocity at any position is $v^2 = (x-1)(x-3)$. [2]

Solution:
$$\begin{aligned} v\frac{dv}{dx} &= x - 2, \\ \int_0^v v\,dv &= \int_3^x (x-2)\,dx, \\ \left[\frac{v^2}{2}\right]_0^v &= \left[\frac{x^2}{2} - 2x\right]_3^x, \\ \frac{v^2}{2} - 0 &= \frac{x^2}{2} - 2x - \left(\frac{9}{2} - 6\right), \\ v^2 &= x^2 - 4x + 3, \\ &= (x-1)(x-3). \end{aligned}$$

(ii) Find its acceleration when its velocity is $2\sqrt{6}\,\text{ms}^{-1}$. [2]

Solution: Method 1—

$$\begin{aligned}(2\sqrt{6})^2 &= x^2 - 4x + 3,\\ x^2 - 4x - 21 &= 0,\\ (x-7)(x+3) &= 0,\\ x &= 7 \text{ or } -3.\end{aligned}$$

Initial acceleration positive and increasing so take $x = 7$,

$$\begin{aligned}\ddot{x} &= 7 - 2,\\ &= 5\,\text{ms}^{-2}.\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}\text{Put } a = x - 2 \text{ in } v^2 &= (x-1)(x-3),\\ i.e.\ v^2 &= (a+1)(a-1),\\ (2\sqrt{6})^2 &= a^2 - 1,\\ a^2 &= 25,\\ a &= \pm 5.\end{aligned}$$

Initial acceleration positive and increasing so we have $a = 5$,

i.e. $\ddot{x} = 5\,\text{ms}^{-2}$.

(c) Prove by mathematical induction that $7^n - 6n - 1$ is divisible by 36 for all positive integers $n \geqslant 2$. [3]

Solution: Test for $n = 2$, $\text{L.H.S.} = 49 - 12 - 1$,

$= 36$.

$\therefore$ True for $n = 2$.

Assume true for some $n = k$,

i.e. $7^k - 6k - 1 = 36p,\ p \in \mathbb{Z}$.

Test for $n = k + 1$, *i.e.* $7^{k+1} - 6(k+1) - 1 = 36q,\ q \in \mathbb{Z}$.

$$\begin{aligned}\text{L.H.S.} &= 7.7^k - 6k - 7,\\ &= 7.7^k - 42k - 7 + 36k,\\ &= 7(7^k - 6k - 1) + 36k,\\ &= 7.36p + 36k,\\ &= 36(7p + k).\end{aligned}$$

$\therefore$ True for $n = k + 1$ if true for $n = k$.

As true for $n = 2$, so true for all integral $n \geqslant 2$ by the principle of mathematical induction.

(d) When the polynomial $P(x)$ is divided by $(x-4)(x+1)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Why is the most general form of $R(x)$ given by $R(x) = ax + b$? [1]

Solution: When dividing polynomials, the remainder must be at least one degree less than the divisor or the division could continue. In this example, the divisor is of the second degree so the remainder can be at most of the first degree: thus $R(x) = ax + b$.

(ii) Given that $P(4) = -5$, show that $R(4) = -5$. [1]

Solution:

$$\begin{aligned} P(x) &= (x-4)(x+1)Q(x) + R(x), \\ P(4) &= (4-4)(4+1)Q(4) + R(4), \\ &= 0 \times 5 \times Q(4) + R(4), \\ &= R(4). \\ \therefore \text{ If } P(4) &= -5,\ R(4) = -5. \end{aligned}$$

(iii) Further, when $P(x)$ is divided by $x+1$, the remainder is 5. Find $R(x)$. [2]

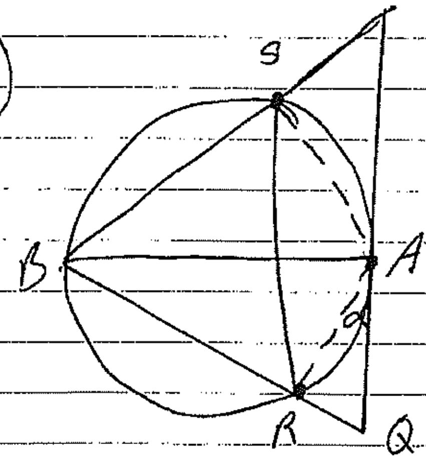
Solution:

$$\begin{aligned} P(4) &= 4a + b = -5 \ldots\ldots \boxed{1} \\ P(-1) &= -a + b = 5 \ldots\ldots\ldots \boxed{2} \\ \boxed{1} - \boxed{2}: \quad 5a &= -10, \\ a &= -2, \\ \text{subst. in } \boxed{2}: \quad 2 + b &= 5, \\ b &= 3. \\ \therefore R(x) &= -2x + 3. \end{aligned}$$

Ext 1 Trial

Question 13

(a)



Join AR, AS.

AB is the diameter.

$\therefore \angle BSA = \angle BRA = 90^\circ$

(Angle in a semi-circle is 90°)

$\therefore \angle ARQ = 90^\circ$ (adjacent $\angle$'s on a straight line)

Let $\angle RAQ = \alpha$

In $\triangle ARQ$, $\angle AQR = 90 - \alpha$ (Angle sum of $\triangle$)

Also, $\angle RSA = \alpha$ (alternate segment theorem)

$\therefore \angle BSR = 90 - \alpha$. ($\angle ASB = 90^\circ$ proven)

ie. $\angle BSR = \angle AQR = 90 - \alpha$.

ie. The exterior angle of quadrilateral PQRS is equal to interior opposite angle

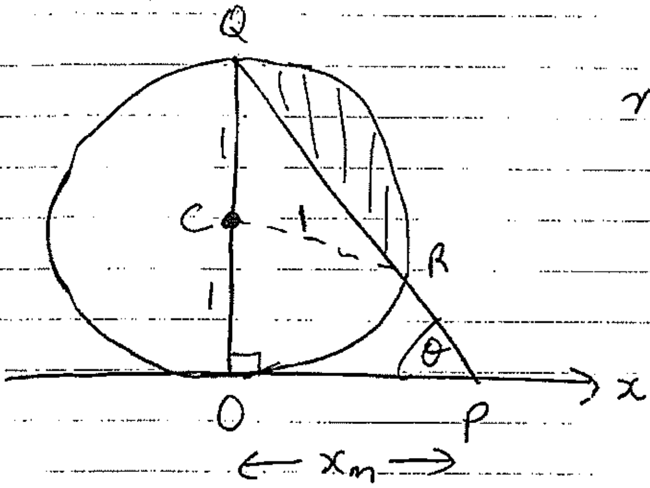
$\Rightarrow$ PQRS is a cyclic quadrilateral.

(3)

13 (b)

$\frac{\theta}{2\pi} \times \pi r^2 \quad \frac{\theta r^2}{2}$

$\frac{1}{2} \times 1 \times 1 \sin 2\theta$



$r = 1$, Area segment $= A\,m^2$.

(i) Find $\angle QCR$.

$\angle QOP = 90°$ (tangent to circle is ⊥ to radius at pt of contact).

Then $\angle PQO = 90 - \theta$ (Angle Sum of a triangle $= 180°$)

$\therefore \angle CRQ = 90 - \theta$ (Isosceles Δ, $CQ = CR =$ radii)

$\therefore \angle QCR = 180 - (90-\theta) - (90-\theta)$ ✓

$\angle QCR = 2\theta$

Area of segment = Area sector − Area triangle

$= \frac{\theta r^2}{2} - \frac{1}{2} ab \sin C$

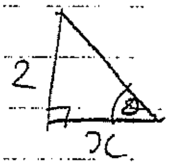
$= \frac{(2\theta)1^2}{2} - \frac{1}{2} \sin 2\theta$ ✓ (2)

$A = \theta - \frac{1}{2} \sin 2\theta$ ✓

(ii) $\frac{dA}{d\theta} = 1 - \frac{1}{2} \cos 2\theta \cdot 2$

$= 1 - \cos 2\theta$ ✓ (1)

(iii) Express x in terms of θ.



$\tan\theta = \frac{2}{x}$ (1)

$x = \frac{2}{\tan\theta}$ ✓ $= 2\cot\theta$

13(b) (iv) P moves towards O at $v = -2$ m/s

Find $\frac{d\theta}{dt}$ when $x = 2\sqrt{3}$ $\Rightarrow \frac{dx}{dt} = -2$ m/s.

Now $x = \frac{2}{\tan\theta} = 2(\tan\theta)^{-1}$

$$\frac{dx}{d\theta} = -2(\tan\theta)^{-2}\sec^2\theta$$

$$= \frac{-2}{\tan^2\theta} \cdot \frac{1}{\cos^2\theta}$$

$$= \frac{-2\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos^2\theta}$$

$$\frac{dx}{d\theta} = \frac{-2}{\sin^2\theta}$$ ✓

$$\Rightarrow \frac{d\theta}{dx} = \frac{-\sin^2\theta}{2}$$

So $\boxed{\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt}}$

$$= -\frac{\sin^2\theta}{2} \times -2$$

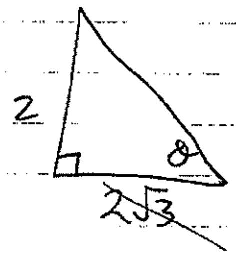
$$\frac{d\theta}{dt} = \sin^2\theta$$ ✓

(2)

When $x = 2\sqrt{3}$,

$\theta = \tan^{-1}\frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$. ✓



Then at $x = 2\sqrt{3}$, $\frac{d\theta}{dt} = \sin^2\frac{\pi}{6} = \frac{1}{4}$ rad/sec. ✓

13(b)(v) Find $\frac{dA}{dt}$ when $x=2\sqrt{3}$.

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt}$$

✓

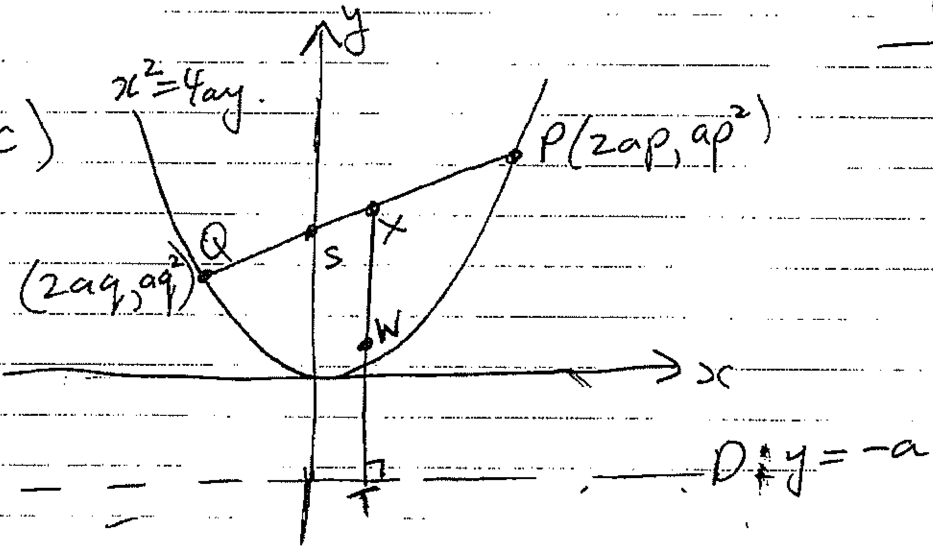
When $\theta = \frac{\pi}{6}$, $\frac{dA}{d\theta} = 1-\cos\frac{\pi}{3}$

$= 1-\frac{1}{2} = \frac{1}{2}$

Then $\frac{dA}{dt}\left(\text{at } x=2\sqrt{3}\right) = \frac{1}{2}\times\frac{1}{4}$ (2)

$= \frac{1}{8}\ m^2/s$ ✓

13(c)



Eqn of PQ

$y=\left(\frac{p+q}{2}\right)x-apq$

(i) Show if PQ passes through $S(0,a) \Rightarrow pq=-1$.

Then $(0,a)$ satisfies eqn

$\Rightarrow a = -apq$

$\Rightarrow -1 = pq$ ✓ (1)

(ii) X is midpt of PQ. XT ⊥ Directrix. W is midpt of XT
Find locus of W

$$X = \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$$

and $T = \left(a(p+q), -a\right)$

Then $W = \left(a(p+q), \dfrac{\frac{a(p^2+q^2)}{2} - a}{2}\right)$

$$= \left(a(p+q), \frac{a(p^2+q^2) - 2a}{4}\right)$$

$$W = \left(a(p+q), \frac{a(p^2+q^2-2)}{4}\right) \checkmark$$

Then $x = a(p+q)$ ①

$$y = \frac{a}{4}\left(p^2+q^2-2\right)$$

$$= \frac{a}{4}\left(p^2+2pq+q^2-2pq-2\right)$$

$$= \frac{a}{4}\left((p+q)^2 - 2(pq+1)\right)$$

But $pq = -1$

$$\Rightarrow y = \frac{a}{4}(p+q)^2 \quad ②$$ ✓

From ① $p+q = \frac{x}{a}$

Sub into ② $\Rightarrow y = \frac{a}{4}\left(\frac{x}{a}\right)^2$

$$y = \frac{x^2}{4a}$$ ✓

$\Rightarrow x^2 = 4ay$. ∴ W is on parabola.

3

XI THSC SHS 2012

Question 14

(a) (i) Trajectory is a straight line.

$y = mx + b$ Point: $(0, h)$

ie $y = x \cdot \tan 30^\circ + h$

$\therefore y = \frac{x}{\sqrt{3}} + h$ —— ① [1]

(ii) Let u be the speed of projection of the missile.

It will hit the plane when they have travelled the same distance horizontally, at time t:

$\therefore ut\cos\alpha = \sqrt{8gh}\, t\cos 30^\circ$

ie $u\cos\alpha = \sqrt{8gh} \cdot \frac{\sqrt{3}}{2}$

$(\because t \neq 0)$

$\therefore u\cos\alpha = \sqrt{6gh}$

Trajectory of the missile (assumed ballistic):

$y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$

ie $y = x\tan\alpha - \frac{gx^2}{2(6gh)}$

$\therefore y = x\tan\alpha - \frac{x^2}{12h}$ —— ②

Substitute ① into ②:

$\frac{x}{\sqrt{3}} + h = x\tan\alpha - \frac{x^2}{12h}$

ie $\frac{x^2}{12h} + \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)xh + h^2 = 0$ —— ③ [3]

(iii) (α) The equation (3) is quadratic in x, and the discriminant is:

$\Delta = \left(\frac{1}{\sqrt{3}} - \tan\alpha\right)^2 h^2 - 4\left(\frac{1}{12}\right)h^2$

$= h^2\left(\frac{1}{3} - \frac{2}{\sqrt{3}}\tan\alpha + \tan^2\alpha\right) - \frac{h^2}{3}$

$= h^2\left(\tan^2\alpha - \frac{2}{\sqrt{3}}\tan\alpha\right)$

$= h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)$

But $\tan\alpha > \frac{2}{\sqrt{3}}$

$\therefore \Delta > 0$

$\therefore$ (3) has two distinct real roots, say x_1 and x_2. [2]

Thus missile can hit plane in two places, B and C.

(β) Now $x_1 + x_2 = 12h\left(\tan\alpha - \frac{1}{\sqrt{3}}\right)$

and $x_1 x_2 = 12h^2$

$\therefore (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$= \left[12h\left(\tan\alpha - \frac{1}{\sqrt{3}}\right)\right]^2 - 4(12h^2)$

$= 144h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)$

$\therefore$ Time elapsed between B & C is the time the plane takes:

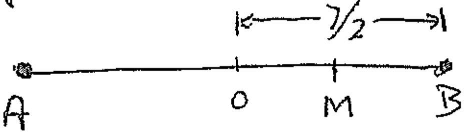
$T = \frac{|x_1 - x_2|}{u\cos\alpha}$

$= \frac{\sqrt{144h^2\tan\alpha\left(\tan\alpha - \frac{2}{\sqrt{3}}\right)}}{\sqrt{6gh}}$

$= \sqrt{\frac{8h\tan\alpha}{g}\left(3\tan\alpha - 2\sqrt{3}\right)}$ [2]

Q14 (Cont'd)

(b)



Period $T = 3$ sec $\quad a = \frac{7}{2}$

$= \frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{3}$

SHM has

$v^2 = n^2(a^2 - x^2)$

and $\ddot{x} = -n^2 x$

Assume $x = 0$ when $t = 0$

$\therefore v^2 = \frac{4\pi^2}{9}\left(\frac{49}{4}\right)$

$v = \frac{7\pi}{3}$ m/s and $\ddot{x} = 0$

At M, $x = \frac{7}{4}$

$\therefore v^2 = \frac{4\pi^2}{9}\left(\frac{49}{4} - \frac{49}{16}\right)$

$= \frac{4\pi^2 \times 3 \times 49}{9 \times 16}$

$\therefore v = \frac{7\sqrt{3}\pi}{6}$ $(\doteqdot 6.35$ m/s$)$

$\ddot{x} = -\frac{4\pi^2}{9}\left(\frac{7}{4}\right)$

$\ddot{x} = -\frac{7\pi^2}{9}$ $(\doteqdot 7.7$ m/s$^2)$

[3]

(c) (i) Each ball can be placed in either urn

Hence $\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n \text{ factors}} = 2^n$ ways

But 2 of these would have one urn empty.

$\therefore 2^n - 2$ ways [1]

(ii) Method I

- There $3^6 = 729$ ways with no restrictions.
- For one empty urn there are $3 \times (2^6 - 2) = 186$
- For two empty urns, there are 3 ways

$\therefore$ There are $729 - 186 - 3$

$= 540$ ways with no empty urns.

[3]

Method II

Possible contents:

$4,1,1 : {}^6C_4 \times {}^2C_1 \times 3 = 90$

$3,2,1 : {}^6C_3 \times {}^3C_2 \times 6 = 360$

$2,2,2 : {}^6C_2 \times {}^4C_2 = 90$

$\therefore$ Total number of ways is 540