

2013



# Mathematics

## Extension 2

### General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

### Total Marks – 100

#### Section I – Pages 2 – 5

10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

#### Section II – Pages 6 – 13

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q14(b) to be detached and placed in Q14 answer booklet.

**Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.**

Section I

10 marks

Marks

Attempt Questions 1 - 10

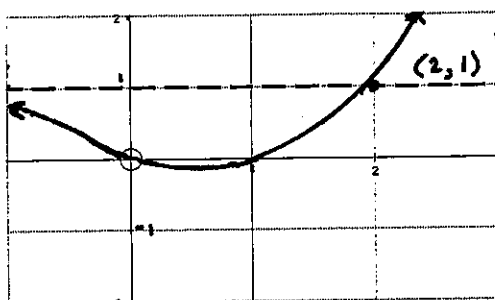
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1. Let  $z = 1 + 2i$  and  $w = -2 + i$ . What is the value of  $\bar{z} \cdot w$

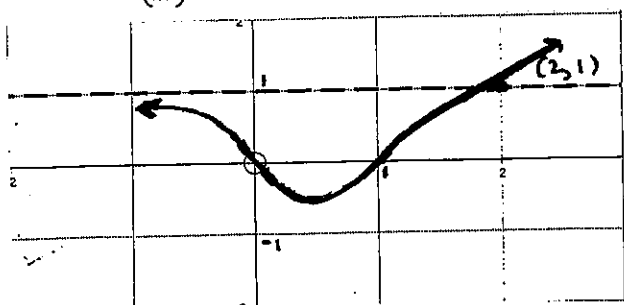
- (A)  $5i$
- (B)  $-4 + 5i$
- (C)  $-3i$
- (D)  $-4 - 3i$

2. The graph of  $y = f(x)$  is shown below

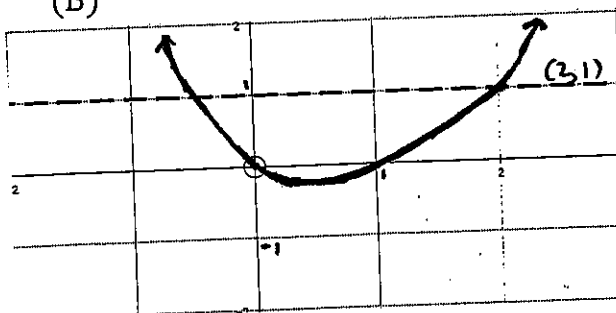


Which of the following is the graph of  $y = \sqrt{f(x)}$ ?

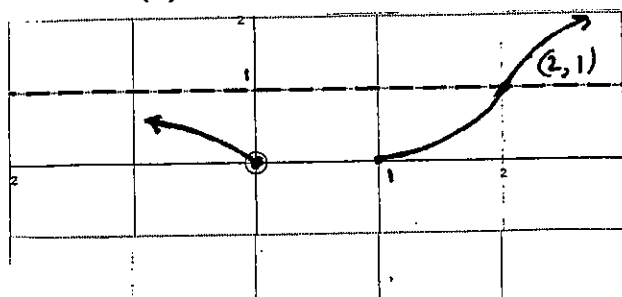
(A)



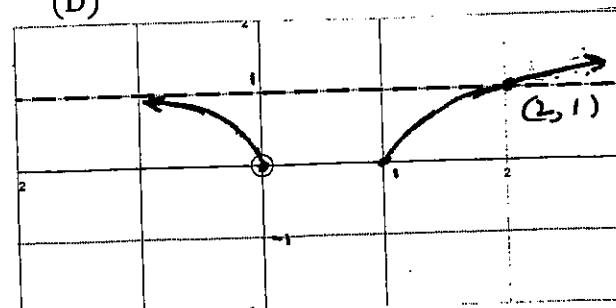
(B)



(C)

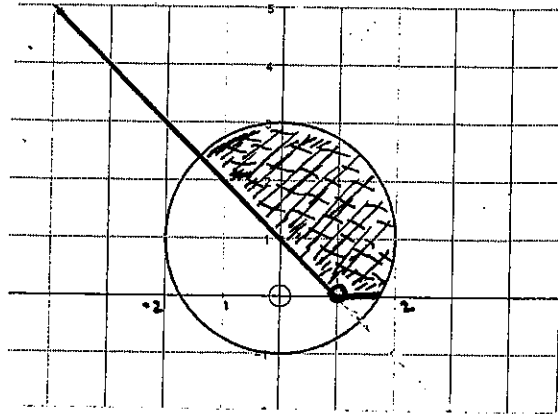


(D)



Section I (cont'd)

3. Consider the Argand diagram below.



Which inequality could define the shaded area?

- (A)  $|z - i| \leq 2$  and  $0 \leq \arg(z - 1) \leq \frac{3\pi}{4}$
- (B)  $|z + i| \leq 2$  and  $0 \leq \arg(z - 1) \leq \frac{3\pi}{4}$
- (C)  $|z - i| \leq 2$  and  $0 \leq \arg(z - 1) \leq \frac{\pi}{4}$
- (D)  $|z + i| \leq 2$  and  $0 \leq \arg(z - 1) \leq \frac{\pi}{4}$
4. The points  $P(a \cos \theta, b \sin \theta)$  and  $Q(a \cos \phi, b \sin \phi)$  lie on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and the chord  $PQ$  subtends a right angle at  $(0, 0)$ . Which of the following is the correct expression?
- (A)  $\tan \theta \cdot \tan \phi = -\frac{b^2}{a^2}$
- (B)  $\tan \theta \cdot \tan \phi = -\frac{a^2}{b^2}$
- (C)  $\tan \theta \cdot \tan \phi = \frac{b^2}{a^2}$
- (D)  $\tan \theta \cdot \tan \phi = \frac{a^2}{b^2}$

Section I (cont'd)

5. Which of the following is an expression for  $\int \frac{\sin x \cdot \cos x}{4 + \sin x} dx$

Use the substitution  $u = 4 + \sin x$

- (A)  $-4 \ln |4 + \sin x| + C$   
(B)  $4 \ln |4 + \sin x| + C$   
(C)  $-\sin x - 4 \ln |4 + \sin x| + C$   
(D)  $4 + \sin x - 4 \ln |4 + \sin x| + C$
6. The polynomial  $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$  has  $x = 1$  as a root of multiplicity 3 and  $x = i$  is a root. Which of the following expressions is a factorised form of  $P(x)$  over the complex numbers?

- (A)  $P(x) = (x + 1)^3 (x - 1)(x + 1)$   
(B)  $P(x) = (x - 1)^3 (x - 1)(x + 1)$   
(C)  $P(x) = (x + 1)^3 (x - i)(x + i)$   
(D)  $P(x) = (x - 1)^3 (x - i)(x + i)$

7. What is the eccentricity of the hyperbola

$$\frac{(x - 1)^2}{10} - \frac{(y + 1)^2}{4} = 1$$

- (A)  $\frac{\sqrt{6}}{2}$       B.  $\sqrt{\frac{7}{5}}$       C.  $\frac{2}{\sqrt{6}}$       D.  $\frac{\sqrt{14}}{2}$
8. The polynomial equation  $x^3 - 3x^2 - x + 2 = 0$  has roots  $\alpha, \beta$  and  $\gamma$ . Which of the following polynomial equations have roots  $\frac{1}{\alpha}, \frac{1}{\beta}$  and  $\frac{1}{\gamma}$

- (A)  $2x^3 - 2x^2 - 3x + 1 = 0$   
(B)  $2x^3 - x^2 - 3x + 1 = 0$   
(C)  $x^3 - 2x^2 - 3x + 1 = 0$   
(D)  $x^3 - x^2 - 3x + 1 = 0$

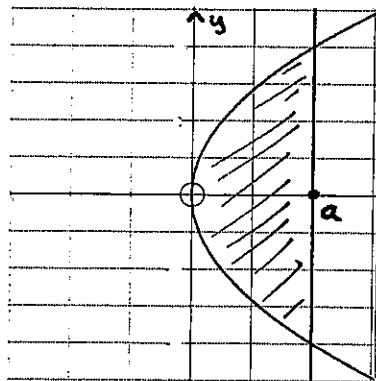
Section I (cont'd)

9. A particle of mass  $m$  is moving in a straight line under the action of a force

$$F = \frac{m}{x^3}(6 - 10x)$$

What of the following is an expression for its velocity in any position, if the particle starts from rest at  $x = 1$ ?

- (A)  $v = \pm \frac{1}{x} \sqrt{(-3 + 10x - 7x^2)}$   
(B)  $v = \pm x \sqrt{(-3 + 10x - 7x^2)}$   
(C)  $v = \pm \frac{1}{x} \sqrt{2(-3 + 10x - 7x^2)}$   
(D)  $v = \pm x \sqrt{2(-3 + 10x - 7x^2)}$
10. A solid is formed by rotating the region enclosed by the parabola  $y^2 = 4ax$ , its vertex  $(0, 0)$  and the line  $x = a$ , about the  $y$ -axis.



What is the volume of this solid using the method of slicing.

- (A)  $8\pi a^3$   
(B)  $\frac{16\pi a^3}{5}$   
(C)  $\frac{8\pi a^3}{5}$   
(D)  $4\pi a^3$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

Question 11 – Start A New Booklet – (15 marks)

Marks

- a) Evaluate  $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5 \cos x} dx$  using the substitution  $\tan\left(\frac{x}{2}\right) = t$  3
- b) Let  $z = 3 - 2i$  and  $w = 1 + \sqrt{2}i$
- (i) Find  $|z|$  1
- (ii) Express  $\frac{w}{z}$  in the form  $a + bi$  where  $a$  and  $b$  are real numbers. 2
- c) Find  $\int \frac{dx}{20 - 4x + x^2}$  2
- d) (i) Write  $z = 1 + i$  in modulus-argument form. 2
- (ii) Hence express  $z^{-4}$  in the form  $x + yi$ , where  $x$  and  $y$  are real. 2
- e) The area bounded by the curve  $y = \frac{1}{x+1}$ , the  $x$ -axis, the line  $x = 2$  and the line  $x = 8$ , is rotated about the  $y$ -axis.
- Find the volume of the solid generated using the method of cylindrical shells. 3

Question 12 – Start A New Booklet – (15 marks)

Marks

a) Find  $\int \frac{x}{(1-x)(1+x^2)} dx$

3

b) Let  $f(x) = \frac{1-x}{x}$

On separate diagrams sketch the graph of the following functions. For each graph label any asymptotes and critical points.

(i)  $y = |f(x)|$

2

(ii)  $y = e^{f(x)}$

2

(iii)  $y^2 = f(x)$

2

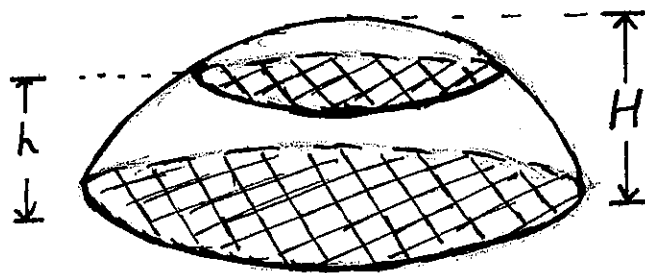
c) (i) Verify that  $\int_0^a \sqrt{a^2 - x^2} dx = \frac{1}{4}\pi a^2$

1

(ii) Deduce that the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\pi ab$

2

(iii)



The diagram shows a mound of height  $H$ . At height  $h$  above the horizontal base, the horizontal cross-section of the mound is an ellipse with equation

3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2 \text{ where } \lambda = 1 - \frac{h^2}{H^2}$$

( $x, y$  are appropriate co-ordinates in the plane of the cross-section).

Show that the volume of the mound is  $\frac{8\pi abH}{15}$

Question 13 – Start A New Booklet – (15 marks)

Marks

- a) On a school camp, one of the girls of mass  $M$  kg jumps vertically (feet first) from a rock ledge into a river below.

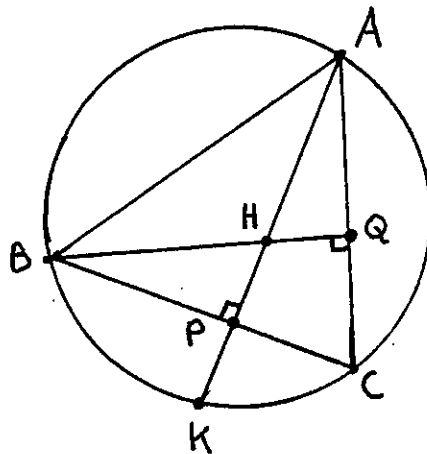
When she is falling at  $v$  m/s, she encounters air resistance equal to  $\frac{Mv}{20}$  Newtons. She hits the water at a speed of  $V$  m/s. Let  $x$  be the displacement below the rock ledge at time  $t$  seconds after jumping.

(i) Show that  $\ddot{x} = g - \frac{v}{20}$ , where  $g$  is the acceleration due to gravity. 1

(ii) If it takes two seconds for her feet to hit the water, using  $g = 10 \text{ m/s}^2$  show that  $V = 200 \left(1 - e^{-\frac{1}{10}}\right)$  3

(iii) Find the height of the rock ledge above the water (to the nearest 0.1 metre) 3

b)



The altitudes  $AP$  and  $BQ$  of the acute triangle  $ABC$  intersect at  $H$ . 3

$AP$  produced cuts the circle at  $K$ .

Prove that  $HP = PK$



Question 13 (cont'd)

Marks

- c) Show that the equation of the tangent to the ellipse

1

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

at the point  $P(a \cos \theta, b \sin \theta)$  is  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$

- (i) The tangent at  $P$  meets the  $x$ -axis at  $S$  and the  $y$ -axis at  $T$ .

Find the area of  $\Delta OST$ .

2

- (ii) If  $A$  is the point  $(a, 0)$  and  $B$  is the point  $(0, b)$ , the area of  $\Delta APB$  is  $\frac{1}{2} ab(\cos \theta + \sin \theta - 1)$  [do not prove this]

Prove that, as  $\theta$  varies in the interval  $0 < \theta < \frac{\pi}{2}$ , the area of  $\Delta APB$  is a maximum when the tangent to the ellipse is parallel to  $AB$ .

2

Question 14 – Start A New Booklet – (15 marks)

Marks

- a) On an Argand diagram, sketch the locus of the points  $z$  such that

2

$$\arg \left\{ \frac{(z-1)}{(z+1)} \right\} = \frac{\pi}{2}$$

- b)  $P \left( cp, \frac{c}{p} \right)$  and  $Q \left( cq, \frac{c}{q} \right)$  are two points on the rectangular hyperbola  $xy = c^2$ .

Given that the equation of the tangent at  $P$  is

$$x + p^2y = 2cp.$$

- (i) The tangents at  $P$  and  $Q$  meet in  $T$ . Find the co-ordinates of  $T$  in terms of  $c, p$  and  $q$ .

2

- (ii) If  $T$  lies on the hyperbola  $xy = k^2$  for all positions of  $P$  and  $Q$ , prove that

1

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

- c) (i) Given  $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$  let  $w$  be a solution to  $z^5 + 1 = 0$  where  $w \neq -1$ . Prove that  $1 + w^2 + w^4 = w + w^3$

1

- (ii) Hence show that

3

$$\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$$

Question 14 (cont'd)

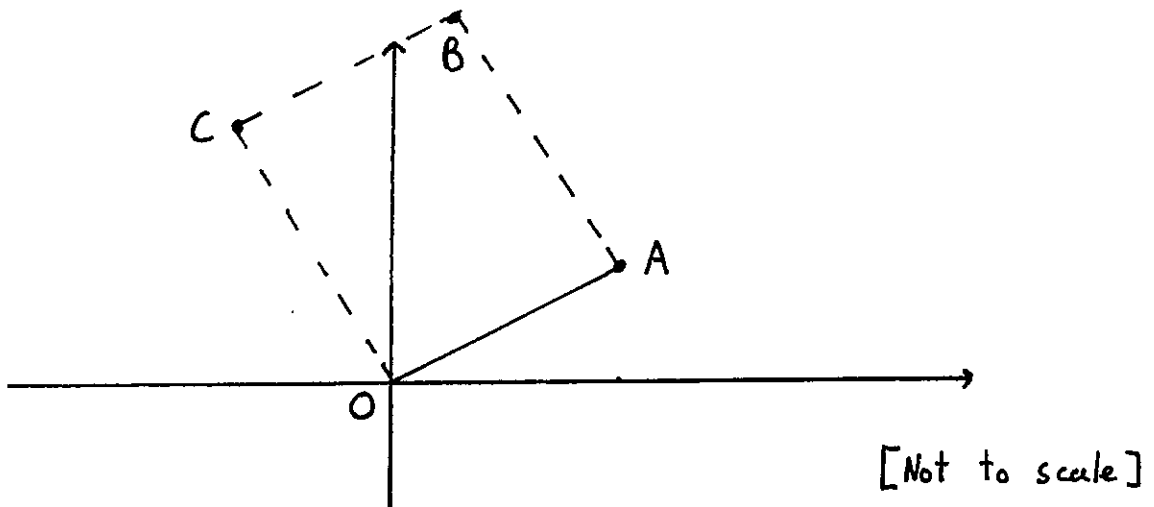
Marks

- d) (i) If  $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$  where  $n \geq 1$ , use integration by parts to prove that 3

$$I_n = \frac{n-1}{n} I_{n-2}$$

- (ii) Hence show that  $I_5 = \frac{8}{15}$  1

- e)  $OABC$  is a rectangle in an Argand diagram where  $O$  is the Origin and point  $A$  corresponds to the complex number  $2 + i$



Given that the length of the rectangle is twice its breadth and  $OA$  is one of the shorter sides, find the complex number representing  $C$ .

2

Question 15 - Start A New Booklet - (15 marks)

Marks

a) Suppose  $\alpha, \beta$  and  $\gamma$  are the roots of the polynomial equation

$$x^3 + x + 12 = 0$$

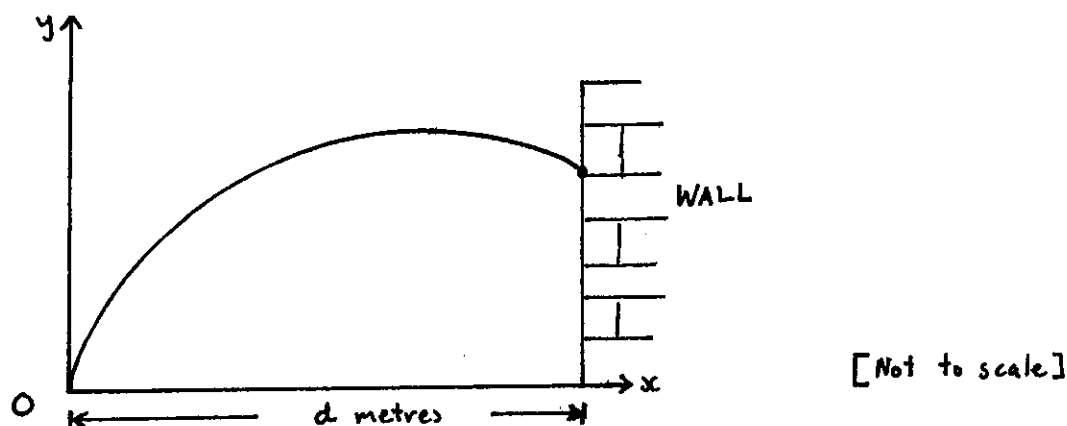
(i) Find  $\alpha^2 + \beta^2 + \gamma^2$  2

(ii) Hence explain why only one of the roots is real. 2

(iii) Let the real root be denoted by  $\alpha$ . Prove that  $-3 < \alpha < -2$  1

(iv) Hence prove that the modulus of each of the other roots lies between 2 and  $\sqrt{6}$  3

b)



4

In the diagram above, the wall of a building stands on level ground,  $d$  metres from a fire hose located at  $O$ . If water leaves the hose with velocity  $V \text{ms}^{-1}$  at an angle  $\theta$  to the ground:

Given  $V > \sqrt{gd}$ , ( $g$  is acceleration in the vertical plane due to gravity) show that the particle will strike the wall above ground level provided that  $\beta < \theta < \frac{\pi}{2} - \beta$  where  $\beta = \frac{1}{2} \sin^{-1} \left( \frac{gd}{V^2} \right)$

You may assume that the range on the horizontal plane from the point of projection is

$$\frac{V^2 \sin 2\theta}{g}$$

c) Find  $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$  3

[Hint: • choose an appropriate substitution]

•  $\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$  ]

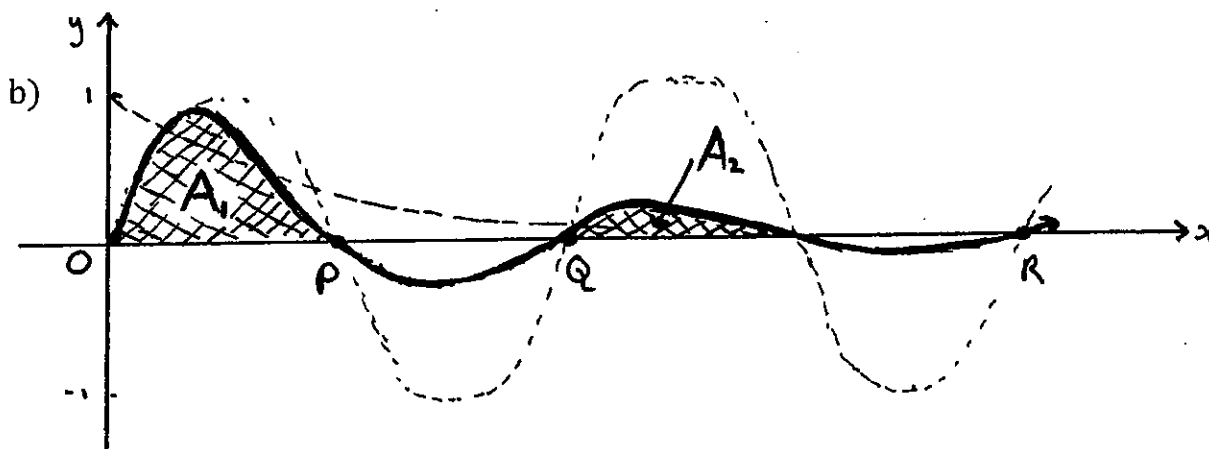
Question 16 – Start A New Booklet – (15 marks)

Marks

a) A polynomial of degree  $n$  is given by  $P(x) = x^n + ax - b$ . It is given that the polynomial has a double root at  $x = \alpha$ .

(i) Find the derived polynomial  $P'(x)$  and show that  $\alpha^{n-1} = -\frac{a}{n}$  2

(ii) Show that  $\left(\frac{a}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$  4



The diagram shows a sketch of part of the curve  $y = f(x)$  with equation

$$y = e^{-x} \cdot \sin x, \quad x \geq 0$$

(i) Find the coordinates of the points  $P$ ,  $Q$  and  $R$  where  $y = f(x)$  cuts the  $x$ -axis. 1

(ii) Integrating by parts gives 3

$$\int e^{-x} \cdot \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

[you do not have to prove this]

If the terms  $A_1, A_2, \dots, A_n$ , represent areas between  $y = f(x)$  and the  $x$ -axis for successive portions of  $y = f(x)$  where  $y$  is positive. (The areas represented by  $A_1$ , and  $A_2$ , are shown as the shaded regions in the diagram).

Show that  $A_n = \frac{1}{2} (e^{(1-2n)\pi} + e^{(2-2n)\pi})$

(iii) Show that  $A_1 + A_2 + A_3 + \dots$  is a geometric series and that 3

$$S_\infty = \frac{e^\pi}{2(e^\pi - 1)}$$

(iv) Given that  $\int_0^\infty e^{-x} \cdot \sin x \, dx = \frac{1}{2}$ , find the exact value of  $\int_0^\infty |e^{-x} \cdot \sin x| \, dx$  2



Student Number: \_\_\_\_\_ Teacher: \_\_\_\_\_

Year 12 Mathematics Extension 2 Trial HSC Examination 2013

**Section I**

Multiple-choice Answer Sheet – Questions 1 – 10

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample  $2 + 4 =$  (A) 2 (B) 6 (C) 8 (D) 9  
A  B  C  D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A  B  C  D

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A  B  C  D   
*correct* ↗

- 
- |     |   |                       |   |                       |   |                       |   |                       |
|-----|---|-----------------------|---|-----------------------|---|-----------------------|---|-----------------------|
| 1.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 2.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 3.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 4.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 5.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 6.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 7.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 8.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 9.  | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |
| 10. | A | <input type="radio"/> | B | <input type="radio"/> | C | <input type="radio"/> | D | <input type="radio"/> |





1

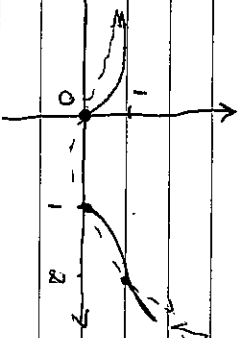
SECTION I

1/  $z = 1 + 2i$      $w = -2 + i$

$\bar{z} = 1 - 2i$      $\bar{w} = (1 - 2i)(-2 + i)$

$= -2 + 5i - 2i^2$   
 $= 5i$     (A)

2/



$\sqrt{f(x)}$  defined only for  $f(x) > 0$ .

where  $f(x) > 1$ ,  $\sqrt{f(x)} < f(x)$

where  $f(x) = 1$ ,  $\sqrt{f(x)} = f(x)$

where  $f(x) < 1$ ,  $\sqrt{f(x)} > f(x)$

(D)

3/  $x^2 + (y-1)^2 \leq 4$

$|z-1| \leq 2$  and  $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$     (A)

4/

$P(a \cos \theta, b \sin \theta)$  and  $Q(a \cos \alpha, b \sin \alpha)$

line on  $\frac{x^2}{a^2} + \frac{y^2}{b^2}$  chord PQ subtends a right angle at  $(0,0)$

$\sin \theta \sin \alpha = -1$

$\frac{b \sin \theta \times b \sin \alpha}{a \cos \theta \times a \cos \alpha} = -1$

$b^2 \tan \theta \times \tan \alpha = -1$      $\therefore \tan \theta \tan \alpha = -\frac{a^2}{b^2}$     (B)

5/  $\int \frac{\sin x \cdot \cos x}{4 + \sin^2 x} dx$

$u = 4 + \sin^2 x$      $\therefore \sin^2 x = u - 4$   
 $\frac{du}{dx} = \cos 2x$      $\therefore du = \cos 2x dx$

$\int \frac{(u-4) du}{u}$

$= \int 1 - \frac{4}{u} du$

$= u - 4 \ln u + c$

$= 4 + \sin^2 x - 4 \ln |4 + \sin^2 x| + c$     (D)

2

A

D

A

B

D

B

D

D

B

B

B

B

C

B

B

B

6/  $f(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$

$= (x-1)^3(x-i)(x+i)$

$(-1)^3 x(i) \times (i) = -1x^{-1}i^2$

$= -1$     (D)

7/  $\frac{(x-1)^2}{10} = \frac{(y+1)^2}{4}$

hyperbola

$e = \sqrt{1 + \frac{b^2}{a^2}}$

$= \sqrt{1 + \frac{14}{10}}$   
 $= \sqrt{\frac{24}{5}}$     (B)

8/  $x^3 - 3x^2 - x + 2 = 0$

$x = \frac{1}{x}$ ,  $x = 1$      $(\frac{1}{x})^3 - 3(\frac{1}{x})^2 - (\frac{1}{x}) + 2 = 0$

$\frac{1}{x^3} - \frac{3}{x^2} - \frac{1}{x} + 2 = 0$

$1 - 3x - x^2 + 2x^3 = 0$     (B)

9/  $F = \frac{m}{x^2} (6 - 10x)$

$F = mg$

$\frac{d}{dx} (\frac{1}{2} v^2) = 6x^{-2} - 10x^{-2}$

$\frac{1}{2} v^2 = \int (6x^{-2} - 10x^{-2}) dx$   
 $\frac{1}{2} v^2 = -3x^{-1} + 5x^{-1} + c$

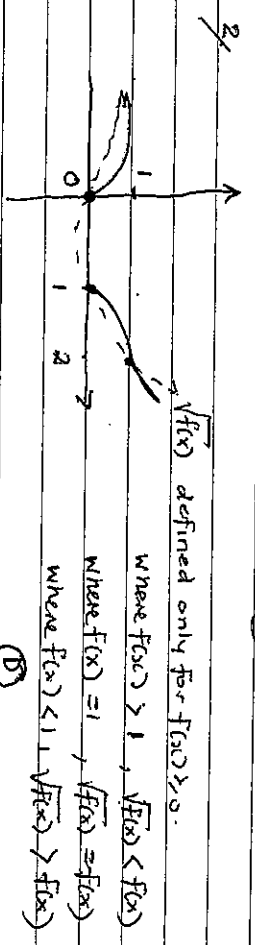
when  $x=1$ ,  $v=0$      $\therefore c = -7$

$\frac{1}{2} v^2 = -3x^{-1} + 5x^{-1} - 7$   
 $v^2 = 2(-3x^{-1} + 5x^{-1} - 7)$

$v = \pm \frac{1}{x} \sqrt{2(-3+5x-7x^2)}$     (C)

SECTION I

1/  $Z = 1+2i$      $w = -2+i$   
 $\bar{Z} = 1-2i$      $\bar{Z} \cdot w = (1-2i)(-2+i)$   
 $= -2 + 5i - 2i^2$   
 $= 5i$     (A)



3/  $x^2 + (y-1)^2 \leq 4$   
 $|z-1| \leq 2$  and  $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$     (A)

4/  $P(a \cos \theta, b \sin \theta)$  and  $Q(a \cos \phi, b \sin \phi)$   
 lie on  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  chord PQ subtends a right angle at  $(0,0)$   
 $m_1 \times m_2 = -1$   
 $\frac{b a \sin \theta \times b a \sin \phi}{a^2 \cos \theta \cos \phi} = -1$   
 $b^2 \tan \theta \times \tan \phi = -1$      $\therefore \tan \theta \tan \phi = -\frac{a^2}{b^2}$     (B)

5/  $\int \frac{\sin x \cos x}{4 + \sin^2 x} dx$      $u = 4 + \sin^2 x$      $\therefore \sin^2 x = u - 4$   
 $\frac{du}{2 dx} = \cos^2 x$      $\therefore du = 2 \cos^2 x dx$   
 $\int \frac{(u-4) du}{u}$   
 $= \int \frac{u-4}{u} du$   
 $= \int (1 - \frac{4}{u}) du$   
 $= u - 4 \ln |u| + c$   
 $= 4 + \sin^2 x - 4 \ln |4 + \sin^2 x| + c$     (D)

A

6/  $f(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$   
 $= (x-1)^2(x-i)(x+i)$   
 $(-i)^3 x(-i) \times (i) = -1x^{-1}i^2$   
 $= -1$     (D)

7/  $\frac{(x-1)^2}{10} - \frac{(y+1)^2}{4} = 1$     hyperbola     $e = \sqrt{\frac{1+b^2}{a^2}}$   
 $b^2 = 4$   
 $a^2 = 10$   
 $e = \sqrt{\frac{14}{10}}$   
 $= \sqrt{\frac{7}{5}}$     (B)

8/  $x^3 - 3x^2 - x + 2 = 0$   
 $x = \frac{1}{x}$ ,  $x = \frac{1}{x}$   
 $(\frac{1}{x})^3 - 3(\frac{1}{x})^2 + (\frac{1}{x}) + 2 = 0$   
 $1 - 3x - x^2 + 2x^3 = 0$     (B)

9/  $F = \frac{m}{x^3} (6-10x)$      $F = mg$   
 $a = 6x^{-3} - 10x^{-2}$   
 $\frac{d}{dx} (\frac{1}{2} v^2) = 6x^{-3} - 10x^{-2}$   
 $\frac{1}{2} v^2 = \int (6x^{-3} - 10x^{-2}) dx$   
 $\frac{1}{2} v^2 = -3x^{-2} + 10x^{-1} + c$   
 when  $x=1$ ,  $v=0$      $\therefore c = -7$   
 $\frac{1}{2} v^2 = -3x^{-2} + 10x^{-1} - 7$   
 $v^2 = 2(-3x^{-2} + 10x^{-1} - 7)$   
 $= \frac{2}{x^2} (-3 + 10x - 7x^2)$   
 $v = \pm \frac{1}{x} \sqrt{2(-3+10x-7x^2)}$     (C)

## QUESTION 11

(a) let  $t = \tan\left(\frac{x}{2}\right)$

Gives on substitution

then  $\tan^{-1}(t) = \frac{x}{2}$

$$\therefore \frac{2 dt}{1+t^2} = dx$$

When  $x=0$ ,  $t=0$

$x = \frac{\pi}{2}$ ,  $t=1$

$$\int_0^1 \frac{1}{3 + 5 \frac{(1-t^2)}{1+t^2}} \cdot \frac{2 dt}{(1+t^2)}$$

$$= \int_0^1 \frac{2 dt}{3(1+t^2) + 5(1-t^2)}$$

$$* = \int_0^1 \frac{dt}{4-t^2}$$

$$= \int_0^1 \frac{\frac{1}{4}}{(2-t)} + \frac{\frac{1}{4}}{(2+t)} dt$$

$$= \frac{1}{4} \left[ -\ln(2-t) + \ln(2+t) \right]_0^1$$

$$= \frac{1}{4} \left[ \ln \left[ \frac{2+t}{2-t} \right] \right]_0^1$$

$$= \frac{1}{4} (\ln 3 - \ln 1)$$

$$= \frac{1}{4} \ln 3$$

\* Now:  $\frac{1}{4-t^2} = \frac{A}{2-t} + \frac{B}{2+t}$

So  $1 = 2(A+B) + t(A-B)$

$$A - B = 0 \quad \dots \text{(I)}$$
$$A + B = \frac{1}{2} \quad \dots \text{(II)}$$

(I) + (II)  $2A = \frac{1}{2}$

$$A = \frac{1}{4}, \quad B = \frac{1}{4}$$

(b) (i)  $|z| = \sqrt{3^2 + (-2)^2}$

$$= \sqrt{13}$$

(ii)  $\frac{w}{z} = \frac{1 + \sqrt{2}i}{3 - 2i} \times \left( \frac{3 + 2i}{3 + 2i} \right)$

$$= \frac{3 + 2i + 3\sqrt{2}i - 2\sqrt{2}}{3^2 + 4}$$

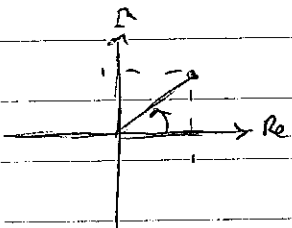
$$= \frac{3-2\sqrt{2}}{13} + \frac{2+3\sqrt{2}}{13} i$$

$$(c) \int \frac{dx}{2x^2 - 4x + x^2} = \int \frac{dx}{x^2 - 4x + 4 + 16}$$

$$= \int \frac{dx}{(x-2)^2 + 4^2}$$

$$= \frac{1}{4} \tan^{-1} \left( \frac{x-2}{4} \right) + C$$

(d) (i) Mod  $|z| = \sqrt{1^2 + 1^2}$  and Arg  $\arg(z) = \tan^{-1} \left( \frac{1}{1} \right)$   
 $= \sqrt{2}$   $= \frac{\pi}{4}$



$$\therefore z = r \operatorname{cis} \theta$$

$$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

$$= \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

(ii) Then  $z^{-4} = \left( \sqrt{2} \cdot \operatorname{cis} \frac{\pi}{4} \right)^{-4}$

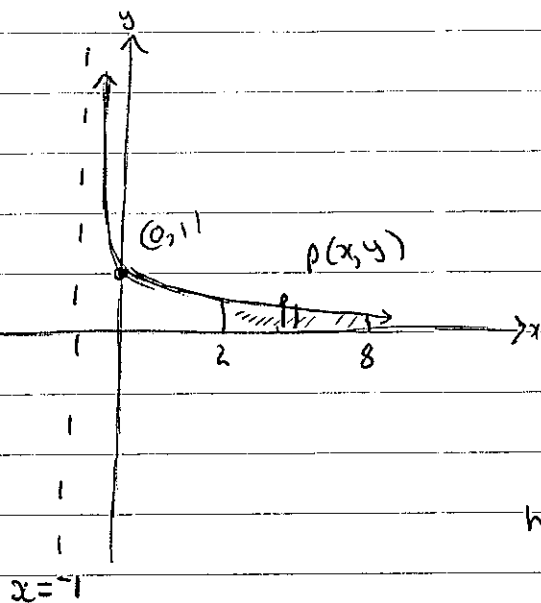
$$= 2^{-2} \cdot \operatorname{cis} \left( -4 \times \frac{\pi}{4} \right) \quad \left( \text{by De Moivre's Theorem} \right)$$

$$= \frac{1}{4} \left( \cos(-\pi) + i \sin(-\pi) \right)$$

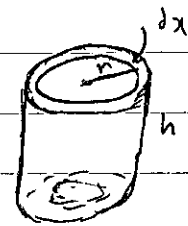
$$= \frac{1}{4} (-1 - 0 \cdot i)$$

$$= -\frac{1}{4} \quad \left( \text{or } -\frac{1}{4} + 0 \cdot i \right)$$

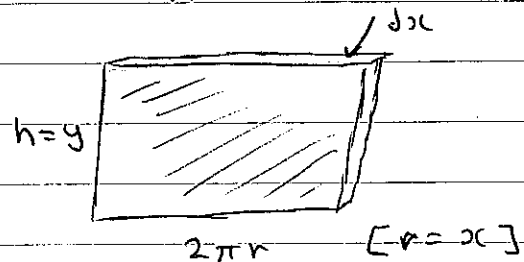
(e)



"Typical" Shell



"Flattened out"



$$\begin{aligned} \text{So } \delta V &= A \cdot \delta x \\ &= 2\pi x \cdot y \cdot \delta x \\ &= 2\pi x \cdot \frac{1}{x+1} \delta x \end{aligned}$$

$$\text{Now } V = \lim_{\delta x \rightarrow 0} \sum_{x=2}^8 2\pi \cdot \frac{x}{x+1} \delta x$$

$$= 2\pi \int_2^8 \frac{x}{x+1} dx$$

$$= 2\pi \int_2^8 \left( \frac{x+1}{x+1} - \frac{1}{x+1} \right) dx$$

$$= 2\pi \left[ x - \ln |(x+1)| \right]_2^8$$

$$= 2\pi \left[ (8 - \ln(9)) - (2 - \ln(3)) \right]$$

$$= 2\pi \left( 6 - \ln\left(\frac{9}{3}\right) \right)$$

$$= 2\pi (6 - \ln 3) u^2$$

# QUESTION 12

(a) Now, partial fractions gives

$$\frac{x}{(1-x)(1+x^2)} \equiv \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$x = A + Ax^2 + Bx - Bx^2 + C - Cx$$

$$= (A-B)x^2 + (B-C)x + (A+C)$$

Gives:  $A-B=0$  --- (I)

$B-C=1$  --- (II)

$A+C=0$  --- (III)

$\therefore A=B$

so  $A-C=1$

+  $A+C=0$

$2A=1$

$A=\frac{1}{2}$

Have  $A=\frac{1}{2}, B=\frac{1}{2}, C=-\frac{1}{2}$

Integral

$$* \int \frac{x}{(1-x)(1+x^2)} dx$$

$$= \frac{1}{2} \int \left( \frac{1}{1-x} + \frac{x-1}{1+x^2} \right) dx$$

$$= \frac{1}{2} \int \left( \frac{1}{1-x} + \frac{x}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{1}{2} \left[ -\ln|1-x| + \frac{1}{2} \ln(1+x^2) - \tan^{-1}x \right] + C$$

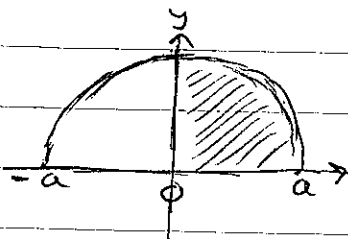
\*  $x < 1$

$$= \frac{1}{2} \left[ \ln \left( \frac{(1+x^2)^{\frac{1}{2}}}{|1-x|} \right) - \frac{1}{2} \tan^{-1}x \right] + C$$

(b) See attached sheet.

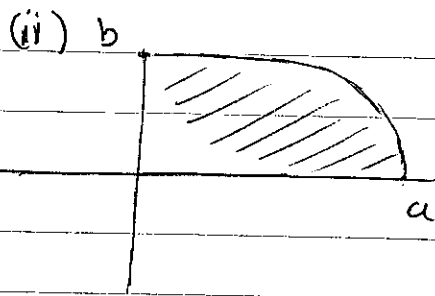
(c) (i)  $\int_0^a \sqrt{a^2-x^2} dx$

describes the area of a quadrant of the circle radius 'a' units, centre (0,0)



So,  $A = \frac{1}{4} \pi r^2$

Given  $\int_0^a \sqrt{a^2-x^2} dx = \frac{1}{4} \pi a^2$



Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

gives  $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$

Area of one quadrant is

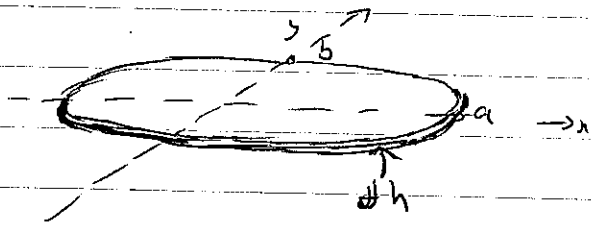
$$= \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$= \frac{b}{a} \cdot \frac{1}{4} \pi a^2 \quad [\text{from (i)}]$$

$$= \frac{1}{4} \pi ab.$$

$\therefore$  Area of four quadrants  $4 \times \frac{1}{4} \pi ab$   
 $= \pi ab.$

(iii) "Typical" cross section is an ellipse in the  $x$ - $y$  plane



Given  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$

then  $\frac{x^2}{\lambda^2 a^2} + \frac{y^2}{\lambda^2 b^2} = 1$

and from part (ii) Area =  $\pi (a\lambda) \cdot (b\lambda)$

$$= \pi ab \lambda^2$$

So  $\delta V = A \cdot \delta h$

$$V = \lim_{\delta h \rightarrow 0} \sum_0^H \pi \cdot ab \lambda^2 \cdot dh$$

$$= \int_0^H \pi ab \lambda^2 dh$$

$$= \pi ab \int_0^H \left[ 1 - \frac{h^2}{H^2} \right]^2 dh$$

$$= \pi ab \int_0^H \left( 1 - 2 \frac{h^2}{H^2} + \frac{h^4}{H^4} \right) dh$$

$$= \pi ab \left[ h - \frac{2}{3} \cdot \frac{h^3}{H^2} + \frac{1}{5} \cdot \frac{h^5}{H^4} \right]_0^H$$

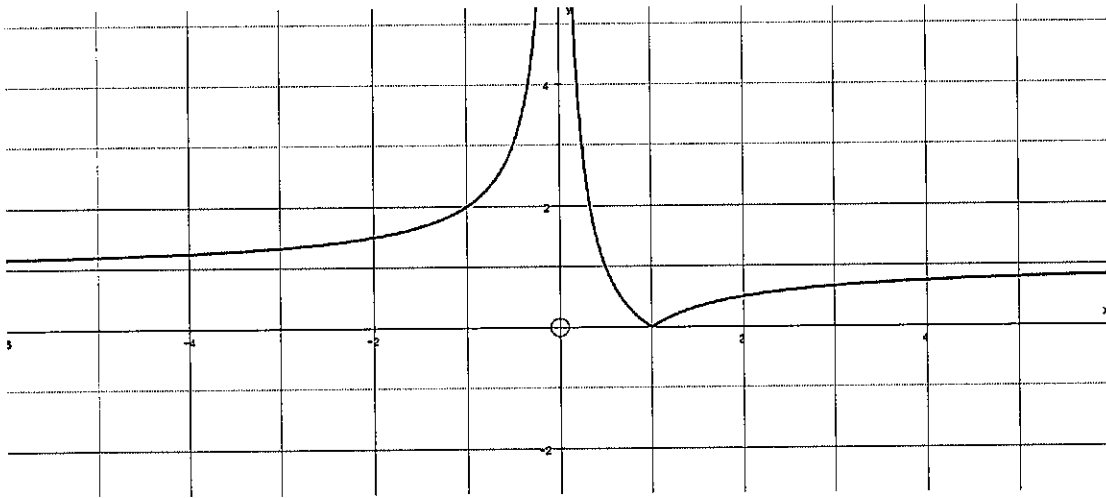
$$= \pi ab \left( H - \frac{2}{3} \cdot H + \frac{1}{5} \cdot H \right)$$

$$V = \frac{8\pi ab H}{15} \omega^3$$

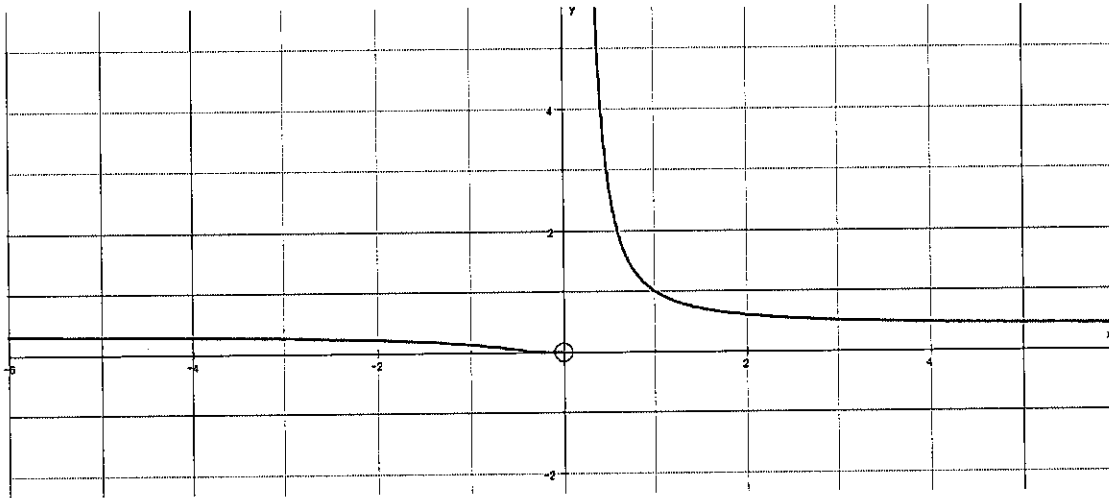


# Graphs

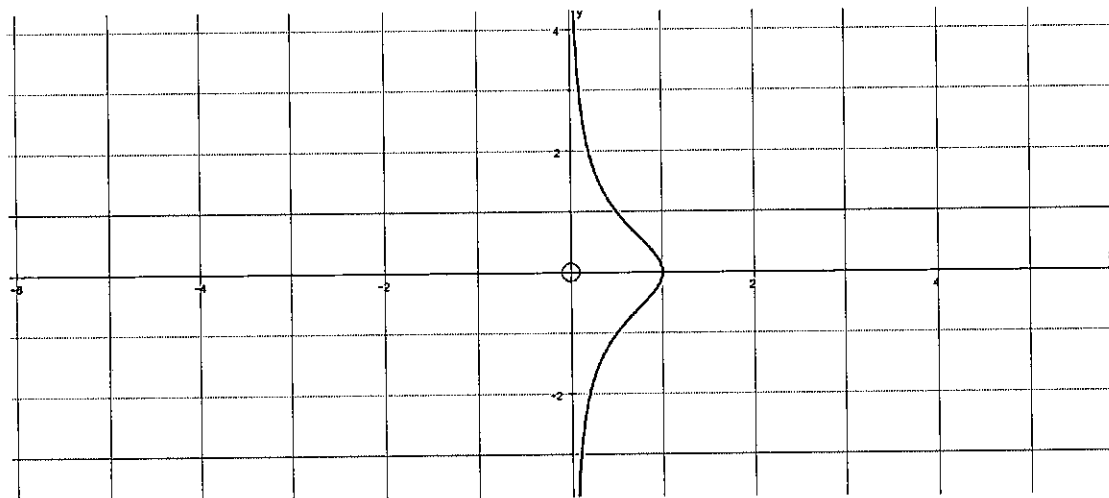
(i).  $y = \frac{|1-x|}{x}$



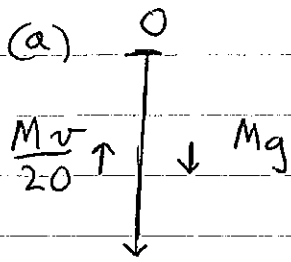
(ii).  $y = e^{\frac{1-x}{x}}$



(iii).  $y^2 = \frac{1-x}{x}$



### QUESTION 13



Direction diagram

$$(i) F = Mg - \frac{Mv}{20}$$

$$\therefore M\ddot{x} = Mg - \frac{Mv}{20}$$

$$\ddot{x} = g - \frac{v}{20}$$

$$(ii) \ddot{x} = \frac{dv}{dt} = \frac{20g - v}{20}$$

$$\frac{dt}{dv} = \frac{20}{20g - v}$$

Integrate

$$\frac{dt}{20} = \frac{dv}{20g - v}$$

$$\frac{t}{20} = -\ln(20g - v) + C$$

at  $t=0, v=0$  so  $C = \ln(20g)$

Then  $\frac{t}{20} = -\ln\left[\frac{20g - v}{20g}\right]$

$$e^{-\frac{t}{20}} = \frac{20g - v}{20g}$$

$$20g e^{-\frac{t}{20}} = 20g - v$$

\*  $v = 20g(1 - e^{-\frac{t}{20}})$

at  $t=2$  gives  $V = 20g(1 - e^{-\frac{1}{10}})$   
 $v = V$

(ii) Also  $\ddot{x} = v \frac{dv}{dx} = \frac{20g-v}{20}$

$$\frac{dv}{dx} = \frac{20g-v}{20}$$

$$= \frac{20g-v}{20v}$$

So  $\frac{dx}{dv} = \frac{20v}{20g-v}$

$$dx = \frac{20v}{20g-v} dv$$

$$\therefore x = -20 \int \frac{20g-v-20g}{20g-v} dv$$

$$= -20 \int \frac{20g-v}{20g-v} - \frac{20g}{20g-v} dv$$

Crosses  $x = -20 \int \left( 1 - \frac{20g}{20g-v} \right) dv$

$$= -20 \left[ v + 20g \ln(20g-v) \right] + C$$

at  $x=0, v=0$  So  $C = 20 \left[ 20g \ln 20g \right]$

$$\therefore x = -20 \left[ v + 20g \ln \left[ \frac{20g-v}{20g} \right] \right]$$

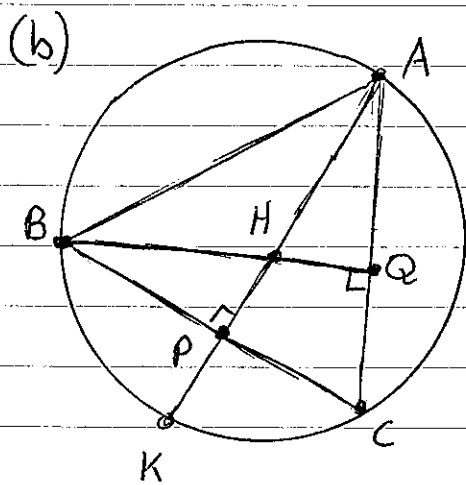
at  $t=2$ ,  $x=h$  and  $v = V = 20g(1 - e^{-\frac{1}{10}})$

We have  $h = -20 \left[ 20g(1 - e^{-\frac{1}{10}}) + 20g \ln(e^{-\frac{1}{10}}) \right]$

( $g=10$ )

$$= -20 \left[ 200(1 - e^{-\frac{1}{10}}) - 20 \right]$$

$\approx 19.3$  metres is height rock ledge.



Let  $\hat{BKA} = \theta$

Then  $\hat{ACB} = \theta$

[angles at the circumference standing on the same arc are equal]

and  $\hat{PAC} = 90^\circ - \theta$

[angle sum of  $\triangle APC$ ]

$\therefore \hat{KBC} = 90^\circ - \theta$  [angles at circumference standing on the same arc are equal]

gives  $\hat{QBC} = 90^\circ - \theta$  [angle sum of  $\triangle QBC$ ]

Then,  $\hat{BPK} = 90^\circ = \hat{BPH}$  [given in data]

$\hat{KBP} = 90^\circ - \theta = \hat{HBC}$  [from above]

PB is common

$\triangle KBP \equiv \triangle HBP$  by congruency test  
AAS

In congruent  $\triangle$ 's corresponding sides are equal,  
 $HP = PK$

(c) Gradient function

$$\frac{\partial x}{\partial^2} + \frac{\partial y}{\partial^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

at  $P(a \cos \theta, b \sin \theta)$

$$m = \frac{-b^2 a \cos \theta}{a^2 b \sin \theta}$$
$$= \frac{-b \cos \theta}{a \sin \theta}$$

So Equation of tangent

$$y - b \sin \theta = \frac{-b \cos \theta}{a \sin \theta} (x - a \cos \theta)$$

$$\frac{y \sin \theta}{b} - \sin^2 \theta = \frac{-x \cos \theta}{a} + \cos^2 \theta$$

$$\therefore \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = \sin^2 \theta + \cos^2 \theta$$
$$= 1$$

(i) Let  $y = 0$ , then  $\frac{x \cos \theta}{a} = 1$

$$x = \frac{a}{\cos \theta}$$

$S\left(\frac{a}{\cos \theta}, 0\right)$

and  $x = 0$ , then  $\frac{y \sin \theta}{b} = 1$

$$y = \frac{b}{\sin \theta}$$

$T\left(0, \frac{b}{\sin \theta}\right)$

$$\therefore \text{Area of } \triangle OST = \frac{1}{2} \cdot \frac{a}{\cos \theta} \cdot \frac{b}{\sin \theta}$$
$$= \frac{ab}{\sin 2\theta}$$

(ii) Let area be  $L$  for  $\triangle APB$

$$\text{then } \frac{dL}{d\theta} = \frac{1}{2} ab [-\sin \theta + \cos \theta]$$

$$\text{and } \frac{d^2L}{d\theta^2} = \frac{1}{2} ab [-\cos \theta - \sin \theta]$$

$$= -\frac{1}{2} ab [\cos \theta + \sin \theta]$$

$$\text{if } ab > 0 \quad \text{and} \quad 0 \leq \theta \leq \frac{\pi}{2}$$
$$\cos \theta + \sin \theta \geq 0$$

$\therefore$  concave down.

$$\text{Let } \frac{dL}{d\theta} = 0, \Rightarrow -\sin \theta + \cos \theta = 0 \quad [ab \neq 0]$$
$$\sin \theta = \cos \theta$$
$$\tan \theta = 1 \quad [\theta \neq \frac{\pi}{2}]$$
$$\theta = \frac{\pi}{4}$$

Maximum Value at  $\theta = \frac{\pi}{4}$

$$\text{So } P \left( a \cos \frac{\pi}{4}, b \sin \frac{\pi}{4} \right) = P \left( \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$$

Gradient of tangent ; Gradient of AB

$$m = - \frac{b^2 \frac{a}{\sqrt{2}}}{a^2 \frac{b}{\sqrt{2}}}$$
$$= - \frac{b}{a}$$
$$m_{AB} = \frac{b-0}{0-a}$$
$$= - \frac{b}{a}$$

Gradient of tangent equals gradient of AB, lines parallel for maximum area.

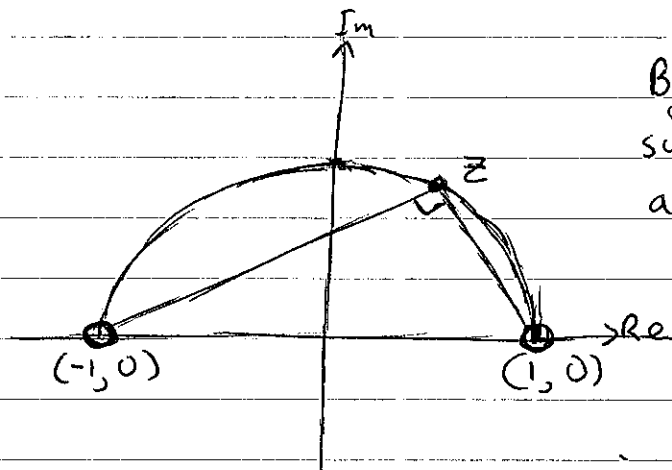
### QUESTION 14

$$(a) \arg \left( \frac{z-1}{z+1} \right) = \arg(z-1) - \arg(z+1)$$

$$\text{So } \arg(z-1) - \arg(z+1) = \frac{\pi}{2} \quad (\text{data})$$

$$\therefore \arg(z-1) = \frac{\pi}{2} + \arg(z+1)$$

$$\begin{aligned} \text{Let } \arg(z-1) &= \theta & \Rightarrow & \theta = \frac{\pi}{2} + \beta \\ \arg(z+1) &= \beta \end{aligned}$$



By exterior angle equals sum of opposite interior angles  $\Rightarrow z$  moves on a semi-circle [radius 1] above the x-axis.

[\* Angle in a semi-circle is a right angle]

$$(b) \quad (i) \text{ Data: tangent at } P \Rightarrow x + p^2 y = 2cp \quad \dots (I)$$

$$\text{by symmetry at } Q \Rightarrow x + q^2 y = 2cq \quad \dots (II)$$

$$(I) - (II) \Rightarrow (p^2 - q^2)y = 2c(p - q)$$

$$y = \frac{2c}{p+q}$$

$$\text{Substitute } x + p^2 \cdot \frac{2c}{p+q} = 2cp$$

$$x = 2cp \left( 1 - \frac{p}{p+q} \right)$$

$$x = \frac{2cp}{p+q} (p+q - p)$$

$$x = \frac{2cpq}{p+q}$$

$$\therefore T \left( \frac{2cpq}{p+q}, \frac{2c}{p+q} \right)$$

(ii) By substitution  
in  $xy = k^2$   $\frac{pq}{(p+q)^2} \Rightarrow \frac{2cpq}{p+q} \times \frac{2c}{p+q} = k^2$

$$\frac{4c^2 pq}{(p+q)^2} = k^2$$

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

(c) (i) If  $\omega$  is a solution and  $\omega \neq 1$   
then  $\omega + 1 \neq 0$

Given  $\omega^5 + 1 = 0$   
then  $(\omega + 1)(\omega^4 - \omega^3 + \omega^2 - \omega + 1) = 0$

$\omega + 1 \neq 0 \quad \therefore \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$   
 $\omega^4 + \omega^2 + 1 = \omega^3 + \omega$

(ii) Let  $z = r \operatorname{cis} \theta$

now  $|z| = r$ ,  $|z^5| = |z|^5 = 1$

So  $r = 1$

gives  $z = \cos \theta + i \sin \theta$



$$z^5 = \cos 5\theta + i \sin 5\theta \quad [\text{De Moivre}]$$

Equating  $\cos 5\theta + i \sin 5\theta = -1 + 0.i$

then  $\cos 5\theta = -1$  and  $\sin 5\theta = 0$

$$\therefore 5\theta = \pm\pi, \pm 3\pi, \pm 5\pi \dots$$

$$\theta = \pm \frac{\pi}{5}, \pm \frac{3\pi}{5}, \pm \pi, \pm \frac{7\pi}{5}, \pm \frac{9\pi}{5} \dots$$

(i)  $\theta = \pi$  ,  $\cos \pi + i \sin \pi = -1$

(ii)  $\theta = \frac{\pi}{5}$  ,  $\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} = \omega$

(iii)  $\theta = -\frac{\pi}{5}$  ,  $\cos \frac{\pi}{5} + i \sin -\frac{\pi}{5} = \bar{\omega} = \cos \frac{\pi}{5} - i \sin \frac{\pi}{5}$

(iv)  $\theta = \frac{3\pi}{5}$  ,  $\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} = \omega^3$

(v)  $\theta = -\frac{3\pi}{5}$  ,  $\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} = \bar{\omega}^3 = \cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5}$

Sum of roots for  $z^4 - z^3 + z^2 - z + 1 = 0$

i)  $\omega + \bar{\omega} + \omega^3 + \bar{\omega}^3 = 1$

$$2 \left( \cos \frac{\pi}{5} \right) + 2 \cos \left( \frac{3\pi}{5} \right) = 1$$

$$\cos \left( \frac{\pi}{5} \right) + \cos \left( \frac{3\pi}{5} \right) = \frac{1}{2}$$

(d) (i) Given  $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \cdot dx$   $n \geq 1$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cdot \sin^2 x \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cdot dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin^{n-2} x \cdot dx$$

$$\begin{aligned}
 I_n &= I_{n-2} - \int_0^{\frac{\pi}{2}} \underbrace{\cos x}_u \left( \underbrace{\cos x \cdot \sin^{n-2} x}_{dv} \right) dx \\
 &= I_{n-2} - \left[ \left( \cos x \cdot \frac{1}{n-1} \sin^{n-1} x \right) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\sin x \cdot \frac{1}{n-1} \sin^{n-1} x dx \right] \\
 &= I_{n-2} - \left[ 0 + \frac{1}{n-1} \int_0^{\frac{\pi}{2}} \sin^n x dx \right]
 \end{aligned}$$

Now  $I_n = I_{n-2} - \frac{1}{n-1} I_n$

$$I_n \left( 1 + \frac{1}{n-1} \right) = I_{n-2} \quad \left[ 1 + \frac{1}{n-1} = \frac{n-1+1}{n-1} \right]$$

$$\frac{n}{n-1} I_n = I_{n-2} = \frac{n}{n-1} I_n$$

$$I_n = \left( \frac{n-1}{n} \right) I_{n-2}$$

(ii) From (i)  $I_5 = \frac{4}{5} I_3$

$$I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x dx$$

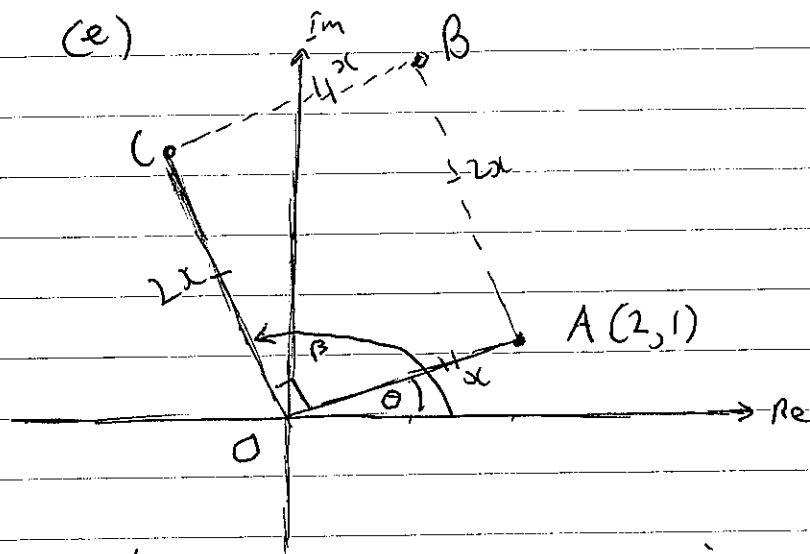
$$= \left[ -\cos x \right]_0^{\frac{\pi}{2}}$$

$$= -(0 - 1)$$

$$= 1$$

Then  $I_5 = \frac{4}{5} \left( \frac{2}{3} \times 1 \right)$

$$= \frac{8}{15}$$



For A,  $|2+i| = \sqrt{5}$

$\therefore OA = \sqrt{5}$  and  $OC = 2\sqrt{5}$

Now, OC is given by rotating OA through  $\frac{\pi}{2}$  and doubling the length

So  $\beta = \arg(OC) = \arg(2+i) + \frac{\pi}{2}$

$= \arg(2+i) + \arg(ki)$

$= \arg[k(-1+2i)]$

Where  $|k(-1+2i)| = \sqrt{k^2 \cdot 5}$

$= k\sqrt{5}$

$= 2\sqrt{5}$

$\therefore k = 2$

Then C corresponds to the complex number  $2(-1+2i)$

$= -2 + 4i$

## Question 15

$$(a) (i) (\alpha^2 + \beta^2 + \gamma^2) = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$\begin{aligned} * \text{ Sum roots } &= -\frac{b}{a} &= 0^2 - 2 \times 1 \\ \alpha + \beta + \gamma &= -\frac{0}{1} &= -2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} * \text{ Sum two is } &= \frac{c}{a} \\ \alpha\beta + \alpha\gamma + \beta\gamma &= \frac{c}{a} \\ &= \frac{1}{1} \end{aligned}$$

(ii) From (i), Sum of squares on roots  $< 0$   
Then at least one of the roots is complex, since  $(z^2) = -1$ .

Coefficients are real for  $x^3 + x + 12$ , so complex roots appear as conjugate pairs. Product of complex conjugates is real, so to get product of roots ( $\alpha\beta\gamma = -12$ ) as real the other root must be real.

$$\text{or let } P(x) = x^3 + x + 12$$

$$P'(x) = 3x^2 + 1$$

Since  $x^2 > 0$ ,  $P'(x) \neq 0$ , no turning points on this curve and since it is odd degree 3 it only cuts the  $x$ -axis at one point. Therefore, only one real root.

(iii) Let complex conjugate roots be  $a+bi$  and  $a-bi$   
[ $b \neq 0$ ]

$$\text{then } x^3 + x + 12 = (x - \alpha) [x - (a+bi)] [x - (a-bi)]$$

$$x^3 + x + 12 = (x - \alpha) [x^2 - 2\alpha x + (a^2 + b^2)]$$

Given  $(a^2 + b^2), \alpha = -12$

$$(a^2 + b^2) > 0 \quad \text{so} \quad \alpha < 0$$

With  $\alpha$  as the real root, the function will change sign as it passes through  $\alpha$ .

Let  $\alpha = -3$ ,  $(-3)^3 + -3 + 12 = -18 < 0$

Let  $\alpha = -2$ ,  $(-2)^3 + -2 + 12 = 2 > 0$

Sign change gives  $-3 < \alpha < -2$ \*

(b) Let point where the water strikes the wall be  $(d, h)$  [h is height above ground]

Given range is  $\frac{v^2 \sin 2\theta}{g}$

then  $0 < d < \frac{v^2 \sin 2\theta}{g}$

\* max distance at  $\theta = \frac{\pi}{4}$  is  $\frac{v^2}{g}$

Now  $0 < \frac{gd}{v^2} < \sin 2\theta$  [ $0 < \theta < \frac{\pi}{2}$   
so  $0 < 2\theta < \pi$ ]

(i)  $0 < 2\theta < \frac{\pi}{2}$  ie  $0 < \theta < \frac{\pi}{4}$

then  $\sin^{-1}\left(\frac{gd}{v^2}\right) < 2\theta$

$$\frac{1}{2} \sin^{-1}\left(\frac{gd}{v^2}\right) < \theta \quad \text{let } \beta = \frac{1}{2} \sin^{-1}\left(\frac{gd}{v^2}\right)$$

gives  $\beta < \theta$  [Note: given  $V > \sqrt{gd}$   
then  $0 < \frac{gd}{V^2} < 1$ ]

(ii) If  $\frac{\pi}{2} < 2\theta < \pi$  then  $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

and angle  $\sin 2\theta = \sin(\pi - 2\theta)$

So  $0 < d < \frac{V^2 \sin(\pi - 2\theta)}{g}$

$$\frac{gd}{V^2} < \sin(\pi - 2\theta)$$

$$\sin^{-1}\left(\frac{gd}{V^2}\right) < \pi - 2\theta$$

Given  $2\theta < \pi - \sin^{-1}\left(\frac{gd}{V^2}\right)$

$$\theta < \frac{\pi}{2} - \frac{1}{2} \sin^{-1}\left(\frac{gd}{V^2}\right)$$

$$\theta < \frac{\pi}{2} - \beta$$

From (i) & (ii) we have

$$\beta < \theta < \frac{\pi}{2} - \beta$$

(c) let  $u^6 = x$

then

$$\begin{aligned} (u^6)^{\frac{1}{2}} &= x^{\frac{1}{2}} \\ u^3 &= x^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} (u^6)^{\frac{1}{3}} &= x^{\frac{1}{3}} \\ u^2 &= x^{\frac{1}{3}} \end{aligned}$$

also  $6u^5 \cdot du = dx$

on substitution

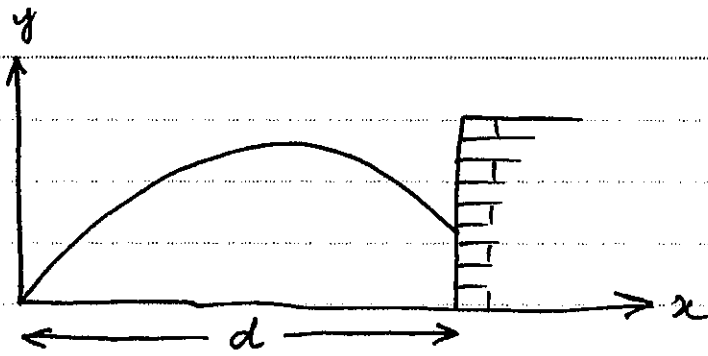
$$\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = 6 \int \frac{u^5}{u^3 + u^2} du$$

$$= 6 \int \frac{u^3}{u+1} du$$

$$= 6 \int u^3 - u + 1 - \frac{1}{u+1} du$$

$$= 6 \left[ \frac{1}{4} u^4 - \frac{1}{2} u^2 + u - \ln(u+1) \right] + C$$

$$= 6 \left( \frac{x^{\frac{2}{3}}}{4} - \frac{x^{\frac{1}{3}}}{2} + x^{\frac{1}{6}} - \ln(x^{\frac{1}{6}} + 1) \right) + C$$



$$\begin{aligned} \ddot{y} &= -g & \text{--- (1)} \\ \dot{y} &= -gt + V \sin \theta & \text{--- (2)} \\ y &= -\frac{1}{2}gt^2 + Vt \sin \theta & \text{--- (3)} \end{aligned}$$

$$\begin{aligned} \ddot{x} &= 0 & \text{--- (4)} \\ \dot{x} &= V \cos \theta & \text{--- (5)} \\ x &= Vt \cos \theta & \text{--- (6)} \end{aligned}$$

When  $x = d$ :  $Vt \cos \theta = d$

$$t = \frac{d}{V \cos \theta} \text{ sub in (3)}$$

$$\begin{aligned} \therefore y &= -\frac{g}{2} \cdot \frac{d^2}{V^2 \cos^2 \theta} + V \sin \theta \cdot \frac{d}{V \cos \theta} \\ &= -\frac{gd^2}{2V^2 \cos^2 \theta} + d \tan \theta \end{aligned}$$

when  $x = d$  we want  $y > 0$

$$\therefore -\frac{gd^2}{2V^2 \cos^2 \theta} + d \tan \theta > 0$$

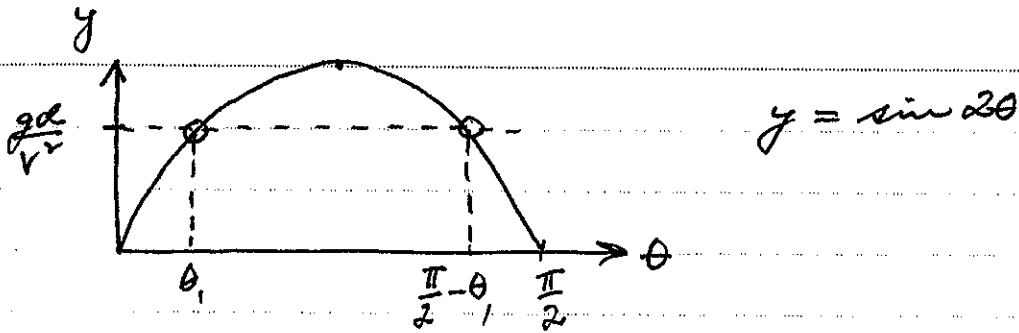
$$-d \left( \frac{gd}{2V^2 \cos^2 \theta} - \tan \theta \right) > 0$$

$$\text{i.e. } \frac{gd}{2V^2 \cos^2 \theta} - \tan \theta < 0$$

$$\Rightarrow \frac{gd}{2V^2 \cos^2 \theta} < \frac{\sin \theta}{\cos \theta}$$

$$x \cos^2 \theta \Rightarrow \frac{gd}{2V^2} < \sin \theta \cos \theta$$

$$\therefore \frac{gd}{V^2} < \sin 2\theta$$



when  $\sin 2\theta = \frac{gd}{v^2}$

$$2\theta = \sin^{-1} \frac{gd}{v^2}, \quad (\text{ALSO } \pi - \sin^{-1} \frac{gd}{v^2})$$

$$\Rightarrow \theta = \frac{1}{2} \sin^{-1} \frac{gd}{v^2}, \quad \frac{\pi}{2} - \frac{1}{2} \sin^{-1} \frac{gd}{v^2}$$

Then  $\sin 2\theta > \frac{gd}{v^2}$  from above graph

$$\Rightarrow \frac{1}{2} \sin^{-1} \frac{gd}{v^2} < \theta < \frac{\pi}{2} - \frac{1}{2} \sin^{-1} \frac{gd}{v^2}$$



## QUESTION 16

$$(a) (i) P'(x) = n \cdot x^{n-1} + a$$

Given  $x = \alpha$  is a double root

$$\text{then } P(\alpha) = 0 = P'(\alpha)$$

$$\underline{\text{So}} \quad n \cdot \alpha^{n-1} + a = 0$$

$$\therefore \alpha^{n-1} = -\frac{a}{n} \quad \dots (I)$$

$$(ii) \quad \text{From (i)} \quad P(\alpha) = 0$$

$$\underline{\text{so}} \quad \alpha^n + a\alpha - b = 0 \quad \dots (II)$$

$$\text{From (i)} \quad \alpha^n = \left(-\frac{a}{n}\right)\alpha$$

$$\text{So} \quad \alpha \left(-\frac{a}{n}\right) + a\alpha - b = 0$$

$$a\alpha \left(1 - \frac{1}{n}\right) = b$$

$$a\alpha = \frac{nb}{(n-1)}$$

$$\alpha = \frac{nb}{a(n-1)} \quad *$$

Sub into II, \*

$$\left(\frac{nb}{a(n-1)}\right)^n + \left(\frac{nb}{a(n-1)}\right)a - b = 0$$

$$(nb)^n + nb [a^n(n-1)^{n-1}] - b [a^n(n-1)^n] = 0$$

$$(nb)^n = a^n b [(n-1)^n - n(n-1)^{n-1}]$$

$$= a^n b (n-1)^{n-1} [(n-1) - n]$$

$$n^n b^n = -a^n b (n-1)^{n-1}$$

gives  $\frac{b^{n-1}}{(n-1)^{n-1}} = \frac{-a^n}{n^n}$

So  $\left(\frac{a}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$

(b) (i)  $y = e^{-x} \cdot \sin x$

Now,  $e^{-x} > 0$  for all  $x$

Then  $y = 0$  when  $\sin x = 0$  [ $x \geq 0$ ]

true at  $x = \pi, 2\pi, 3\pi, 4\pi, \dots$

So points  $P(\pi, 0)$ ;  $Q(2\pi, 0)$ ;  $R(4\pi, 0)$ ...

(ii) Given  $\int e^{-x} \cdot \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x)$

we have  $A_1 = -\frac{1}{2} \left[ e^{-x} (\sin x + \cos x) \right]_0^\pi$

$$= -\frac{1}{2} \left[ (e^{-\pi} (0 + -1)) - (e^0 (0 + 1)) \right]$$

$$= \frac{1}{2} (e^{-\pi} + 1)$$

$$A_2 = -\frac{1}{2} \left[ e^{-x} (\sin x + \cos x) \right]_{2\pi}^{3\pi}$$

$$= -\frac{1}{2} \left[ e^{-3\pi} (0 + -1) - e^{-2\pi} (0 + 1) \right]$$

$$= \frac{1}{2} (e^{-3\pi} + e^{-2\pi})$$

⋮

$$\begin{aligned}
 A_n &= -\frac{1}{2} \left[ e^{-x} (\sin x + \cos x) \right]_{(2n-2)\pi}^{(2n-1)\pi} \\
 &= -\frac{1}{2} \left[ e^{-(2n-1)\pi} (0+1) - e^{-(2n-2)\pi} (0+1) \right] \neq \\
 &= \frac{1}{2} \left( e^{(1-2n)\pi} + e^{(2-2n)\pi} \right)
 \end{aligned}$$

\* Since  $\sin(2n-1)\pi = 0 = \sin(2n-2)\pi$   
 and  $\cos(2n-1)\pi = -1$  ;  $\cos(2n-2)\pi = 1$

We can write  $A_n = \frac{1}{2} \left( e^{(1-2n)\pi} + e^{(2-2n)\pi} \right)$

(ii) For G.P.  $\frac{A_2}{A_1} = \frac{\frac{1}{2} (e^{-3\pi} + e^{-2\pi})}{\frac{1}{2} (e^{-\pi} + e^0)}$   
 $= \frac{e^{-2\pi} (e^{-\pi} + 1)}{(e^{-\pi} + 1)}$   
 $= e^{-2\pi}$

and  $\frac{A_3}{A_2} = \frac{\frac{1}{2} e^{-4\pi} (e^{-\pi} + 1)}{\frac{1}{2} e^{-2\pi} (e^{-\pi} + 1)}$   
 $= e^{-2\pi}$

Show  $\frac{A_2}{A_1} = \frac{A_3}{A_2}$  has common ratio  $\frac{1}{e^{2\pi}}$

and so it represents a G.P. ,  $a = \frac{1}{2} (e^{-\pi} + 1)$   
 $r = e^{-2\pi}$

Now,  $\left| \frac{1}{e^{2\pi}} \right| < 1$

With  $|r| < 1$  ,  $S_\infty = \frac{\frac{1}{2} (e^{-\pi} + 1)}{1 - e^{-2\pi}}$

$$S_{\infty} = \frac{1+e^{\pi}}{2e^{\pi}} \bigg/ \frac{e^{2\pi}-1}{e^{2\pi}}$$

$$= \frac{(1+e^{\pi})e^{\pi}}{2(e^{2\pi}-1)}$$

$$= \frac{e^{\pi}(1+e^{\pi})}{2(e^{\pi}+1)(e^{\pi}-1)}$$

$$S_{\infty} = \frac{e^{\pi}}{2(e^{\pi}-1)}$$

(iv) Let  $B_1, B_2, B_3, \dots$  be areas "below" the curve.

Then  $A_1 + A_2 + A_3 + \dots + A_n + |B_1 + B_2 + B_3 + \dots + B_n|$  gives total area bounded by the curve.

$$\int_0^{\infty} e^{-x} \cdot \sin x \, dx = A_1 + A_2 + \dots - (B_1 + B_2 + \dots)$$

$$= \frac{e^{\pi}}{2(e^{\pi}-1)} - (B_1 + B_2 + \dots)$$

From data  $\frac{1}{2} = \frac{e^{\pi}}{2(e^{\pi}-1)} - (B_1 + B_2 + \dots)$

Thus  $B_1 + B_2 + \dots + B_n = \frac{e^{\pi} - (e^{\pi}-1)}{2(e^{\pi}-1)}$

$$= \frac{1}{2(e^{\pi}-1)}$$

Exact value  $\int_0^{\infty} |e^{-x} \cdot \sin x| \, dx = \frac{e^{\pi}}{2(e^{\pi}-1)} + \frac{1}{(e^{\pi}-1)}$

$$= \frac{e^{\pi} + 1}{2(e^{\pi}-1)}$$

~~QED~~