Trig Integrals

(1) Standard Integrals

$$\int \sin ax dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos ax dx = \frac{1}{a}\sin ax + c$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax + c$$

$$\int \tan ax dx = \int \frac{\sin ax}{\cos ax} dx$$

$$= -\frac{1}{a}\log \cos ax + c \qquad OR \qquad \frac{1}{a}\log \sec ax + c$$

$(2)\sin^n x \quad \text{or} \quad \cos^n x$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

$$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx$$

$$= \int \sin x(1-\cos^2 x)\,dx$$

$$u = \cos x$$

$$du = -\sin x\,dx$$

$$= -\int (1-u^2)\,du$$

$$= \frac{1}{3}u^3 - u + c$$

$$= \frac{1}{3}\cos^3 x - \cos x + c$$

Odd Power

Factorise as $\sin x(\sin^2 x)^{\text{some power}}$

Substitute $\sin^2 x = 1-\cos^2 x$

Use $u = \cos x$

Even Power

Factorise as $\left(\sin^2 x\right)^{some\ power}$

Substitute $\sin^2 x = \frac{1}{2}\left(1-\cos 2x\right)$

$$\int \sin^4 x dx = \int \left(\sin^2 x\right)^2 dx$$

$$= \frac{1}{4}\int \left(1-\cos 2x\right)^2 dx$$

$$= \frac{1}{4}\int \left(1-2\cos 2x+\cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left[1-2\cos 2x+\frac{1}{2}\left(1+\cos 4x\right)\right] dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right)+c$$

$$\int \sin^5 x dx = \int \sin x\left(\sin^2 x\right)^2 dx$$

$$= \int \sin x\left(1-\cos^2 x\right)^2 dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$= -\int \left(1-u^2\right)^2 du$$

$$= -\int \left(1-2u^2+u^4\right) du$$

$$= -\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)+c$$

$$= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c$$

$(3) \sin^n x$ and $\cos^n x$

Usually done by substitution $u = \sin x$ or $u = \cos x$

e.g. $(i) \int \cos^5 x \sin^3 x dx$

$$= \int \cos^5 x (1 - \cos^2 x) \sin x dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$= -\int u^5 (1 - u^2) du$$

$$= \int (u^7 - u^5) du$$

$$= \frac{1}{8} u^8 - \frac{1}{6} u^6 + c$$

$$= \frac{1}{8} \cos^8 x - \frac{1}{6} \cos^6 x + c$$

Both powers odd

Choose either as u

Usually the higher power

$$(ii)\int \sin^6 x \cos^3 x dx$$

$$= \int \sin^6 x(1-\sin^2 x)\cos x dx$$

$$u = \sin x$$

$$du = \cos x dx$$

$$= \int u^6(1-u^2)du$$

$$= \int (u^6 - u^8)du$$

One power odd & one power even

Choose even as u

$$= \frac{1}{7}u^7 - \frac{1}{9}u^9 + c$$

$$= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + c$$

$$(iii)\int \sin^2 x \cos^2 x dx = \int \sin^2 x(1-\sin^2 x)dx$$

$$= \int (\sin^2 x - \sin^4 x)dx$$

Both powers even

Use $\sin^2 x = 1-\cos^2 x$

or $\cos^2 x = 1-\sin^2 x$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{3}{8}x + \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x + c$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

(4) $\tan^n x$ or $\cot^n x$

$$\int \tan x\,dx = -\log\cos x + c$$

$$\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)dx$$

$$= \tan x - x + c$$

$$\int \tan^3 x\,dx = \int \tan x\left(\sec^2 x - 1\right)dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$u = \tan x$$

$$du = \sec^2 x\,dx$$

$$= \int u\,du - \int \tan x\,dx$$

$$= \frac{1}{2}u^2 + \log\cos x + c$$

$$= \frac{1}{2}\tan^2 x + \log\cos x + c$$

$$\int \tan^4 x\,dx = \int \tan^2 x\left(\sec^2 x - 1\right)dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$u = \tan x$$

$$du = \sec^2 x\,dx$$

$$= \int u^2 du - \int \tan^2 x\,dx$$

$$= \frac{1}{3}u^3 - \tan x + x + c$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + c$$

(5) $\sec^n x$ or $\operatorname{cosec}^n x$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$$= \log(\sec x + \tan x) + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$$

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x\,dx \qquad dv = \sec^2 x\,dx$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \log(\sec x + \tan x)$$

$$\therefore 2\int \sec^3 x\,dx = \sec x \tan x + \log(\sec x + \tan x) + c$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log(\sec x + \tan x) + c$$

Odd powers

Done by parts

$$\int \sec^4 x\,dx = \int \sec^2 x(1+\tan^2 x)\,dx$$

$$= \int (1+u^2)\,du$$

$$= u + \frac{1}{3}u^3 + c$$

$$= \tan x + \frac{1}{3}\tan^3 x + c$$

$u = \tan x$

$du = \sec^2 x\,dx$

Even Power

Factorise as $\sec^2 x(\sec^2 x)^{\text{some power}}$

Substitute $\sec^2 x = 1 + \tan^2 x$

Use $u = \tan x$

***Patel* Exercise 2C; 1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 18**

***Cambridge* Exercise 2F; 2abdf, 5bf, 6cdf, 7c, 8bd, 13c, 14, 15ac**