

Complex Numbers & Trig Identities

(i) Express $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos \theta$ and $\sin \theta$

$$\begin{aligned}\cos 2\theta + i \sin 2\theta &= (\cos \theta + i \sin \theta)^2 \\ &= \cos^2 \theta + 2i \sin \theta \cos \theta - \sin^2 \theta\end{aligned}$$

By equating real and imaginary parts

$$\underline{\cos 2\theta = \cos^2 \theta - \sin^2 \theta}$$

$$\underline{\sin 2\theta = 2 \sin \theta \cos \theta}$$

(ii) Express $\sin 3\theta$, $\cos 3\theta$ and $\tan 3\theta$ in terms of $\sin \theta$, $\cos \theta$ and $\tan \theta$

$$\begin{aligned}\cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 \\ &= \cos^3 \theta + 3i \sin \theta \cos^2 \theta - 3 \sin^2 \theta \cos \theta - i \sin^3 \theta\end{aligned}$$

By equating real and imaginary parts

$$\begin{aligned}\sin 3\theta &= 3 \sin \theta \cos^2 \theta - \sin^3 \theta & \tan 3\theta &= \frac{\sin 3\theta}{\cos 3\theta} \\ &= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta & &= \frac{3 \sin \theta \cos^2 \theta - \sin^3 \theta}{\cos^3 \theta - 3 \sin^2 \theta \cos \theta} \\ &= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta & &= \frac{3 \sin \theta \cos^2 \theta - \sin^3 \theta}{\cos^3 \theta} \\ &= \underline{3 \sin \theta - 4 \sin^3 \theta} & &= \frac{\cos^3 \theta}{\cos^3 \theta - 3 \sin^2 \theta \cos \theta} \\ \cos 3\theta &= \cos^3 \theta - 3 \sin^2 \theta \cos \theta & &= \frac{\cos^3 \theta}{\cos^3 \theta - 3 \sin^2 \theta \cos \theta} \\ &= \cos^3 \theta - 3(1 - \cos^2 \theta) \cos \theta & &= \frac{3 \sin \theta}{\cos \theta} - \frac{\sin^3 \theta}{\cos^3 \theta} \\ &= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta & &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \\ &= \underline{4 \cos^3 \theta - 3 \cos \theta}\end{aligned}$$

(iii) Show that $x = \cos \frac{\pi}{9}$ is a solution to $8x^3 - 6x - 1 = 0$

$$8x^3 - 6x - 1 = 0$$

$$2(4x^3 - 3x) - 1 = 0$$

$$\text{let } x = \cos \theta$$

$$2(4\cos^3 \theta - 3\cos \theta) - 1 = 0$$

$$2\cos 3\theta - 1 = 0$$

$$\cos 3\theta = \frac{1}{2}$$

$$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

$$x = \cos \frac{\pi}{9}, \cos \frac{5\pi}{9}, \sin \frac{7\pi}{9}$$

$\therefore x = \cos \frac{\pi}{9}$ is a solution

(iii) Evaluate $\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{5\pi}{9}$

$x = \cos \frac{\pi}{9}, \cos \frac{5\pi}{9}, \sin \frac{7\pi}{9}$ are the roots of $8x^3 - 6x - 1 = 0$

$\alpha\beta\gamma = \frac{1}{8}$ product of roots

$$\cos \frac{\pi}{9} \cos \frac{5\pi}{9} \cos \frac{7\pi}{9} = \frac{1}{8}$$

$$\text{but } \cos \frac{7\pi}{9} = -\cos \frac{2\pi}{9}$$

$$\therefore \cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{5\pi}{9} = -\frac{1}{8}$$

(iv) If $z = \cos \theta + i \sin \theta$, find $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$

$$z^n = \cos n\theta + i \sin n\theta$$

$$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i \sin(-n\theta)$$

$$= \cos n\theta - i \sin n\theta \quad \left\{ \begin{array}{l} \cos \text{ is even function } \Rightarrow \cos(-x) = \cos x \\ \sin \text{ is odd function } \Rightarrow \sin(-x) = -\sin x \end{array} \right\}$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

(v) Express $\cos^3 \theta$ in terms of $\cos n\theta$

$$(2 \cos \theta)^3 = \left(z + \frac{1}{z} \right)^3$$

$$\begin{aligned} 8 \cos^3 \theta &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \\ &= \left(z^3 + \frac{1}{z^3} \right) + 3 \left(z + \frac{1}{z} \right) \end{aligned}$$

$$8 \cos^3 \theta = 2 \cos 3\theta + 6 \cos \theta$$

$$\therefore \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

Cambridge: Exercise 7B; 2 to 5, 7 to 12