MATHEMATICS (EXTENSION 1)

2013 HSC Course Assessment Task 3 (Trial Examination)

June 26, 2013

General instructions

- Working time – 2 hours.
  (plus 5 minutes reading time)

- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.

- Board approved calculators may be used.

- Attempt **all** questions.

- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 9)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... # BOOKLETS USED: .....

Class (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Lin

- ◯ 12M4A – Mr Fletcher
- ◯ 12M4B – Ms Beevers
- ◯ 12M4C – Ms Ziaziaris

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{70}$	

Section I

10 marks
Attempt Question 1 to 10
**Allow approximately 15 minutes for this section**

Mark your answers on the answer sheet provided.

Questions   **Marks**

1. What is the acute angle between the lines $y = 2x - 3$ and $3x + 5y - 1 = 0$, correct   **1**
to the nearest degree?

   (A) $32°$ (B) $50°$ (C) $82°$ (D) $86°$

2. The coordinates of the points that divide the interval joining $(-7, 5)$ and $(-1, -7)$ **1**
externally in the ratio $1 : 3$ are

   (A) $(-10, 8)$ (B) $(-10, 11)$ (C) $(2, 8)$ (D) $(2, 11)$

3. CT is a tangent to the circle. AB is a secant, intersecting the circle at A and B.   **1**
AB intersects CT at T.
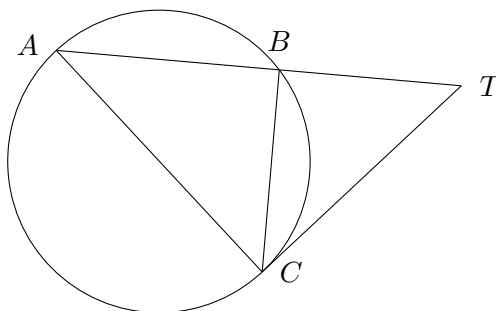
Which of the following statements is correct?

   (A) $CT^2 = AC \times BC$ (C) $CT^2 = AC \times BT$

   (B) $CT^2 = AB \times BC$ (D) $CT^2 = AT \times BT$

4. What is the domain and range of $y = \cos^{-1}\left(\dfrac{3x}{2}\right)$?   **1**

   (A) $D = \left\{ x : -\dfrac{2}{3} \leq x \leq \dfrac{2}{3} \right\}$, $R = \{ y : 0 \leq y \leq \pi \}$

   (B) $D = \{ x : -1 \leq x \leq 1 \}$, $R = \{ y : 0 \leq y \leq \pi \}$

   (C) $D = \left\{ x : -\dfrac{2}{3} \leq x \leq \dfrac{2}{3} \right\}$, $R = \{ y : -\pi \leq y \leq \pi \}$

   (D) $D = \{ x : -1 \leq x \leq 1 \}$, $R = \{ y : -\pi \leq y \leq \pi \}$

5. Which of the following is equivalent to $\int \dfrac{dx}{4x^2 + 9}$, ignoring the constant of integration?   **1**

(A) $\tan^{-1} \dfrac{2x}{3}$
                                  (C) $\dfrac{2}{3} \tan^{-1} \dfrac{2x}{3}$

(B) $\dfrac{1}{6} \tan^{-1} \dfrac{2x}{3}$
                               (D) $\dfrac{3}{2} \tan^{-1} \dfrac{2x}{3}$

6. Which of the following is the general solution to $3\tan^2 x - 1 = 0$?   **1**

(A) $n\pi \pm \dfrac{\pi}{6}$
                                  (C) $n\pi \pm \dfrac{\pi}{3}$

(B) $2n\pi \pm \dfrac{\pi}{6}$
                              (D) $2n\pi \pm \dfrac{\pi}{3}$

7. What is the value of $\sin(\alpha + \beta)$ if $\sin \alpha = \dfrac{8}{17}$ and $\sin \beta = \dfrac{4}{5}$?   **1**

(A) $\dfrac{36}{85}$
                                  (C) $\dfrac{84}{85}$

(B) $\dfrac{77}{85}$
                                  (D) $\dfrac{88}{85}$

8. Which of the following represents the exact value of $\int_0^{\frac{\pi}{8}} \cos^2 x \, dx$?   **1**

(A) $\dfrac{\pi - 2\sqrt{2}}{16}$
                            (C) $\dfrac{\pi + 2\sqrt{2}}{16}$

(B) $\dfrac{\pi - 2\sqrt{2}}{8}$
                            (D) $\dfrac{\pi + 2\sqrt{2}}{8}$

9. Which of the following represents the derivative of $y = \cos^{-1}\left(\dfrac{1}{x}\right)$?   **1**

(A) $-\dfrac{1}{\sqrt{x^2 - 1}}$
                            (C) $\dfrac{1}{\sqrt{x^2 - 1}}$

(B) $-\dfrac{1}{x\sqrt{x^2 - 1}}$
                           (D) $\dfrac{1}{x\sqrt{x^2 - 1}}$

10. Which of the following expressions is true?   **1**

(A) $\tan^{-1} x = \sin^{-1} \dfrac{1}{\sqrt{1 - x^2}}$
             (C) $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1 - x^2}}$

(B) $\tan^{-1} x = \sin^{-1} \dfrac{1}{\sqrt{1 + x^2}}$
             (D) $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1 + x^2}}$

Examination continues overleaf...

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)   Commence a NEW page.   **Marks**

(a)   Solve for x: $\dfrac{5}{x-1} > 2$ **2**

(b)   Find the value of θ, such that   **4**

$$\sqrt{3}\cos\theta - \sin\theta = 1$$

where $0 \le \theta \le 2\pi$.

(c)   A freshly baked cake is cooling in a room of constant temperature of $20°$C. At time t minutes, its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k(T - 20)$$

where k is a positive constant.

The initial temperature of the cake is $150°$C and it cools to $100°$C after 15 minutes.

     i.   Show that $T = 20 + Ae^{-kt}$ (where A is a constant) is a solution to this equation.   **1**

     ii.   Find the values A and k, giving k correct to 3 decimal places.   **2**

     iii.   How long will it take for the cake to cool to $25°$C?   **2**
         (Use the value of k obtained in the previous part)

(d)   A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O on the line. Its velocity v ms^{-1} is given by

$$v = \sqrt{x}$$

Initially, the particle is 1 m to the right of O.

     i.   Show that its acceleration a is independent of x.   **1**

     ii.   Express x in terms of t.   **2**

     iii.   Find the distance travelled by the particle during the third second of its motion.   **1**

Question 12 (15 Marks)   Commence a NEW page.   **Marks**

(a)   Show that $\dfrac{\sin x}{1 - \cos x} = \cot \dfrac{x}{2}$.   **2**

(b)   Use the substitution $u = \sin^2 x$ to evaluate   **4**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1 + \sin^2 x}\, dx$$

Give your answer in simplest exact form.

(c)   Let $f(x) = x^2 - 2x$ for $x \geq 1$.

   i.   On the same set of axes, sketch the graphs of $y = f(x)$, $y = x$ and the   **3**
        inverse $y = f^{-1}(x)$.

   ii. Find an expression for $f^{-1}(x)$.   **2**

   iii. Evaluate $f^{-1}(2)$.   **1**

(d)   Use mathematical induction to show that for all positive integers $n \geq 2$,   **3**

$$2 \times 1 + 3 \times 2 + 4 \times 3 + \cdots + n(n-1) = \frac{n(n^2 - 1)}{3}$$

Question 13 (15 Marks)   Commence a NEW page.   **Marks**

(a)   i.   Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.   **2**

       ii.   Hence evaluate $\displaystyle\int 2\sin^3\theta\,d\theta$ **1**

(b) $P(x) = ax^3 - 7x^2 + kx + 4$ has $x - 4$ as a factor. When $P(x)$ is divided by $(x-1)$, the remainder is -6.

       i.   Determine the values of a and k.   **2**

       ii.   Evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.   **2**

(c) AB is a diameter of the circle and C is a point on the circle. The tangent to the circle at A meets BC produced at D. E is a point on AD and F is a point on CD such that $EF \parallel AC$.
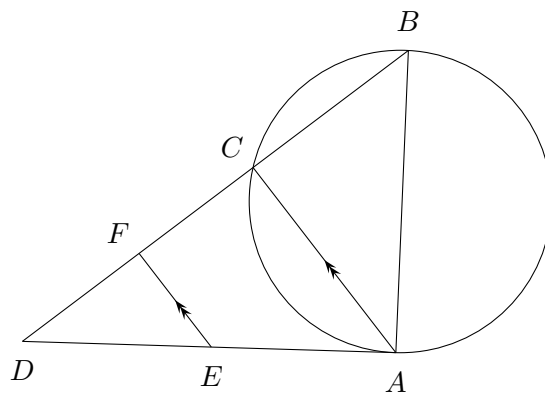
       i.   State why $\angle EAC = \angle ABC$.   **1**

       ii.   Hence show that $EABF$ is a cyclic quadrilateral.   **2**

       iii.   Show that BE is a diameter of the circle through E, A, B and F.   **1**

(d) $P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P cuts the y axis at N. M is the midpoint of PN.
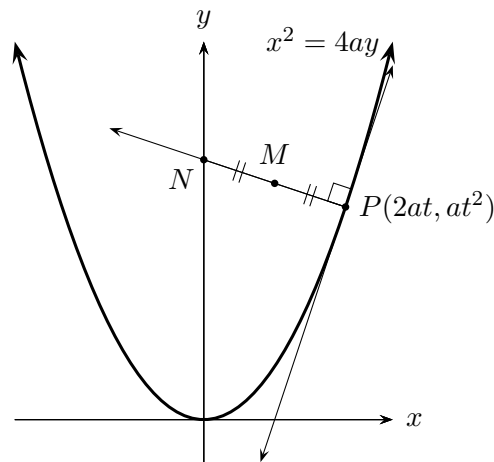
       i.   Show that the normal has equation $x + ty = 2at + at^3$.   **2**

       ii.   Find the equation of the locus of M as P moves on the parabola.   **2**

Question 14 (15 Marks)   Commence a NEW page.   **Marks**

(a)   The radius of a spherical balloon is expanding at a constant rate of $6\,\text{cms}^{-1}$.

        i.   At what rate is the volume of the balloon expanding when its radius is   **2**
4 cm?

        ii.   At what rate is the surface area of the balloon expanding if the rate of   **3**
change of volume is $15\,\text{cm}^3\text{s}^{-1}$?

(b)   From a point X due east of a tower, the angle of elevation of the top of the tower
H is $31°$. From another point Y due south of X, the angle of elevation is $25°$.
$XY = 150\,\text{m}$.
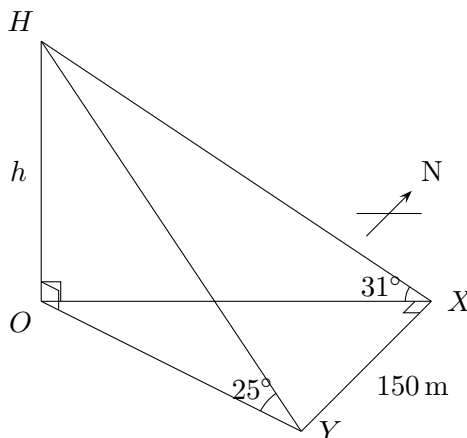
        i.   Show that $h = \dfrac{150\tan 25° \tan 31°}{\sqrt{\tan^2 31° - \tan^2 25°}}$.   **3**

        ii.   Hence, find the height of the tower.   **1**

(c) AOB is the diameter of the circle with centre O and radius 1 metre. AC is a
chord of the circle such that $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$.
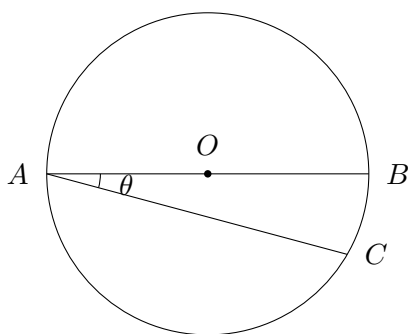
The area of the part of the circle contained between AB and AC is equal to one
quarter of the area of the circle.

        i.   Show that $\theta + \dfrac{1}{2}\sin 2\theta - \dfrac{\pi}{4} = 0$.   **2**

        ii.   Show that $0.4 < \theta < 0.5$.   **2**

        iii. Use one application of Newton's Method with an initial approximation of   **2**
$\theta_0 = 0.4$ to find the next approximation to θ, correct to 2 decimal places.

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x$, $x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

◯ 12M3A – Mr Lam   ◯ 12M4A – Mr Fletcher

◯ 12M3B – Mr Berry   ◯ 12M4B – Ms Beevers

◯ 12M3C – Mr Lin   ◯ 12M4C – Ms Ziaziaris

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

6 – Ⓐ Ⓑ Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ Ⓓ

8 – Ⓐ Ⓑ Ⓒ Ⓓ

9 – Ⓐ Ⓑ Ⓒ Ⓓ

10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (D) **2.** (B) **3.** (D) **4.** (A) **5.** (B) **6.** (A) **7.** (C)
8. (C) **9.** (D) **10.** (D)

Question 11 (Ziaziaris)

(a) (2 marks)

$$\frac{5}{x-1} > 2$$
$$5(x-1) > 2(x-1)^2$$
$$5(x-1) - 2(x-1)^2 > 0$$
$$(x-1)\left(5 - 2(x-1)\right) > 0$$
$$(x-1)(7-2x) > 0$$
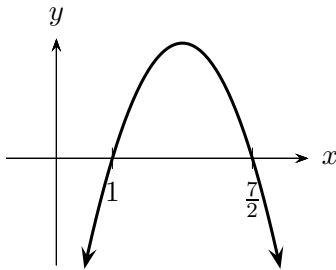
$$\therefore 1 < x < \frac{7}{2}$$

(b) (4 marks)

$$\sqrt{3}\cos\theta - \sin\theta = 1$$
$$\equiv R\cos(\theta + \alpha)$$
$$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$
$$\begin{cases} R\cos\alpha = \sqrt{3} & (1) \\ R\sin\alpha = 1 & (2) \end{cases}$$

$(1)^2 + (2)^2$:

$$R^2 = 3 + 1 = 4$$
$$\therefore R = 2$$

$(2) \div (1)$:

$$\tan\alpha = \frac{1}{\sqrt{3}}$$
$$\therefore \alpha = \frac{\pi}{6}$$
$$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right) = 1$$
$$\therefore \cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$$
$$\theta + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$\therefore \theta = \frac{\pi}{6}, \frac{3\pi}{2}$$

(c) i. (1 mark)

$$T = 20 + Ae^{-kt}$$
$$\therefore T - 20 = Ae^{-kt}$$
$$\frac{dT}{dt} = -k \cdot Ae^{-kt}$$
$$= -k(T - 20)$$

ii. (2 marks)

$$T = 20 + Ae^{-kt}$$

When $t = 0$, $T = 150$:

$$150 = 20 + Ae^0$$
$$\therefore A = 130$$
$$\therefore T = 20 + 130e^{-kt}$$

When $t = 15$, $T = 100$

$$100 = 20 + 130e^{-15k}$$
$$\frac{80}{130} = e^{-15k}$$
$$-15k = \log_e \frac{8}{13}$$
$$k = -\frac{1}{15}\log_e \frac{8}{13} \approx 0.032 \text{ (3 dp)}$$

iii. (2 marks)

$$25 = 20 + 130e^{-0.032t}$$
$$\frac{5}{13} = e^{-0.032t}$$
$$-0.032t = \log_e \frac{5}{13}$$
$$t = \frac{1}{-0.032}\log_e \frac{5}{13}$$
$$\approx 100.66 \text{ min}$$

If exact values are used,

$$t \approx 101.82 \, \text{min}$$
$$= 101 \, \text{min} \, 49 \, \text{s}$$
$$\approx 102 \, \text{min}$$

(d) i. (1 mark)

$$v = \sqrt{x}$$
$$\therefore v^2 = x$$
$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}x\right)$$
$$= \frac{1}{2}$$

which is independent of x.

ii. (2 marks)

$$\frac{dx}{dt} = x^{\frac{1}{2}}$$

Separating variables,

$$\frac{dx}{x^{\frac{1}{2}}} = dt$$
$$\int x^{-\frac{1}{2}} \, dx = \int dt$$
$$\therefore t = 2x^{\frac{1}{2}} + C$$

When $t = 0$, $x = 1$.

$$0 = 2 + C$$
$$\therefore C = -2$$
$$\therefore t = 2x^{\frac{1}{2}} - 2$$
$$2\sqrt{x} = t + 2$$
$$\therefore x = \left(\frac{t+2}{2}\right)^2 \quad \left(= \frac{1}{4}t^2 + t + 1\right)$$

iii. (1 mark)

$$x(2) = \left(\frac{2+2}{2}\right)^2 = 4$$
$$x(3) = \left(\frac{3+2}{2}\right)^2 = \frac{25}{4}$$

Dist travelled $= \dfrac{25}{4} - 4 = 2.25 \, \text{m}$

Question 12 (Lin)

(a) (2 marks)
Let $t = \tan\left(\frac{\theta}{2}\right)$:

$$\frac{\sin x}{1 - \cos x} = \frac{\frac{2t}{1+t^2}}{1 - \left(\frac{1-t^2}{1+t^2}\right)}$$
$$= \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1+t^2-(1-t^2)}{\cancel{1+t^2}}}$$
$$= \frac{2t}{2t^2}$$
$$= \frac{1}{t} = \cot \frac{x}{2}$$

(b) (4 marks)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1 + \sin^2 x} \, dx$$

Let $u = \sin^2 x$:

$$\frac{du}{dx} = 2\sin x \cos x = \sin 2x$$
$$\therefore du = \sin 2x \, dx$$
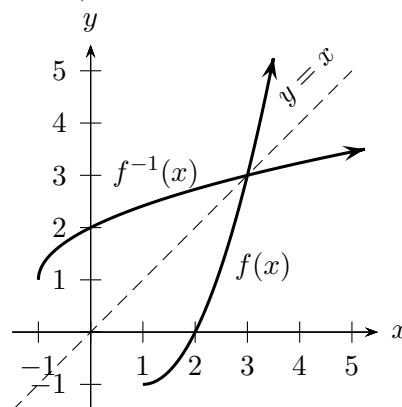$$x = \frac{\pi}{4} \quad \to u = \sin^2 \frac{\pi}{4} = \frac{1}{2}$$
$$x = \frac{\pi}{3} \quad \to u = \sin^2 \frac{\pi}{3} = \frac{3}{4}$$
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1 + \sin^2 x} \, dx = \int_{u=\frac{1}{2}}^{u=\frac{3}{4}} \frac{1}{1+u} \, du$$
$$= \left[\log_e(1+u)\right]_{\frac{1}{2}}^{\frac{3}{4}}$$
$$= \log_e\left(\frac{7}{4}\right) - \log_e\left(\frac{3}{2}\right)$$
$$= \log_e \frac{\frac{7}{4}}{\frac{3}{2}} = \log_e \frac{7}{6}$$

(c) i. (3 marks)

ii. (2 marks)

$$y = x^2 - 2x \quad x \geq 1$$

Interchanging variables,

$$x = y^2 - 2y \quad y \geq 1$$
$$x + 1 = y^2 - 2y + 1$$
$$\therefore (y-1)^2 = x + 1$$
$$y - 1 = \sqrt{x+1} \text{ (as } y \geq 1\text{, take positive root)}$$
$$\therefore y = 1 + \sqrt{x+1}$$

iii. (1 mark)

$$f^{-1}(2) = 1 + \sqrt{2+1} = 1 + \sqrt{3}$$

(d) (3 marks)

Let $P(n)$ be the proposition

$$P(n): \quad 2\times1+3\times2+4\times3+\cdots+n(n-1)$$
$$= \frac{n(n^2-1)}{3}$$

- Base case: $P(2)$ –

$$2(2-1) = 2$$
$$\frac{2\left(2^2-1\right)}{3} = \frac{2 \times 3}{3} = 2$$

Hence $P(2)$ is true.

- Inductive hypothesis: assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$P(k): \quad 2\times1+3\times2+4\times3+\cdots+k(k-1)$$
$$= \frac{k(k^2-1)}{3}$$

Now examine $P(k+1)$:

$$\overbrace{2 \times 1 + 3 \times 2 + 4 \times 3 + \cdots + k(k-1)}^{=P(k)}$$
$$+ (k+1)([k+1]-1)$$
$$= \frac{k\left(k^2-1\right)}{3} + k(k+1)$$
$$= \frac{k(k-1)(k+1) + 3k(k+1)}{3}$$
$$= \frac{(k+1)\left(k^2 - k + 3k\right)}{3}$$
$$= \frac{(k+1)\left(k^2 + 2k\right)}{3}$$
$$= \frac{(k+1)\left(k^2 + 2k + 1 - 1\right)}{3}$$
$$= \frac{(k+1)\left((k+1)^2 - 1\right)}{3}$$

Question 13 (Berry)

(a) i. (2 marks)

$$\sin 3\theta = \sin(2\theta + \theta)$$
$$= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$$
$$= 2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$
$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta(1 - \sin^2\theta) - \sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - \sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

ii. (1 mark)

$$\int 2\sin^3\theta \, d\theta = \frac{1}{2}\int 4\sin^3\theta \, d\theta$$
$$= \frac{1}{2}\int (3\sin\theta - \sin 3\theta) \, d\theta$$
$$= \frac{1}{2}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right] + C$$
$$= \frac{1}{6}\cos 3\theta - \frac{3}{2}\cos\theta + C$$

(b) i. (2 marks)

$$P(x) = ax^3 - 7x^2 + k(4) + 4$$

As $(x-4)$ is a factor, by the factor theorem:

$$P(4) = 0$$
$$\therefore a \times (4^3) - 7(4^2) + 4k + 4 = 0$$
$$64a - 112 + 4k + 4 = 0$$
$$16a + k - 27 = 0$$

Using the remainder theorem,

$$P(1) = -6$$
$$\therefore a(1^3) - 7(1^2) + k(1) + 4 = -6$$
$$a - 7 + k + 4 = -6$$
$$a + k - 3 = -6$$
$$k = -3 - a$$
$$\begin{cases} 16a + k = 27 & (1) \\ k = -3 - a & (2) \end{cases}$$

Substitute (2) to (1):

$$16a + (-3 - a) = 27$$
$$15a = 30$$
$$a = 2$$
$$\therefore k = -5$$

ii. (2 marks)

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$

$$= \frac{\left(\dfrac{c}{a}\right)}{\left(-\dfrac{d}{a}\right)} = \frac{\left(\dfrac{-5}{2}\right)}{\left(-\dfrac{4}{2}\right)}$$

$$= \frac{5}{4}$$

(c) i. (1 mark)

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

ii. (2 marks)
- $\angle AEF = 180° - \angle EAC$
  (Cointerior angles, $AC \parallel EF$)
- $\angle AEF = 180° - \angle ABC$
  (Since $\angle EAC = \angle ABC$)

iii. (1 mark)
- $\angle EAB = 90°$ (The tangent to a circle is perpendicular to the diameter drawn to the point of contact)
- $\therefore BE$ is the diameter of the the circle through E, A, B and F as it subtends a right angle at the circumference of the circle at point A.

(d) i. (2 marks)

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

At the point $P(2at, at^2)$

$$m_{\tan} = \frac{2at}{2a} = t$$
$$\therefore m_\perp = -\frac{1}{t}$$

Apply the point-gradient formula,

$$y - y_1 = m(x - x_1)$$
$$y - at^2 = -\frac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty = 2at + at^3$$

ii. (2 marks)
Find point N by letting $x = 0$:

$$0 + ty = 2at + at^3$$
$$\therefore y = 2a + at^2 \qquad (t \neq 0)$$

Finding the midpoint M of PN

$$M = \left(\frac{0 + 2at}{2}, \frac{2a + at^2 + at^2}{2}\right)$$
$$= \left(\frac{2at}{2}, \frac{2at + 2at^2}{2}\right)$$
$$= (at, a + at^2)$$
$$\begin{cases} x = at & (1) \\ y = a + at^2 & (2) \end{cases}$$

Substitute (1) into (2) after rearranging,

$$y = a + a\left(\frac{x}{a}\right)^2$$
$$= a + \frac{x^2}{a}$$
$$\therefore ay = a^2 + x^2$$
$$x^2 = ay - a^2 = a(y - a)$$

Question 14 (Lam)

(b)   i.   (3 marks)

(a)   i.   (2 marks)

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\left|\begin{array}{l} V = \frac{4}{3}\pi r^3 \quad \frac{dr}{dt} = 6\,\text{cms}^{-1} \\[2mm] \frac{dV}{dt} = 4\pi r^2 \end{array}\right.$$

$$\frac{dV}{dt} = 4\pi r^2 \times 6\Big|_{r=4}$$
$$= 4 \times \pi \times 4^2 \times 6$$
$$= 384\pi$$

- In $\triangle OHX$

$$\frac{h}{x} = \tan 31°$$
$$\therefore x = \frac{h}{\tan 31°}$$

- Similarly in $\triangle OHY$,

$$\frac{h}{y} = \tan 25°$$
$$\therefore y = \frac{h}{\tan 25°}$$

ii.   (3 marks)

$$A = 4\pi r^2 \qquad \frac{dA}{dr} = 8\pi r$$
$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$15 = 4\pi r^2 \times 6$$
$$\frac{15}{24\pi} = r^2$$
$$\therefore r = \sqrt{\frac{5}{8\pi}}$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$= 8\pi r \times 6$$
$$= 8\pi \times \sqrt{\frac{5}{8\pi}} \times 6$$
$$= \sqrt{1440\pi}$$
$$= 67.3\,\text{cm}^2\text{s}^{-1}\ (1\,\text{dp})$$
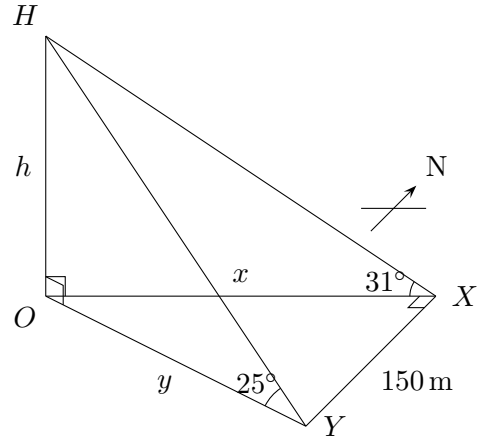
- In $\triangle OXY$,

$$x^2 + 150^2 = y^2$$
$$\left(\frac{h}{\tan 31°}\right)^2 + 150^2 = \left(\frac{h}{\tan 25°}\right)^2$$
$$\left(\frac{h}{\tan 25°}\right)^2 - \left(\frac{h}{\tan 31°}\right)^2 = 150^2$$
$$h^2\left(\frac{1}{\tan^2 25°} - \frac{1}{\tan^2 31°}\right) = 150^2$$
$$h^2\left(\frac{\tan^2 31° - \tan^2 25°}{\tan^2 25° \tan^2 31°}\right) = 150^2$$
$$h^2 = \frac{150^2 \tan^2 25° \tan^2 31°}{\tan^2 31° - \tan^2 25°}$$
$$\therefore h = \frac{150 \tan 25° \tan 31°}{\sqrt{\tan^2 31° - \tan^2 25°}}$$

ii.   (1 mark)

$$h = 110.91\,\text{m}\ (2\,\text{dp})$$

(c) i. (2 marks)
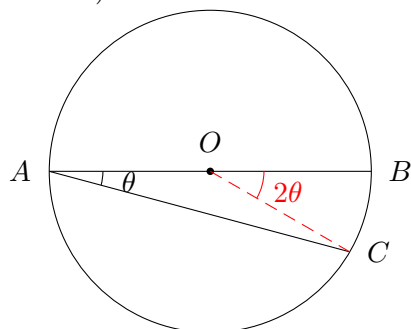
$\angle COB = 2\theta$ as angle at the centre is twice the angle at the circumference, subtended by the same arc. (Can also use angle sum of $\triangle$ and exterior $\angle$)

$$A_{\text{sector } OBC} + A_{\triangle OAC} = \frac{1}{4}\pi r^2$$

$$A_{\text{sector } OBC} = \frac{1}{2}r^2 \times (2\theta)$$
$$= r^2\theta = \theta \quad (r = 1)$$
$$A_{\triangle OBC} = \frac{1}{2}r^2 \times \sin(\pi - 2\theta)$$
$$= \frac{1}{2}r^2 \sin 2\theta = \frac{1}{2}\sin 2\theta$$

(As $\sin(\pi - x) = \sin x$)

Adding areas and equating,

$$\theta + \frac{1}{2}\sin 2\theta = \frac{\pi}{4}$$
$$\therefore \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} = 0$$

ii. (2 marks)

- When $\theta = 0.4$,

$$0.4 + \frac{1}{2}\sin 0.8 - \frac{\pi}{4} \approx -0.026\cdots$$

- When $\theta = 0.5$,

$$0.5 + \frac{1}{2}\sin 1 - \frac{\pi}{4} \approx 0.135$$

If $f(\theta) = \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4}$, then $f(\theta)$ is continuous for all $x \in \mathbb{R}$ as θ and $\sin 2\theta$ are both continuous. From above, $f(0.4) < 0$ and $f(0.5) > 0$, therefore a zero crossing exists for $0.4 < \theta < 0.5$, satisfying the equation found in part (i).

iii. (2 marks)

$$\left| \begin{array}{l} f(\theta) = \theta + \dfrac{1}{2}\sin 2\theta - \dfrac{\pi}{4} \\[2mm] f'(\theta) = 1 + \dfrac{1}{2} \times 2\cos 2\theta \\[2mm] \qquad\quad = 1 + \cos 2\theta \end{array} \right.$$

$$\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$$
$$= 0.4 - \frac{0.4 + \frac{1}{2}\sin 0.8 - \frac{\pi}{4}}{1 + \cos 0.8}$$
$$\left(= 0.4 - \frac{-0.02672011\cdots}{1.696706\cdots} \right)$$
$$= 0.41574\cdots$$
$$= 0.42 \ (2 \text{ dp})$$