Multiple Roots

If $P(x)$, has a root, $x = a$, of multiplicity m,

then $P'(x)$ has a root, $x = a$, of multiplicity $m - 1$

Proof:

$$P(x) = (x-a)^m Q(x) \qquad (m > 1,\ x = a \text{ is not a root of } Q(x))$$

$$P'(x) = (x-a)^m Q'(x) + Q(x)m(x-a)^{m-1}(1)$$

$$= (x-a)^{m-1}[(x-a)Q'(x) + mQ(x)]$$

$$= (x-a)^{m-1}R(x) \qquad (\text{where } x = a \text{ is not a root of } R(x))$$

$\therefore$ <u>$P'(x)$ has a root, $x = a$, of multiplicity $m - 1$</u>

e.g. (i) Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$, given that it has a double root

$$P(x) = x^3 - 4x^2 - 3x + 18$$

$$P'(x) = 3x^2 - 8x - 3$$

$$= (3x+1)(x-3)$$

$\because$ double root is $x = -\frac{1}{3}$ or $x = 3$

NOT POSSIBLE

As (3x + 1) is not a factor

$$x^3 - 4x^2 - 3x + 18 = 0$$

$$(x-3)^2(x+2) = 0$$

$$\underline{x = -2 \text{ or } x = 3}$$

(ii) (1991)

Let $x = \alpha$ be a root of the quartic polynomial;

$$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$$

where $(2+B)^2 \neq 4A^2$

a) show that α cannot be 0, 1 or -1

$$P(0) = 1 \neq 0, \quad \therefore \alpha \neq 0$$

$$P(1) = 1 + A + B + A + 1 \qquad P(-1) = 1 - A + B - A + 1$$

$$= 2A + B + 2 \qquad = -2A + B + 2$$

BUT

$$(2+B)^2 \neq 4A^2$$

$$2 + B \neq \pm 2A$$

$$\pm 2A + B + 2 \neq 0$$

$$\therefore P(1) \neq 0, P(-1) \neq 0$$

$$\text{hence } \alpha \neq \pm 1$$

b) Show that $\frac{1}{\alpha}$ is a root

$$P\left(\frac{1}{\alpha}\right)=\frac{1}{\alpha^4}+\frac{A}{\alpha^3}+\frac{B}{\alpha^2}+\frac{A}{\alpha}+1$$

$$=\frac{1+A\alpha+B\alpha^2+A\alpha^3+\alpha^4}{\alpha^4}$$

$$=\frac{P(\alpha)}{\alpha^4}$$

$$=\frac{0}{\alpha^4} \qquad (\because P(\alpha)=0 \text{ as } \alpha \text{ is a root})$$

$$=0$$

$\therefore \frac{1}{\alpha}$ is a root of $P(x)$

c) Deduce that if α is a multiple root, then its multiplicity is 2 and

$$4B = 8 + A^2$$

If α is a double root of $P(x)$, then so is $\frac{1}{\alpha}$, which accounts for 4 roots

However $P(x)$ is a quartic which has a maximum of 4 roots

Thus no roots can have a multiplicity > 2

$$P'(x) = 4x^3 + 3Ax^2 + 2Bx + A$$

let the roots be $\alpha, \frac{1}{\alpha}$ and β

$$\alpha + \frac{1}{\alpha} + \beta = -\frac{3}{4}A \qquad \text{(sum of roots)}\ldots(1)$$

$$1 + \alpha\beta + \frac{\beta}{\alpha} = \frac{1}{2}B \qquad \left(\sum \alpha\beta\right)\ldots(2)$$

$$\beta = -\frac{1}{4}A \qquad \left(\sum \alpha\beta\gamma\right)\ldots(3)$$

Substitute (3) into (1)

$$\alpha + \frac{1}{\alpha} - \frac{1}{4}A = -\frac{3}{4}A$$

$$\alpha + \frac{1}{\alpha} = -\frac{1}{2}A$$

Substitute (3) into (2)

$$1 - \frac{1}{4}A\alpha - \frac{1}{4}A\frac{1}{\alpha} = \frac{1}{2}B$$

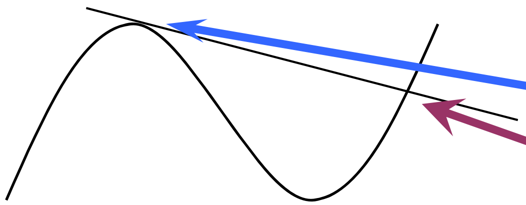
$$1 - \frac{1}{4}A\left(\alpha + \frac{1}{\alpha}\right) = \frac{1}{2}B$$

$$1 + \frac{1}{8}A^2 = \frac{1}{2}B$$

$$8 + A^2 = 4B$$

Cambridge: **Exercise 5B; 1 to 16**

Patel: **Exercise 5B; 6b, 7b, 8 a,c,e,g,h**



Note: tangent to a cubic has two solutions only:

A double root

and a single root