Integration By Partial Fractions

To find; $\int \frac{A(x)}{P(x)}dx$

(1) If $\deg A(x) \geq \deg P(x)$, perform a division

(2) If $\deg A(x) < \deg P(x)$, factorise $P(x)$

a) for linear factor $(x-a)$, write $\frac{A}{x-a}$

b) for multiple linear factors $(x-a)^{\text{n}}$, write

$$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\ldots+\frac{C}{(x-a)^n}$$

c) for polynomial factors e.g. ax^2+bx+c, write $\frac{Ax+B}{ax^2+bx+c}$

e.g. (i) $\int \frac{x^2}{x+1}dx$

$$= \int \left[x - 1 + \frac{1}{x+1} \right] dx$$

$$= \frac{1}{2}x^2 - x + \log(x+1) + c$$

$$\begin{array}{r} x \quad -1 \\ x+1 \overline{) x^2 + 0x + 0} \\ \underline{x^2 + x} \\ -x + 0 \\ \underline{-x - 1} \\ 1 \end{array}$$

(ii) $\int \frac{3dx}{x^2 - x}$

$$= \int \frac{3dx}{x(x-1)}$$

$$= \int \left[\frac{-3}{x} + \frac{3}{(x-1)} \right] dx$$

$$= -3\log x + 3\log(x-1) + c$$

$$= 3\log\left(\frac{x-1}{x} \right) + c$$

$$\frac{A}{x} + \frac{B}{x-1} = \frac{3}{x(x-1)}$$

$$A(x-1) + Bx = 3$$

$\underline{x = 0}$

$-A = 3$

$A = -3$

$\underline{x = 1}$

$B = 3$

$$(iii)\int \frac{x+5}{x^2-3x-10}dx$$

$$=\int \frac{x+5}{(x-5)(x+2)}dx$$

$$=\int \left[\frac{10}{7(x-5)}-\frac{3}{7(x+2)}\right]dx$$

$$=\frac{10}{7}\log(x-5)-\frac{3}{7}\log(x+2)+c$$

$$\frac{A}{(x-5)}+\frac{B}{(x+2)}=\frac{x+5}{(x-5)(x+2)}$$

$$A(x+2)+B(x-5)=x+5$$

$x=-2$ | $x=5$

$-7B=3$ | $7A=10$

$B=\frac{-3}{7}$ | $A=\frac{10}{7}$

$$(iv)\int \frac{dx}{x^3+x}$$

$$=\int \frac{dx}{x(x^2+1)}$$

$$=\int \left[\frac{1}{x}-\frac{x}{x^2+1}\right]dx$$

$$=\log x-\frac{1}{2}\log(x^2+1)+c$$

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{1}{x(x^2+1)}$$

$$A(x^2+1)+(Bx+C)x=1$$

$x=0$ | $x=i$

$A=1$ | $-B+Ci=1$

$B=-1$ | $C=0$

$$(v)\int \frac{x\,dx}{(x+1)^2(x^2+1)}$$

$$\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{Cx+D}{(x^2+1)}=\frac{x}{(x+1)^2(x^2+1)}$$

$$A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2=x$$

$x=-1$

$$2B=-1$$

$$B=\frac{-1}{2}$$

$x=i$

$$-2C+2Di=i$$

$$C=0 \quad D=\frac{1}{2}$$

$x=0$

$$2A+B+D=0$$

$$2A-\frac{1}{2}+\frac{1}{2}=0$$

$$A=0$$

$$=\int\left[\frac{-1}{2(x+1)^2}+\frac{1}{2(x^2+1)}\right]dx$$

$$=\frac{1}{2}\int\left[-(x+1)^{-2}+\frac{1}{(x^2+1)}\right]dx$$

$$=\frac{1}{2}\left\{\frac{-(x+1)^{-1}}{-1}+\tan^{-1}x\right\}+c$$

$$=\frac{1}{2}\left(\frac{1}{x+1}+\tan^{-1}x\right)+c$$

Alternative method for finding constants when denominator is a product of distinct linear factors

$$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}+\ldots=\frac{f(x)}{(x-a)(x-b)(x-c)\ldots}$$

1) To find A substitute a into $\dfrac{f(x)}{(x-b)(x-c)\ldots}$ i.e. $A=\dfrac{f(a)}{(a-b)(a-c)\ldots}$

2) To find B substitute b into $\dfrac{f(x)}{(x-a)(x-c)\ldots}$ i.e. $B=\dfrac{f(b)}{(b-a)(b-c)\ldots}$

3) To find C substitute c into $\dfrac{f(x)}{(x-a)(x-b)\ldots}$ i.e. $C=\dfrac{f(c)}{(c-a)(c-b)\ldots}$

and so on for all of the constants

e.g. (i) $\frac{A}{x}+\frac{B}{x-1}=\frac{3}{x(x-1)}$

$$A=\frac{3}{(0-1)} \qquad\qquad B=\frac{3}{1}$$

$$=-3 \qquad\qquad =3$$

(ii) $\frac{A}{(x-5)}+\frac{B}{(x+2)}=\frac{x+5}{(x-5)(x+2)}$

$$A=\frac{5+5}{(5+2)} \qquad\qquad B=\frac{-2+5}{(-2-5)}$$

$$=\frac{10}{7} \qquad\qquad =-\frac{3}{7}$$

using complex numbers, the idea can be applied to quadratic factors

$$(iii)\quad \frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{1}{x\left(x^2+1\right)}$$

$$A=\frac{1}{\left(0^2+1\right)}$$

$$=1$$

$$Bi+C=\frac{1}{i}$$

$$=-i$$

$$B=-1 \qquad C=0$$

$$(iv)\quad \frac{A}{(x-1)}+\frac{Bx+C}{4x^2+1}=\frac{x^2+4x}{(x-1)\left(4x^2+1\right)}$$

$$A=\frac{1^2+4(1)}{\left(4(1)^2+1\right)}$$

$$=1$$

$$\frac{Bi}{2}+C=\frac{\left(\frac{i}{2}\right)^2+4\left(\frac{i}{2}\right)}{\left(\frac{i}{2}-1\right)}$$

$$=\frac{-\frac{1}{4}+2i}{\frac{i}{2}-1}$$

$$=\frac{-1+8i}{2i-4}\times\frac{2i+4}{2i+4}$$

$$=\frac{-2i-4-16+32i}{-4-16}$$

$$=1-\frac{3}{2}i$$

$$B=-3 \qquad C=1$$

multiple factors require just a little bit more work

$$(v)\ \frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}=\frac{9x}{(x+2)(x-1)^2}$$

$$A=\frac{9(-2)}{(-2-1)^2}$$

$$=-2$$

the constant of the higher power is found the same way

$$C=\frac{9(1)}{(1+2)}$$

$$=3$$

the third constant is found via a simple substitution

let $x=0$

$$\frac{A}{0+2}+\frac{B}{0-1}+\frac{C}{(0-1)^2}=\frac{9(0)}{(0+2)(0-1)^2}$$

$$-1-B+3=0$$

$$B=2$$

$$(vi)\quad \frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{Cx+D}{(x^2+1)}=\frac{x}{(x+1)^2(x^2+1)}$$

$$B=\frac{-1}{((-1)^2+1)}\qquad Ci+D=\frac{i}{(i+1)^2}\qquad \text{let } x=0\qquad A+B+D=0$$

$$A-\frac{1}{2}+\frac{1}{2}=0$$

$$=-\frac{1}{2}\qquad\qquad =\frac{i}{2i}$$

$$A=0$$

$$=\frac{1}{2}$$

$$C=0\qquad D=\frac{1}{2}$$

quadratic denominator that can't be factorised

$$(vii)\quad \frac{A}{x+1}+\frac{Bx+C}{x^2+4x+5}=\frac{2}{(x+1)\left(x^2+4x+5\right)}$$

Complete the square to find an appropriate substitution

$$A=\frac{2}{\left((-1)^2+4(-1)+5\right)}$$

$$=1$$

$$x^2+4x+5=(x+2)^2+1$$

let $x=-2+i$ $\quad B(-2+i)+C=\frac{2}{(-2+i+1)}$

$$-2B+C+Bi=\frac{2}{-1+i}\times\frac{-1-i}{-1-i}$$

$$=\frac{-2-2i}{2}$$

$$=-1-i$$

$B=-1$

$$-2B+C=-1$$

$$2+C=-1$$

$$C=-3$$

Patel **Exercise 2G; odds**

“*Cambridge*” Exercise 2C; 1, 2, 3, 4bc, 5a, 6b, 7a, 8b (*i*), 9b, 10, 13bcef