

Forming Polynomials With The Roots Of Another

If $\alpha, \beta, \gamma, \dots$ are the roots of a polynomial, to form an equation with roots;

(1) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \dots$

let $y = \frac{1}{x}$ and substitute $x = \frac{1}{y}$

(2) $k\alpha, k\beta, k\gamma, \dots$

let $y = kx$ and substitute $x = \frac{y}{k}$

(3) $\alpha + c, \beta + c, \gamma + c, \dots$

let $y = x + c$ and substitute $x = y - c$

(4) $\alpha^2, \beta^2, \gamma^2, \dots$

let $y = x^2$ and substitute $x = y^{\frac{1}{2}}$

e.g. If α, β, γ are the roots of $x^3 + x + 2 = 0$, form an equation whose roots are;

a) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

let $y = \frac{1}{x}$

$x = \frac{1}{y}$

$$\left(\frac{1}{y}\right)^3 + \frac{1}{y} + 2 = 0$$

$$\underline{1 + y^2 + 2y^3 = 0}$$

b) $\alpha + 1, \beta + 1, \gamma + 1$

$$\text{let } y = x + 1$$

$$x = y - 1$$

$$(y - 1)^3 + (y - 1) + 2 = 0$$

$$y^3 - 3y^2 + 3y - 1 + y - 1 + 2 = 0$$

$$y^3 - 3y^2 + 4y = 0$$

c) $\alpha^2, \beta^2, \gamma^2$

$$\text{let } y = x^2$$

$$x = y^{\frac{1}{2}}$$

$$\left(y^{\frac{1}{2}}\right)^3 + y^{\frac{1}{2}} + 2 = 0$$

$$y^{\frac{3}{2}} + y^{\frac{1}{2}} + 2 = 0$$

$$y^{\frac{1}{2}}(y + 1) = -2$$

$$y(y + 1)^2 = 4$$

$$y^3 + 2y^2 + y = 4$$

$$y^3 + 2y^2 + y - 4 = 0$$

$$\text{d) } \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$$

$$\text{let } y = \frac{1}{x^2}$$

$$x = y^{-\frac{1}{2}}$$

$$y^{-\frac{3}{2}} + y^{-\frac{1}{2}} + 2 = 0$$

$$y^{-\frac{3}{2}}(y+1) = -2$$

$$(y+1) = -2y^{\frac{3}{2}}$$

$$(y+1)^2 = 4y^3$$

$$y^2 + 2y + 1 = 4y^3$$

$$\underline{4y^3 - y^2 - 2y - 1 = 0}$$

e) Find $\alpha^2 + \beta^2 + \gamma^2$

$$\begin{aligned} & \alpha^2 + \beta^2 + \gamma^2 \\ &= (\sum \alpha)^2 - 2\sum \alpha\beta \\ &= (0)^2 - 2(1) \\ &= \underline{-2} \end{aligned}$$

OR using equation found in c)

$$\begin{aligned} & \alpha^2 + \beta^2 + \gamma^2 \\ &= \frac{-b}{a} \\ &= \frac{-2}{1} \\ &= \underline{-2} \end{aligned}$$

Cambridge: Exercise 5C; 1 to 11, 13, 14, 15

***Patel: Exercise 5E; 9, 10, 11, 14, 16, 18, 19, 23, 24,
27, 30, 32, 34, 35***