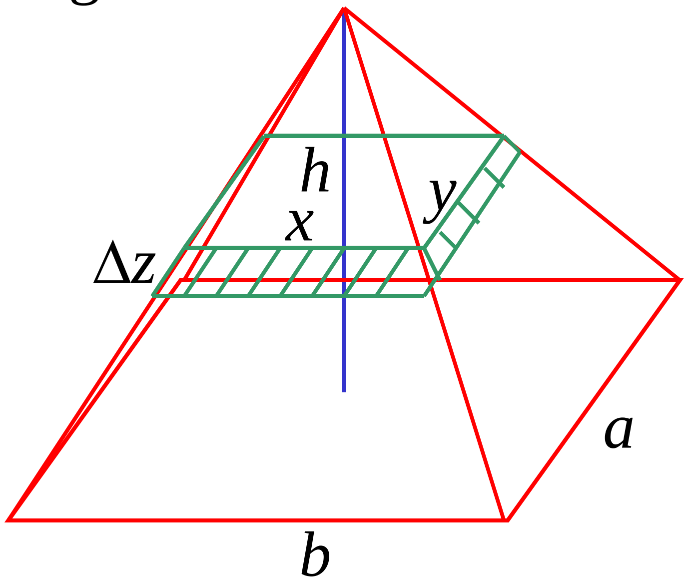
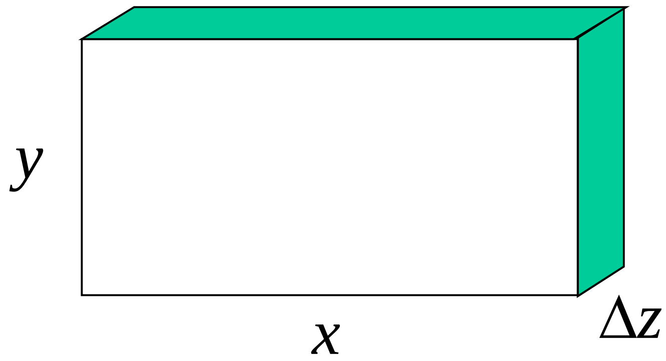
Volumes By Non Circular Cross-Sections

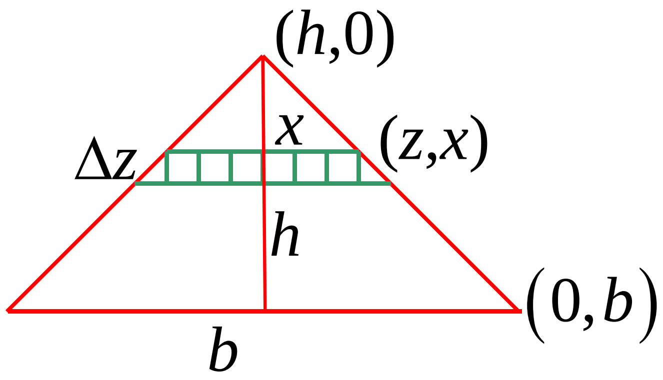
e.g. Find the volume of this rectangular pyramid



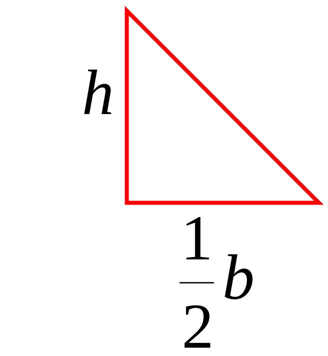


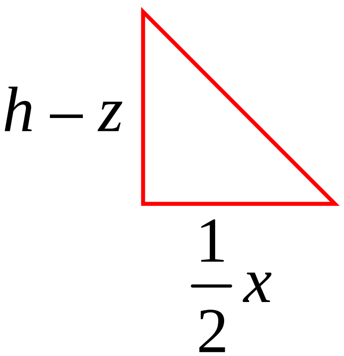
$$A(z) = xy$$

Find x in terms of z



Method 1: using similar Δ's





$$\frac{h}{\frac{1}{2}b}=\frac{h-z}{\frac{1}{2}x}$$

$$\frac{h}{b}=\frac{h-z}{x}$$

$$x=\frac{b(h-z)}{h}$$

Method 2: using coordinate geometry

$$m=\frac{b-0}{0-h}$$

$$=\frac{-b}{h}$$

$$x-0=\frac{-b}{h}(z-h)$$

$$x=\frac{b(h-z)}{h}$$

Similarly;

$$y=\frac{a(h-z)}{h}$$

$$A(z) = \left[\frac{b(h-z)}{h}\right]\left[\frac{a(h-z)}{h}\right]$$

$$= \frac{ab(h-z)^2}{h^2}$$

$$\Delta V = \frac{ab(h-z)^2}{h^2} \cdot \Delta z$$

$$V = \lim_{\Delta z \to 0} \sum_{z=0}^{h} \frac{ab(h-z)^2}{h^2} \cdot \Delta z$$

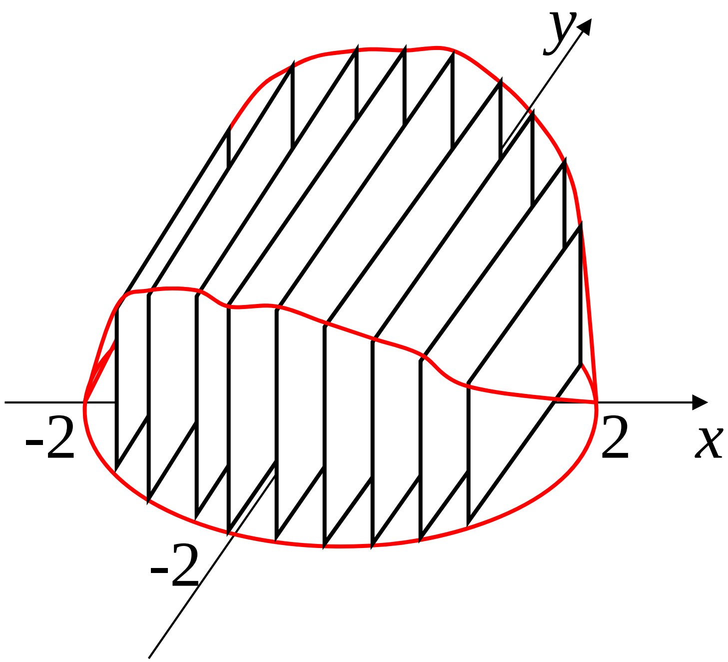
$$= \frac{ab}{h^2} \int_0^h (h-z)^2 \, dz$$

$$= \frac{ab}{h^2} \left[\frac{(h-z)^3}{-3}\right]_0^h$$

$$= \frac{ab}{h^2} \left\{0 + \frac{h^3}{3}\right\}$$

$$= \frac{abh}{3} \text{ units}^3$$

(ii) A solid has a circular base, $x^2 + y^2 = 4$, and each cross section is a square perpendicular to the base.



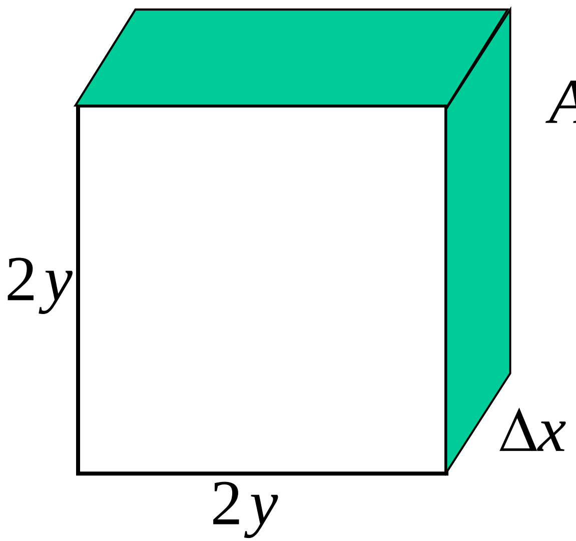
$$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} 4\left(4 - x^2\right) \cdot \Delta x$$

$$= 8\int_0^2 \left(4 - x^2\right) dx$$

$$= 8\left[4x - \frac{1}{3}x^3\right]_0^2$$

$$= 8\left(8 - \frac{8}{3} - 0\right)$$

$$= \frac{128}{3} \text{ units}^3$$

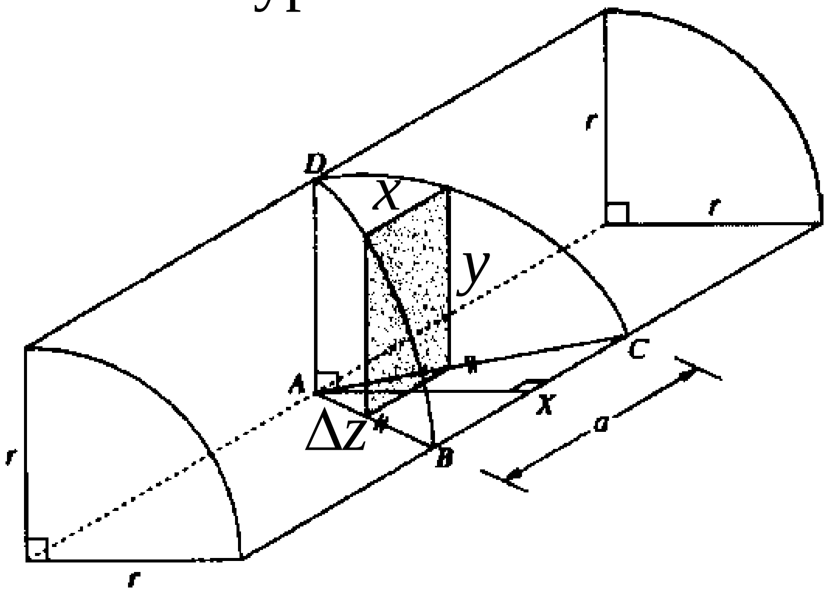


$$A(x) = 4y^2$$

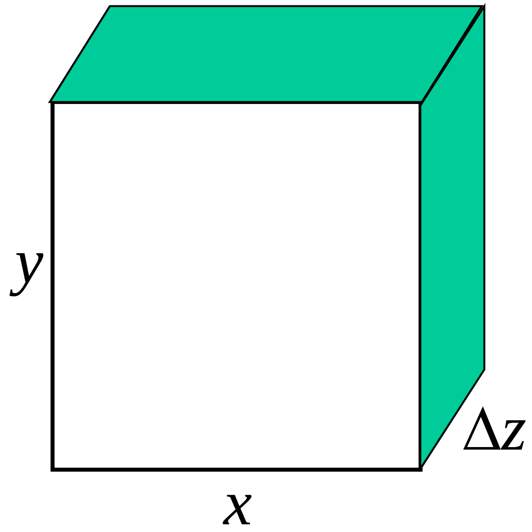
$$= 4\left(4 - x^2\right)$$

$$\Delta V = 4\left(4 - x^2\right) \cdot \Delta x$$

(*iii*) (2012) The solid *ABCD* is cut from a quarter cylinder of radius *r* as shown. Its base is an isosceles triangle *ABC* with *AB* = *AC*. The length of *BC* is *a* and the midpoint of *BC* is *X*. The cross-sections perpendicular to *AX* are rectangles. A typical cross-section is shown shaded in the diagram.

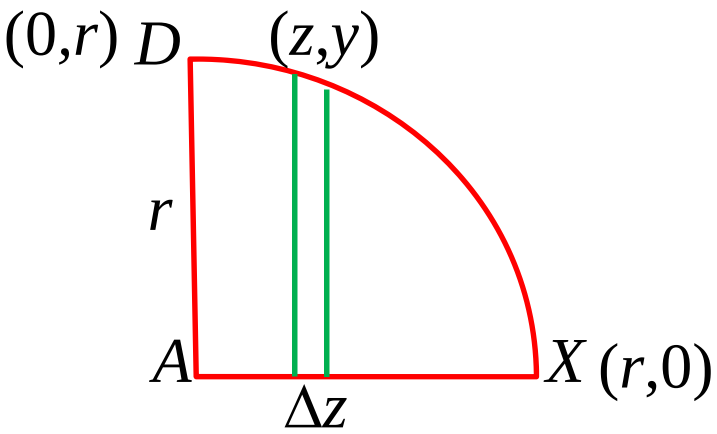


Find the volume of the solid *ABCD*.



$$A(z) = xy$$

Find y in terms of z



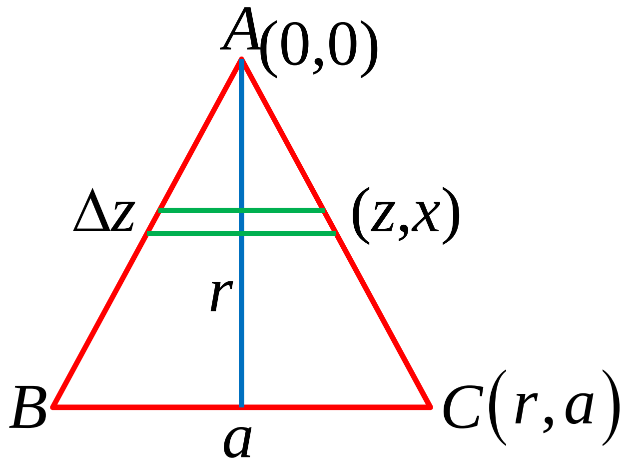
$$y = \sqrt{r^2 - z^2}$$

$$A(z) = xy$$

$$A(z) = \frac{az}{r}\sqrt{r^2 - z^2}$$

$$\Delta V = \frac{az}{r}\sqrt{r^2 - z^2} \cdot \Delta z$$

Find x in terms of z



$$m = \frac{a-0}{r-0}$$

$$= \frac{a}{r}$$

$$x - 0 = \frac{a}{r}(z-0)$$

$$x = \frac{az}{r}$$

$$V = \lim_{\Delta z \to 0} \sum_{z=0}^{r} \frac{az}{r}\sqrt{r^2 - z^2} \cdot \Delta z$$

$$= \frac{a}{r}\int_0^r z\sqrt{r^2 - z^2}\,dz$$

$$= -\frac{a}{2r}\int_0^r -2z\sqrt{r^2 - z^2}\,dz$$

$$= \frac{a}{2r}\left[\frac{2}{3}\left(r^2 - z^2\right)^{\frac{3}{2}}\right]_r^0$$

$$= \frac{a}{3r}\left(r^3 - 0\right)$$

$$= \frac{ar^2}{3}\text{ units}^3$$

Note: Mean = 1.01/4

"*Cambridge*" Exercise 6C; 2, 3, 5 to 8, 10 to 19

***Patel* Exercise 3A; 16ade, 19**

***Patel* Exercise 3C; 2, 7, 8**