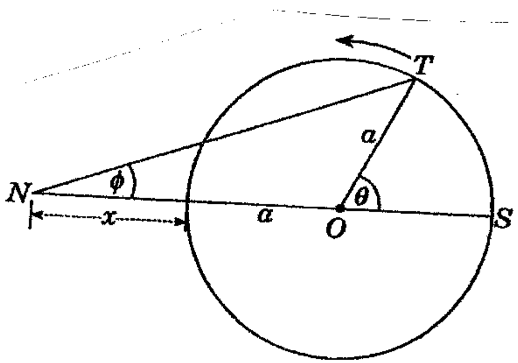
Miscellaneous Dynamics Questions

e.g. (i) (1992) – *variable angular velocity*



The diagram shows a model train T that is moving around a circular track, centre O and radius a metres.
The train is travelling at a constant speed of u m/s. The point N is in the same plane as the track and is x metres from the nearest point on the track. The line NO produced meets the track at S.

Let $\angle TNS = \phi$ and $\angle TOS = \theta$ as in the diagram

a) Express $\dfrac{d\theta}{dt}$ in terms of a and u

$$l = a\theta$$

$$\frac{dl}{dt} = a\frac{d\theta}{dt}$$

$$u = a\frac{d\theta}{dt}$$

$$\underline{\frac{d\theta}{dt} = \frac{u}{a}}$$

b) Show that $a\sin(\theta-\phi)$-$(x+a)\sin\phi = 0$ and deduce that;

$$\frac{d\phi}{dt} = \frac{u\cos(\theta-\phi)}{(x+a)\cos\phi + a\cos(\theta-\phi)}$$

$$\angle NTO + \angle TNO = \angle TOS \qquad (\text{exterior } \angle, \Delta OTN)$$

$$\angle NTO + \phi = \theta$$

$$\angle NTO = \theta - \phi$$

$$\text{In } \Delta NTO; \frac{a}{\sin\phi} = \frac{a+x}{\sin(\theta-\phi)}$$

$$a\sin(\theta-\phi) = (a+x)\sin\phi$$

$$\underline{a\sin(\theta-\phi) - (a+x)\sin\phi = 0}$$

differentiate with respect to t

$$a\cos(\theta-\phi)\left(\frac{d\theta}{dt}-\frac{d\phi}{dt}\right)-(a+x)\cos\phi\cdot\frac{d\phi}{dt}=0$$

$$a\cos(\theta-\phi)\cdot\frac{d\theta}{dt}-[a\cos(\theta-\phi)+(a+x)\cos\phi]\cdot\frac{d\phi}{dt}=0$$

$$[a\cos(\theta-\phi)+(a+x)\cos\phi]\cdot\frac{d\phi}{dt}=a\cos(\theta-\phi)\cdot\frac{u}{a}$$

$$\underline{\frac{d\phi}{dt}=\frac{u\cos(\theta-\phi)}{a\cos(\theta-\phi)+(a+x)\cos\phi}}$$

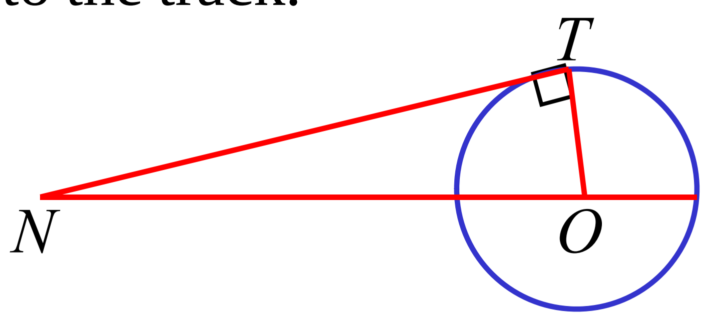
c) Show that $\frac{d\phi}{dt}=0$ when NT is tangential to the track.

when NT is a tangent;

$\angle NTO=90^\circ$ (tangent $\perp$ radius)

$\therefore \theta-\phi=90^\circ$

$$\frac{d\phi}{dt}=\frac{u\cos 90^\circ}{a\cos 90^\circ+(a+x)\cos\phi}$$

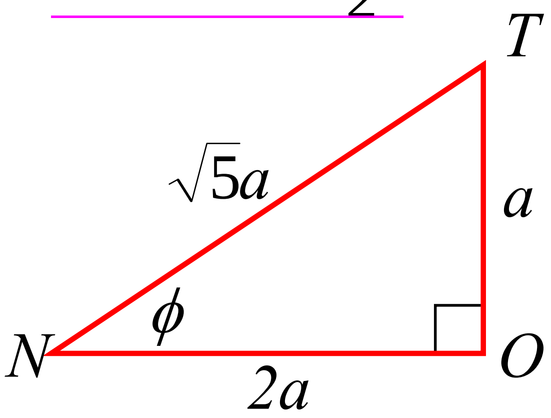


$$\underline{\therefore \frac{d\phi}{dt}=0}$$

d) Suppose that $x = a$

Show that the train's angular velocity about N when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times the angular velocity about N when $\theta = 0$

when $\theta = \frac{\pi}{2}$



$$\cos\phi = \frac{2}{\sqrt{5}}$$

$$\cos\left(\frac{\pi}{2} - \phi\right) = \frac{1}{\sqrt{5}}$$

$$\frac{d\phi}{dt} = \frac{u\cos\left(\frac{\pi}{2} - \phi\right)}{a\cos\left(\frac{\pi}{2} - \phi\right) + 2a\cos\phi}$$

$$= \frac{u\left(\frac{1}{\sqrt{5}}\right)}{a\left(\frac{1}{\sqrt{5}}\right) + (2a)\left(\frac{2}{\sqrt{5}}\right)}$$

$$= \frac{u}{5a}$$

when $\theta = 0$

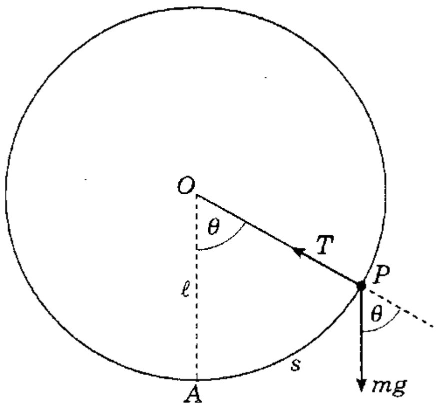
$$\frac{d\phi}{dt} = \frac{u\cos 0}{a\cos 0 + 2a\cos 0}$$

$$= \frac{u}{3a}$$

$$= \frac{5}{3} \cdot \frac{u}{5a}$$

Thus the angular velocity when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times the angular velocity when $\theta = 0$

(*ii*) (2000)



A string of length l is initially vertical and has a mass P of m kg attached to it. The mass P is given a horizontal velocity of magnitude V and begins to move along the arc of a circle in a counterclockwise direction.

Let O be the centre of this circle and A the initial position of P. Let s denote the arc length AP, $v = \frac{ds}{dt}$, $\theta = \angle AOP$ and let the tension in the string be T. The acceleration due to gravity is g and there are no frictional forces acting on P.

For parts a) to d), assume the mass is moving along the circle.

a) Show that the tangential acceleration of P is given by $\frac{d^2s}{dt^2} = \frac{1}{l}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right)$

$$s = l\theta$$

$$v = \frac{ds}{dt}$$

$$= l\frac{d\theta}{dt}$$

$$\frac{d^2s}{dt^2} = \frac{dv}{dt}$$

$$= \frac{dv}{d\theta}\cdot\frac{d\theta}{dt}$$

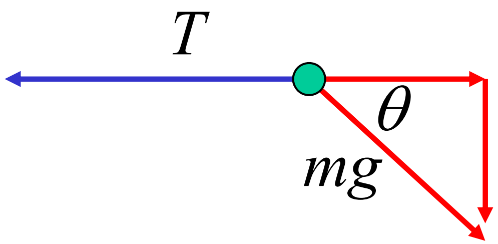
$$= \frac{dv}{d\theta}\cdot\frac{v}{l}$$

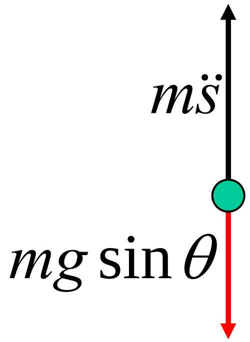
$$= \frac{1}{l}\cdot\frac{dv}{d\theta}\cdot v$$

$$= \frac{1}{l}\cdot\frac{dv}{d\theta}\cdot\frac{d}{dv}\left(\frac{1}{2}v^2\right)$$

$$= \frac{1}{l}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right)$$

b) Show that the equation of motion of P is $\frac{1}{l}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -g\sin\theta$





$$m\ddot{s} = -mg\sin\theta$$

$$\ddot{s} = -g\sin\theta$$

$$\underline{\frac{1}{l}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -g\sin\theta}$$

c) Deduce that $V^2 = v^2 + 2gl(1-\cos\theta)$

$$\frac{1}{l}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -g\sin\theta$$

$$\frac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -gl\sin\theta$$

$$\frac{1}{2}v^2 = gl\cos\theta + c$$

$$v^2 = 2gl\cos\theta + c$$

$$\text{when } \theta = 0, v = V$$

$$\therefore V^2 = 2gl + c$$

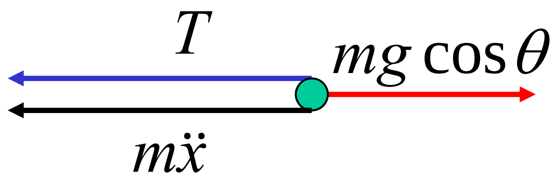
$$c = V^2 - 2gl$$

$$v^2 = 2gl\cos\theta + V^2 - 2gl$$

$$V^2 = v^2 + 2gl - 2gl\cos\theta$$

$$\underline{V^2 = v^2 + 2gl(1-\cos\theta)}$$

d) Explain why $T\text{-}mg\cos\theta = \frac{1}{l}mv^2$



$$m\ddot{x} = T - mg\cos\theta$$

But, the resultant force towards the centre is centripetal force.

$$\frac{mv^2}{l} = T - mg\cos\theta$$

$$T - mg\cos\theta = \frac{1}{l}mv^2$$

e) Suppose that $V^2 = 3gl$. Find the value of θ at which $T = 0$

$$T - mg\cos\theta = \frac{1}{l}m\left[V^2 - 2gl(1-\cos\theta)\right]$$

$$0 - mg\cos\theta = \frac{1}{l}m\left[3gl - 2gl(1-\cos\theta)\right]$$

$$-mg\cos\theta = m(g + 2g\cos\theta)$$

$$3mg\cos\theta = -mg$$

$$\cos\theta = -\frac{1}{3}$$

$$\theta = 1.911\text{radians}$$

f) Consider the situation in part e). Briefly describe, in words, the path of *P* after the tension *T* becomes zero.

When $T = 0$, the particle would undergo projectile motion, i.e. it would follow a parabolic arc.

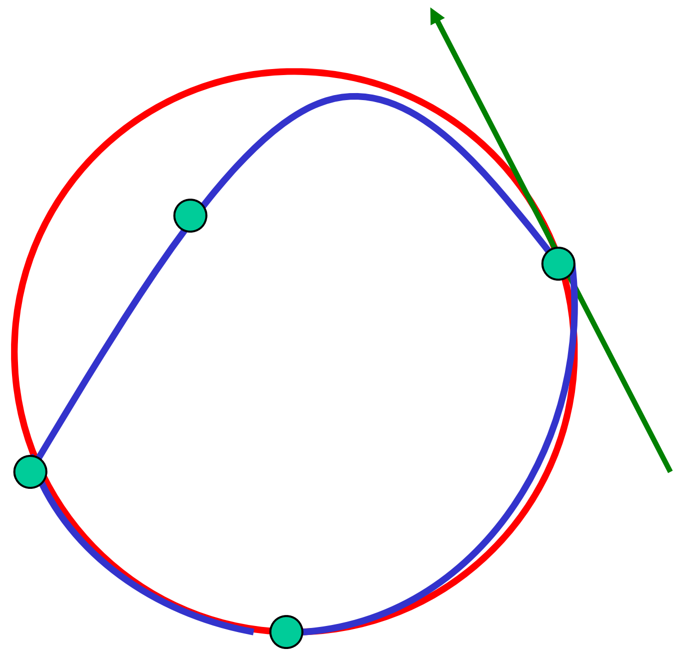
Its initial velocity would be tangential to the circle with magnitude;

$$T - mg\cos\theta = \frac{1}{l}mv^2$$

$$-mg\left(-\frac{1}{3}\right) = \frac{1}{l}mv^2$$

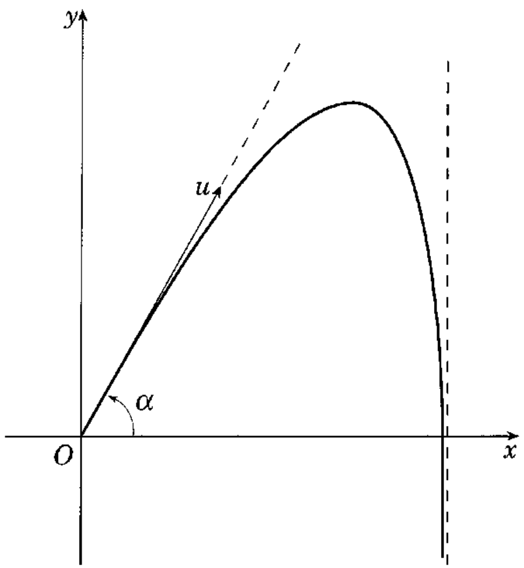
$$v^2 = \frac{gl}{3}$$

$$\underline{v = \sqrt{\frac{gl}{3}}}$$



(iii) (2003)

A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle of α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both directions.



The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g$$

where k is a constant and the acceleration due to gravity is g.

(You are NOT required to show these)

a) Derive the result $\dot{x} = ue^{-kt}\cos\alpha$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$

$$t = -\frac{1}{k}\int_{u\cos\alpha}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$

$$t = -\frac{1}{k}\left[\log \dot{x}\right]_{u\cos\alpha}^{\dot{x}}$$

$$t = -\frac{1}{k}\left[\log \dot{x} - \log(u\cos\alpha)\right]$$

$$t = -\frac{1}{k}\log\left(\frac{\dot{x}}{u\cos\alpha}\right)$$

$$-kt = \log\left(\frac{\dot{x}}{u\cos\alpha}\right)$$

$$\frac{\dot{x}}{u\cos\alpha} = e^{-kt}$$

$$\underline{\dot{x} = ue^{-kt}\cos\alpha}$$

b) Verify that $\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]$ satisfies the appropriate equation of motion and initial condition

$$\frac{d\dot{y}}{dt} = -k\dot{y} - g$$

$$t = -\int_{u\sin\alpha}^{\dot{y}} \frac{d\dot{y}}{k\dot{y} + g}$$

$$t = -\frac{1}{k}\left[\log(k\dot{y} + g)\right]_{u\sin\alpha}^{\dot{y}}$$

$$-kt = \log(k\dot{y} + g) - \log(ku\sin\alpha + g)$$

$$-kt = \log\left(\frac{k\dot{y} + g}{ku\sin\alpha + g}\right)$$

$$\frac{k\dot{y} + g}{ku\sin\alpha + g} = e^{-kt}$$

$$\underline{\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]}$$

c) Find the value of t when the particle reaches its maximum height

Maximum height occurs when $\dot{y} = 0$

$$t = -\frac{1}{k}\left[\log(k\dot{y} + g)\right]_{u\sin\alpha}^{0}$$

$$t = -\frac{1}{k}\left[\log(g) - \log(ku\sin\alpha + g)\right]$$

$$t = \frac{1}{k}\log\left(\frac{ku\sin\alpha + g}{g}\right)$$

d) What is the limiting value of the horizontal displacement of the particle?

$$\dot{x} = ue^{-kt}\cos\alpha$$

$$\frac{dx}{dt} = ue^{-kt}\cos\alpha$$

$$x = \lim_{t\to\infty} u\cos\alpha \int_0^{\infty} e^{-kt}\,dt$$

$$x = \lim_{t\to\infty} u\cos\alpha\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$x = \lim_{t\to\infty}\frac{u\cos\alpha}{k}\left(-e^{-kt} + 1\right)$$

$$x = \frac{u\cos\alpha}{k}$$

“*Cambridge*” Exercise 9D; 1, 2, 4

***Patel* Exercise 9E; 1 to 4, 7**

***Patel* Exercise 9F; 1, 2, 4, 7, 9, 12, 14, 16, 20, 22, 25**