Integration Using Substitution

Try to convert the integral into a "standard integral"

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

"NON" STANDARD INTEGRALS

Not listed in the standard integrals, however will save time if you know

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$$

$$\int \frac{du}{u^2} = -\frac{1}{u}$$

$$\int \ln x dx = x \ln x - x$$

$$\int \tan x dx = \log \sec x$$

$$\int \text{complementary trig ratio} = -\text{complement of the answer}$$

e.g.

$$\int \cot x \, dx = -\log \operatorname{cosec} x$$

$$= \log \sin x$$

cot is the complement of tan

so the answer is minus and the complement of sec is cosec e.g. (i) $\int x\sqrt{x^2+4}\,dx$ $= \frac{1}{2}\int 2x\sqrt{x^2+4}\,dx$

$\frac{1}{2}du$ → $x\,dx$; u → x^2+4

$u = x^2 + 4$

$du = 2x\,dx$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+4\right)^{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+4\right)\sqrt{x^2+4} + c$$

when substituting
$u = f(x)$
make the function
causing the
problem u

OR

e.g. (i) $\int x\sqrt{x^2+4}dx = \int 2\tan\theta\sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta d\theta$

$$= 8\int \tan\theta\sec^3\theta$$

$$= 8\int \frac{\sin\theta d\theta}{\cos^4\theta}$$

$u = \cos\theta$

$du = -\sin\theta d\theta$

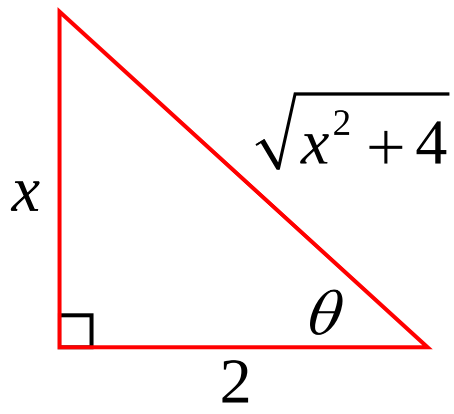
$$= -8\int \frac{du}{u^4}$$

$$= -8\times -\frac{1}{3}u^{-3} + c$$

$$= \frac{8}{3}\sec^3\theta + c$$

$$= \frac{1}{3}\left(\sqrt{x^2+4}\right)^3 + c$$

$$= \frac{1}{3}\left(x^2+4\right)\sqrt{x^2+4} + c$$



if a root of a sum or difference of squares is involved, could try a trig substitution

Keep an eye out for $f'(x)\times f(x)$

$$(ii)\int \frac{x+1}{4x^2+8x-7}\,dx \quad = \frac{1}{8}\int \frac{du}{u}$$

$$u = 4x^2+8x-7$$

$$du = (8x+8)\,dx$$

$$= \frac{1}{8}\log u + c$$

$$= \frac{1}{8}\log\left(4x^2+8x-7\right)+c$$

$$(iii)\int \frac{dx}{x(\log x)^3} \quad = \int \frac{du}{u^3}$$

$$u = \log x$$

$$du = \frac{dx}{x}$$

$$= \int u^{-3}\,du$$

$$= -\frac{1}{2}u^{-2}+c$$

$$= -\frac{1}{2u^2}+c$$

$$= -\frac{1}{2(\log x)^2}+c$$

$$(iv)\int \frac{x}{\sqrt{1-x}}dx = \int \frac{1-u^2}{u}\cdot -2u\,du$$

$$= 2\int (u^2-1)du$$

$$= 2\left(\frac{1}{3}u^3 - u\right) + c$$

$$= \frac{2}{3}\left(\sqrt{1-x}\right)^3 - 2\sqrt{1-x} + c$$

$$= \frac{2}{3}(1-x)\sqrt{1-x} - 2\sqrt{1-x} + c$$

$$x = 1-u^2 \Rightarrow u = \sqrt{1-x}$$

$$dx = -2u\,du$$

substituting
$u = f(x)$
instead of
$x = f(u)$
avoids domain
problems later

$$(v)\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x \cos x\,dx = \int_{\frac{\sqrt{3}}{2}}^{1} u^5\,du$$

$$u = \sin x$$

$$du = \cos x\,dx$$

$$x = \frac{\pi}{3}, u = \sin\frac{\pi}{3}$$

$$u = \frac{\sqrt{3}}{2}$$

$$= \frac{1}{6}\left[u^6\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$= \frac{1}{6}\left\{1^6 - \left(\frac{\sqrt{3}}{2}\right)^6\right\}$$

$$x = \frac{\pi}{2}, u = \sin\frac{\pi}{2}$$

$$u = 1$$

$$= \frac{37}{384}$$

$$(vi)\int_3^4 \frac{x\,dx}{\sqrt{25-x^2}} = -\frac{1}{2}\int_{16}^{9} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\int_9^{16} u^{-\frac{1}{2}}\,du$$

$$= \left[u^{\frac{1}{2}}\right]_9^{16}$$

$$= \sqrt{16} - \sqrt{9}$$

$$= 1$$

$$u = 25 - x^2$$

$$du = -2x\,dx$$

$$x = 3, u = 16$$

$$x = 4, u = 9$$

$$(vii)\int_0^5 \sqrt{25-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{25-25\sin^2 u}\cdot 5\cos u\,du \qquad x = 5\sin u \Rightarrow u = \sin^{-1}\frac{x}{5}$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{25\cos^2 u}\cdot 5\cos u\,du \qquad dx = 5\cos u\,du$$

$$x = 0, u = \sin^{-1} 0$$

$$u = 0$$

$$= 25\int_0^{\frac{\pi}{2}} \cos^2 u\,du \qquad x = 5, u = \sin^{-1} 1$$

$$u = \frac{\pi}{2}$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2u)\,du$$

$$= \frac{25}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}}$$

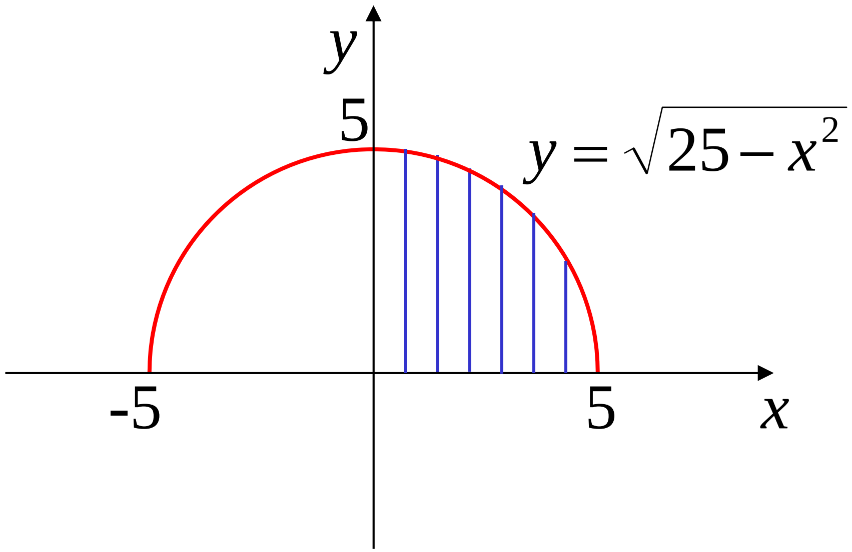
$$= \frac{25}{2}\left\{\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right\}$$

$$= \frac{25\pi}{4}$$

$$(vii)\int_0^5 \sqrt{25-x^2}\,dx$$

$$=\frac{1}{4}\pi(5)^2$$

$$=\frac{25\pi}{4}$$



don't forget about algebraic manipulation

$$(viii)\int x^3\sqrt{4-x^2}\,dx$$

$$=\int\left[-x\left(4-x^2\right)+4x\right]\sqrt{4-x^2}\,dx$$

$$=\int\left[-x\left(4-x^2\right)^{\frac{3}{2}}+4x\left(4-x^2\right)^{\frac{1}{2}}\right]dx$$

$$=\frac{1}{2}\times\frac{2}{5}\left(4-x^2\right)^{\frac{5}{2}}-2\times\frac{2}{3}\left(4-x^2\right)^{\frac{3}{2}}+c$$

$$=\frac{1}{5}\left(4-x^2\right)^2\sqrt{4-x^2}-\frac{4}{3}\left(4-x^2\right)\sqrt{4-x^2}+c$$

Exercise 6C; 1, 2ace, 3, 4ace, 5ace etc, 6, 7 & 8ac, 11, 12
Exercise 6D; 1, 2a, 3, 4b, 5ac, 6ace etc, 7ab(i,ii)
8ab (i,iii,vi), 9ab (ii,iv,vi), 11, 13

Patel Exercise 2A ace in all
Cambridge Exercise 2B; 1b (iii), 3ce, 4cde, 5cd, 6, 7bc,
8ab, 9a, 10bd, 11, 12, 13
NOTE: substitution is not given in Extension 2