

Integrating Quadratic Denominators

$$\left. \begin{array}{l} (1) \int \frac{dx}{a^2 - x^2} = \int \frac{dx}{(a+x)(a-x)} \\ (2) \int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x+a)(x-a)} \end{array} \right\} \begin{array}{l} \text{done via} \\ \text{partial fractions} \end{array} = \frac{1}{2a} \log \left[\frac{a+x}{a-x} \right] + c = \frac{1}{2a} \log \left[\frac{x-a}{x+a} \right] + c$$

$$(3) \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$(5) \int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left\{ x + \sqrt{x^2 - a^2} \right\} + c$$

$$(4) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$(6) \int \frac{dx}{\sqrt{a^2 + x^2}} = \log \left\{ x + \sqrt{a^2 + x^2} \right\} + c$$

$$\begin{aligned}
 \text{e.g. (i)} \int \frac{5dx}{x^2 + 4x + 9} & \\
 &= \int \frac{5dx}{(x+2)^2 + 5} \\
 &= 5 \times \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + c \\
 &= \sqrt{5} \tan^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \int \frac{(3x+2)dx}{\sqrt{x^2 - 4x + 1}} & \\
 &= \frac{3}{2} \int \frac{2x-4}{\sqrt{x^2 - 4x + 1}} dx + \int \frac{8dx}{\sqrt{x^2 - 4x + 1}} \\
 &\quad u = x^2 - 4x + 1 \\
 &\quad du = (2x-4)dx \\
 &= \frac{3}{2} \int \frac{du}{\sqrt{u}} + \int \frac{8dx}{\sqrt{(x-2)^2 - 3}} \\
 &= \frac{3}{2} \times 2\sqrt{u} + 8 \log \left\{ x-2 + \sqrt{x^2 - 4x + 1} \right\} + c \\
 &= 3\sqrt{x^2 - 4x + 1} + 8 \log \left\{ x-2 + \sqrt{x^2 - 4x + 1} \right\} + c
 \end{aligned}$$

$$\begin{aligned}
(iii) \int \sqrt{\frac{x+3}{2-x}} dx &= \int \sqrt{\frac{x+3}{2-x}} \cdot \sqrt{\frac{x+3}{x+3}} dx && u = 6 - x - x^2 \\
&= \int \frac{x+3}{\sqrt{6-x-x^2}} dx && du = (-2x-1)dx \\
&= -\frac{1}{2} \int \frac{-2x-1}{\sqrt{6-x-x^2}} dx + \frac{1}{2} \int \frac{5dx}{\sqrt{6-x-x^2}} \\
&= -\frac{1}{2} \int \frac{du}{\sqrt{u}} + \frac{5}{2} \int \frac{dx}{\sqrt{\frac{25}{4} - \left(x + \frac{1}{2}\right)^2}} \\
&= -\frac{1}{2} \times 2\sqrt{u} + \frac{5}{2} \sin^{-1} \left[\frac{2\left(x + \frac{1}{2}\right)}{5} \right] + c \\
&= -\sqrt{6-x-x^2} + \frac{5}{2} \sin^{-1} \left[\frac{2x+1}{5} \right] + c
\end{aligned}$$

The “missing” standard integrals

integrand
finishes
with x

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left[\frac{a+x}{a-x} \right] + c$$

primitive
finishes
with $a - x$

integrand
begins
with x

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left[\frac{x-a}{x+a} \right] + c$$

primitive
begins
with $x - a$

Patel Exercise 2F; odd

Patel Exercise 2H; 1, 2, 5, 6, 9, 15, 17 to 20

***“Cambridge” Exercise 2D; 2bcef, 3bdf,
4def, 5ce, 6ab, 7ac***