

SYDNEY GIRLS HIGH SCHOOL

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Diagrams are NOT drawn to scale.
- All necessary working should be shown in every question.
- Start each question on a new page.

Total marks – 100

Section I Pages 3-7

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 8-17

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

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[2]

Section I

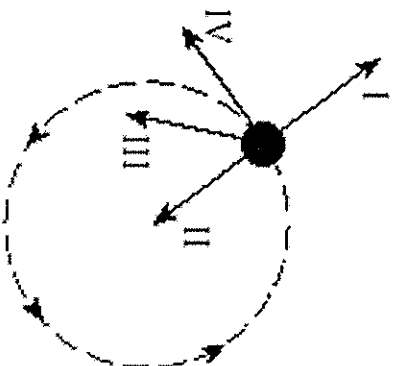
10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

- 1 An object moves in a circular path at a constant speed. Which vector in the diagram below best represents the object's acceleration?

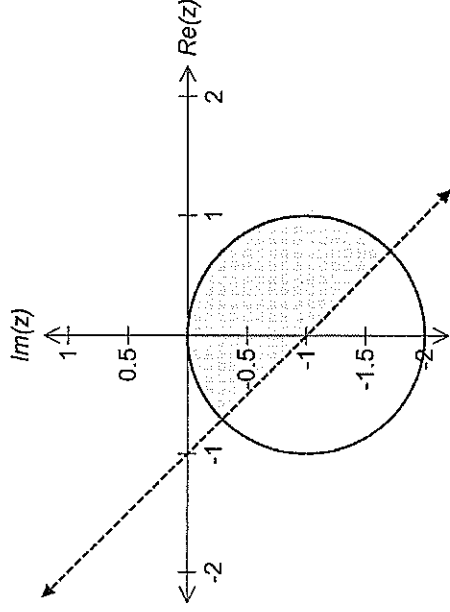


- (A) I
(B) II
(C) III
(D) IV

2. Which of the following cannot be the argument of a complex number z such that $z^9 = -1 + i$?

- (A) $\frac{11\pi}{36}$
(B) $\frac{\pi}{12}$
(C) $\frac{29\pi}{36}$
(D) $\frac{19\pi}{36}$

3. Consider the Argand diagram below:



Which pair of inequalities correctly define the shaded area?

- (A) $|z+i| \leq 1$ and $0 \leq \arg(z+1) < -\frac{\pi}{4}$
- (B) $|z-i| \leq 1$ and $0 \leq \arg(z-1) < -\frac{\pi}{4}$
- (C) $|z-i| \leq 1$ and $0 \leq \arg(z-1) < \frac{\pi}{4}$
- (D) $|z+i| \leq 1$ and $0 \leq \arg(z+1) < \frac{\pi}{4}$

4. A point $P(x, y)$ moves so that the ratio of its distance from the point $(1, 0)$ and the line $x = 2$ is a constant e , where $0 < e < 1$. The locus described by this point would be a:

- (A) Circle
- (B) Parabola
- (C) Hyperbola
- (D) Ellipse.

5. Given that $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$, then if $P(x) = (4x^2 + ax - 1)^2$, the value of

a is:

- (A) 1
- (B) 2
- (C) $\frac{1}{2}$
- (D) 0.

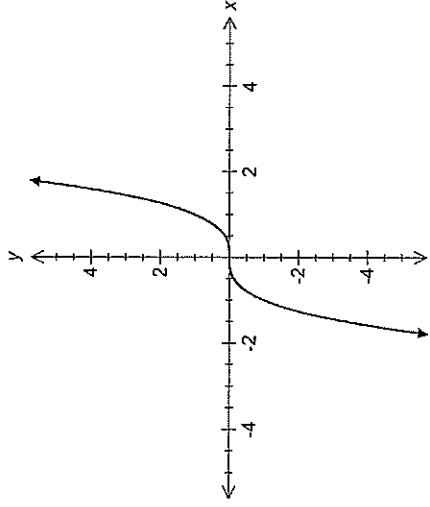
6. $\int_0^x 5 \sin x \cos^4 x \, dx =$

- (A) 0
- (B) 2
- (C) -2
- (D) 20

7. Given that $\int \sec^n x \, dx = \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$ then $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx =$

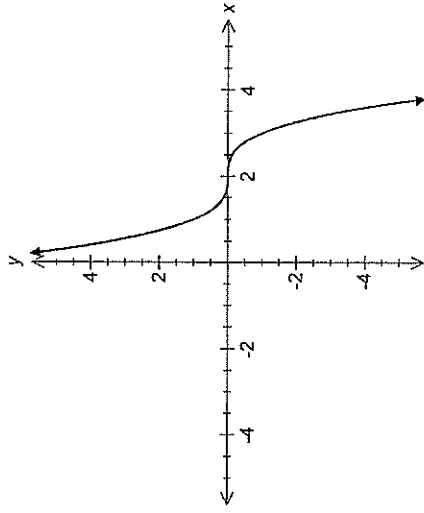
- (A) $\frac{4}{3}$
- (B) 1
- (C) $\frac{5}{6}$
- (D) $\frac{6+4\sqrt{2}}{9}$

8. The graph of $y = f(x)$ is shown below.

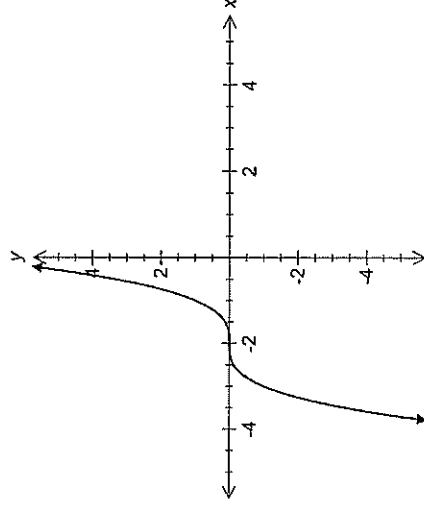


Which graph best represents $y = f(2-x)$?

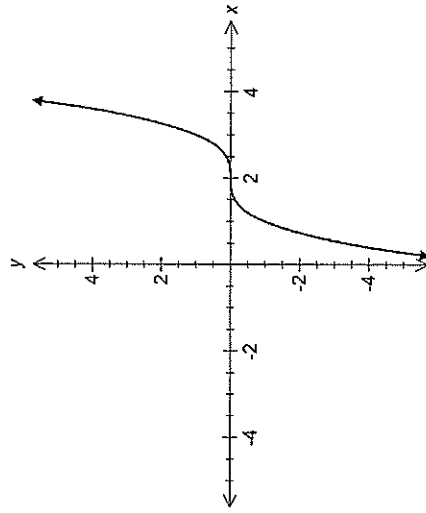
(A)



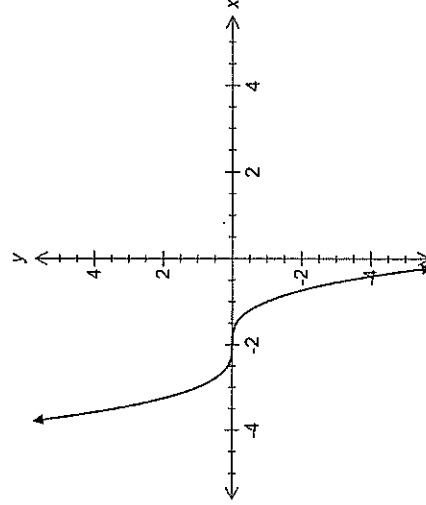
(B)



(C)



(D)



[6]

9. The curve $4y^2 - 9x^2 = 36$ is defined parametrically by the equations:

- (A) $x = 4 \sec \theta, y = 9 \tan \theta$
- (B) $x = 3 \sec \theta, y = 2 \tan \theta$
- (C) $x = 2 \tan \theta, y = 3 \sec \theta$
- (D) $x = 9 \tan \theta, y = 4 \sec \theta$

10. $\frac{3}{(x^2+2)(x-1)} = \frac{Px+Q}{x^2+2} + \frac{R}{x-1}$ where $x \neq 1$, and P, Q, R are constants. Which one of the following statements is false?

- (A) $3 = (Px+Q)(x-1) + R(x^2+2)$
- (B) $R = 1$
- (C) $3 = 2R - Q$
- (D) $3 = P + R$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) The complex number w is given by $w = -1 + \sqrt{3}i$.

(i) Show that $w^2 = 2\bar{w}$. 2

(ii) Evaluate $|w|$ and $\arg w$. 2

(iii) Show that w is a root of the equation $w^3 - 8 = 0$. 1

(b) Sketch the region of the Argand diagram whose points satisfy the inequalities

$|z - \bar{z}| \leq 4$ and $\frac{-\pi}{3} \leq \arg z \leq \frac{\pi}{3}$. 2

Question 11 continues on page 9

- (c) $ABCD$ is a quadrilateral whose diagonals AC and BD are equal and bisect each other at the origin. A is the complex number z and $\angle AOB = 30^\circ$.
- (i) Find the coordinates of B , C , and D in terms of z . 2
- (ii) What type of quadrilateral is $ABCD$? (Justify your answer) 1
- (d) The complex number z is a function of the real number r given by the rule 3
- $$z = \frac{r-i}{r+i}, \quad 0 \leq r \leq 1.$$
- Evaluate $|z|$ and hence describe the locus of z as r varies from 0 to 1.

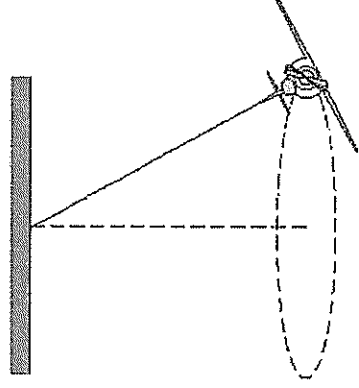
- (e) By completing the square, find $\int \frac{dx}{x^2 + 4x - 1}$. 2

End of Question 11

Question 12 (15 marks)

- (a) (i) Express $\frac{x+7}{x^2(x+2)}$ in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. 2
- (ii) Hence or otherwise find $\int \frac{x+7}{x^2(x+2)} dx$. 2
- (b) Find $\int \frac{dx}{\sqrt{1+4x^2}}$. 2
- (c) Evaluate $\int_0^1 xe^{-x} dx$. 4

- (d) A toy airplane attached to a string 1.5m long moves in a horizontal circle as shown in the diagram below. The string makes an angle of 30° with the vertical. (Use $g = 10 \text{ ms}^{-2}$)



- (i) What is the mass of the plane if the tension in the string is 5N? 3
- (ii) Calculate the period of the plane's motion. 2

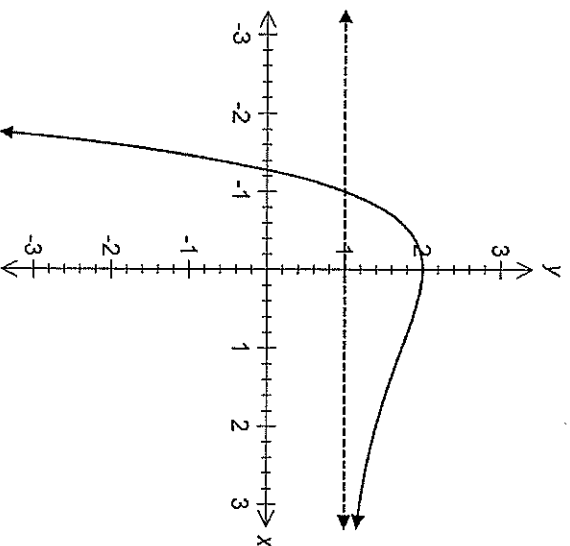
End of Question 12

[10]

Question 13 (15 marks)

- (a) Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, show that $\int_0^{\frac{\pi}{3}} \frac{1}{1 + \sin \theta} d\theta = \sqrt{3} - 1$. 4

- (b) The diagram shows the graph $y = f(x)$.



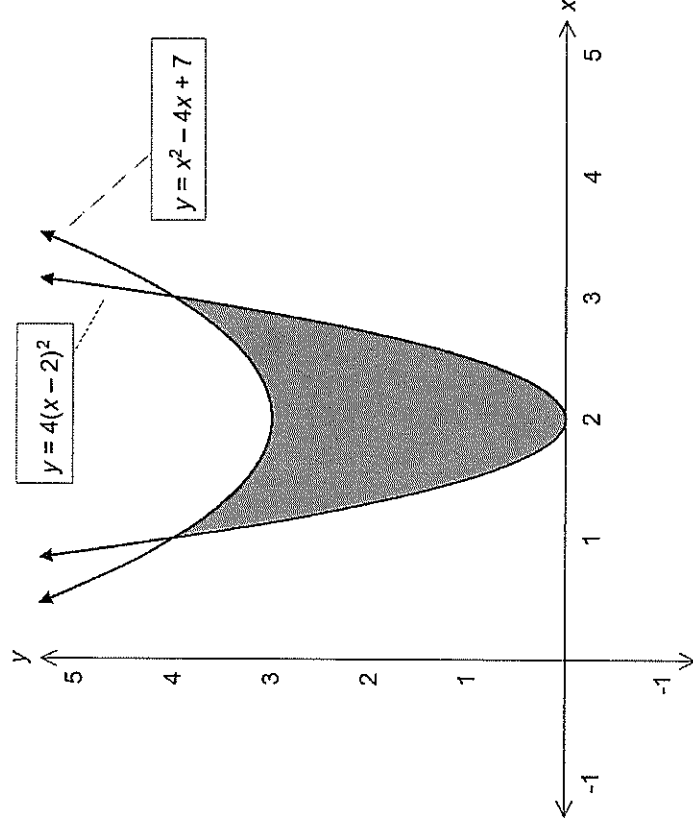
Draw separate one-third page sketches of the graphs of the following:

- | | | |
|-------|----------------------|---|
| (i) | $y = f(x)$ | 1 |
| (ii) | $y = \frac{1}{f(x)}$ | 2 |
| (iii) | $y = [f(x)]^2$ | 2 |
| (iv) | $y = \ln[f(x)]$. | 2 |

Question 13 continues on page 12

- (c) Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region between the curves $y = 4(x-2)^2$ and $y = x^2 - 4x + 7$ about the y axis.

4



End of Question 13

[12]

Question 14 (15 marks)

- (a) The Earth's orbit around the sun is elliptical with the Sun as one of the foci. The semi major axis of the Earth's orbit is 1.486×10^8 kilometres and its eccentricity is 0.017.
- (i) How close to the sun does the earth come? 2
- (ii) What is the greatest possible distance between the sun and the earth? 2
- (b) Determine all the roots of the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ given that there is a root of multiplicity 3. 3
- (c) The polynomial $P(x) = x^3 + ax^2 + bx + 6$ where a, b are real numbers has $1 - i$ as one zero.
- (i) Find a and b . 2
- (ii) Factorise $P(x)$ over the complex field. 1
- (iii) Factorise $P(x)$ over the real field. 1
- (d) Show that the equation $3x^5 + 20x^3 + 45x = c$ can have only one real root, and find the value of the constant c , if the sum of the other (complex) roots is -7 . 4

End of Question 14

[13]

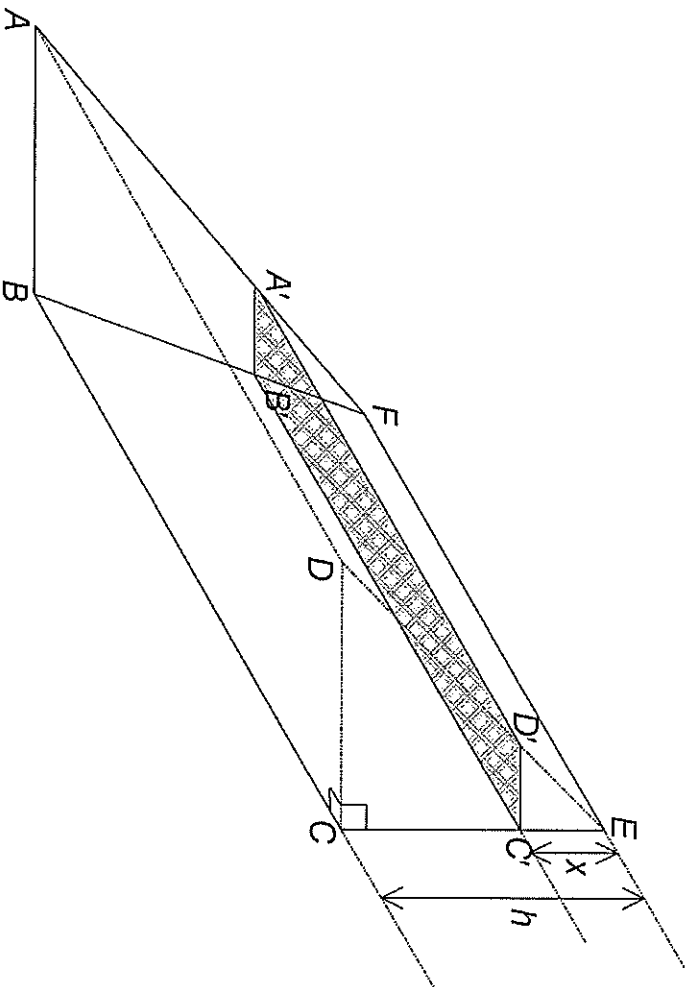
Question 15 (15 Marks)

(a) For the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, find:

- | | | |
|-------|---|---|
| (i) | its eccentricity | 1 |
| (ii) | the coordinates of its foci | 1 |
| (iii) | the equations of its directrices | 1 |
| (iv) | the equations of its asymptotes | 1 |
| (v) | the equation of the chord of contact of tangents drawn from the point (1,2) | 1 |
| (b) | Consider the region enclosed by the curves $x = y^2$ and $x = 2 - y^2$. Find the volume of the solid formed when this region is rotated about the line $x = 3$ by taking slices perpendicular to the axis of rotation. | 4 |

Question 15 continues on page 15

- (c) Consider a solid $ABCDEF$ whose height is h , and whose base is a rectangle $ABCD$, where $AB = a$, $BC = b$, and the top edge $EF = c$



Consider a rectangular slice $A'B'C'D'$ (parallel to the base $ABCD$) x units from the top edge, with width Δx .

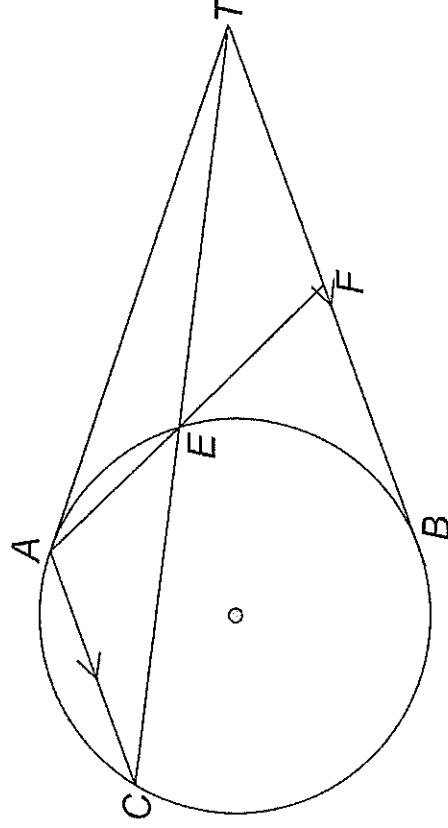
- (i) Show that the volume of the slice is $\Delta V = \left(\frac{x}{h}a\right)\left(c + \frac{b-c}{h}x\right)\Delta x$. 3
- (ii) Hence show that the volume of the solid is $\frac{ha}{6}(2b+c)$. 3

End of question 15

[15]

Question 16 (15 marks)

- (a) Two tangents TA , TB are drawn from a point T to a given circle. Through A , a chord AC is drawn parallel to the other tangent TB and TC meets the circle at E .



- (i) Prove $\triangle AFT$ is similar to $\triangle EFT$. 2
- (ii) Hence show that $TF^2 = AF \times EF$. 1
- (iii) Hence or otherwise prove that AE extended bisects TB . 2
- (b) (i) Prove that the equation of the tangent to the curve $x = \frac{1}{1+t^4}$, $y = \frac{t^5}{1+t^4}$ at the point with parameter t is $4y + (5t + t^5)x = 5t$. 3
- (ii) This tangent meets the coordinate axes OX and OY in the points P and Q . 3
 Show that the area of the triangle OPQ never exceeds $\left(\frac{15}{32}\right)\left(\frac{5}{3}\right)^{\frac{1}{4}}$.

Question 16 continues on page 17

(c) Prove by mathematical induction for all positive integers n ,

4

$$\tan^{-1}\left(\frac{1}{2 \times 1^2}\right) + \tan^{-1}\left(\frac{1}{2 \times 2^2}\right) + \dots + \tan^{-1}\left(\frac{1}{2 \times n^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2n+1}\right).$$

End of paper

[17]

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$



Extension 2 Trial HSC 2013

Solutions.

Multiple Choice.

- 1) B
- 2) C
- 3) All responses marked correct
- 4) D
- 5) B
- 6) B
- 7) ~~X~~A
- 8) A
- 9) C
- 10) D

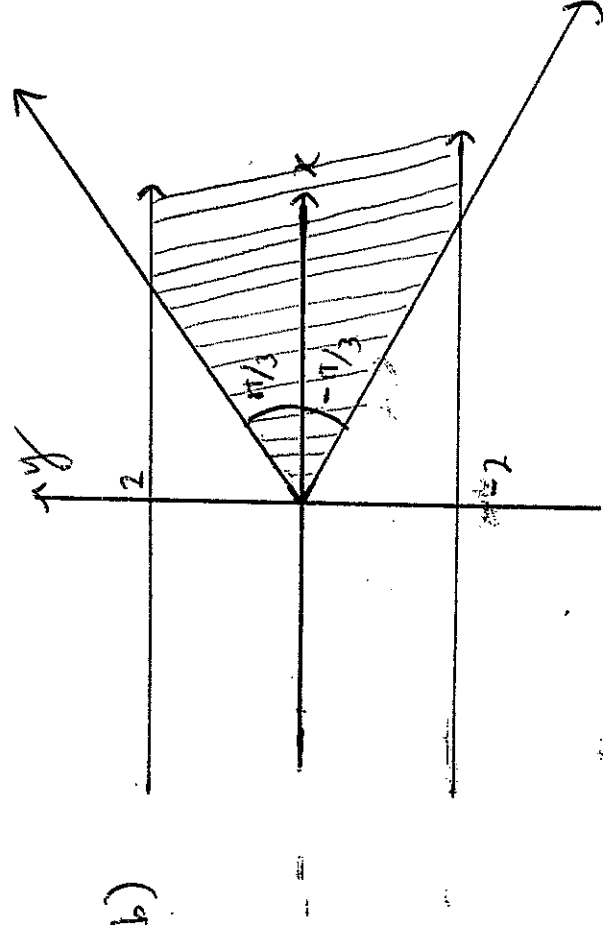
ii)

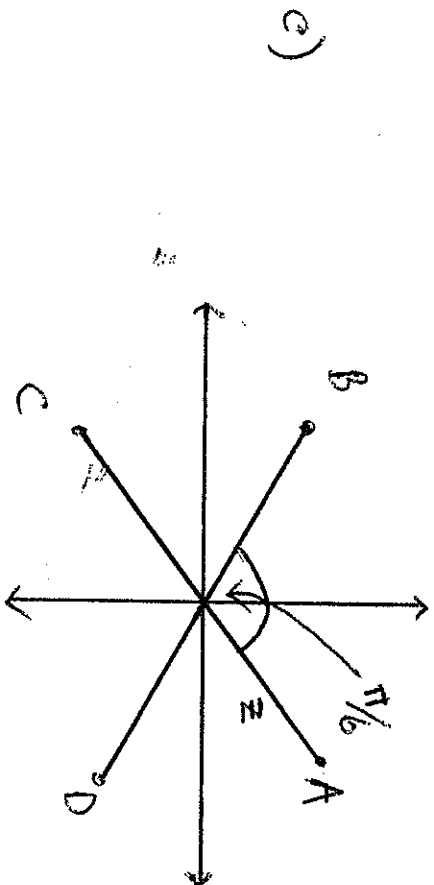
$$\begin{aligned} a) \quad i) \quad w^2 &= (-1 + \sqrt{3}i)^2 \\ &= 1 - 2\sqrt{3}i - 3 \\ &= -2 - 2\sqrt{3}i \\ \bar{w} &= 2(-1 - \sqrt{3}i) \\ &= -2 - 2\sqrt{3}i \\ &= w^2 \end{aligned}$$

$$\begin{aligned} ii) \quad |w| &= \sqrt{(-1)^2 + \sqrt{3}^2} \\ &= 2 \end{aligned}$$

$$\arg w = 2\pi/3$$

$$\begin{aligned} iii) \quad w^3 - 8 &= [2 \operatorname{cis} 2\pi/3]^3 - 8 \\ &= 8 \operatorname{cis} 2\pi - 8 \\ &= 8 - 8 \\ &= 0 \end{aligned}$$





i) $B = z \operatorname{cis}\left(\frac{\pi}{6}\right)$

$C = -z$

$D = -z \operatorname{cis}\left(\frac{\pi}{6}\right)$

ii) ABCD is a rectangle.
(equal diagonals that bisect each other).

d) $|z| = \frac{|r-i|}{|r+i|}$
 $= \frac{\sqrt{r^2+1}}{\sqrt{r^2+1}}$
 $= 1$

When $r=0$ $z = \frac{-i}{i} = -1$

$r=1$ $z = \frac{1-i}{1+i} \times \frac{(1-i)}{(1-i)}$
 $= -i$

\therefore locus is a quarter of the circle $x^2 + y^2 = 1$
 in the 3rd quadrant.

$$\begin{aligned} \text{e) } \int \frac{dx}{x^2 + 4x - 1} &= \int \frac{dx}{(x+2)^2 - 5} \\ &= \frac{1}{2\sqrt{5}} \left[\ln \left(\frac{x+2-\sqrt{5}}{x+2+\sqrt{5}} \right) \right] + C \end{aligned}$$

$$12 \text{ (a) (i) } 3x+7 = Ax(x+2) + B(x+2) + Cx^2$$

$$\text{when } x=0 \quad \text{when } x=2$$

$$7 = 2B \quad 5 = 4C$$

$$B = \frac{7}{2} \quad C = \frac{5}{4}$$

$$\text{when } x=1$$

$$8 = 3A + 3B + C$$

$$= 3A + 2\frac{7}{2} + \frac{5}{4}$$

$$A = -\frac{5}{4}$$

$$\begin{aligned} \text{(ii) } \int \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{\frac{5}{4}}{4(x+2)} \right) dx \\ = -\frac{5}{4} \log x - \frac{7}{2x} + \frac{5}{4} \log(x+2) + c \end{aligned}$$

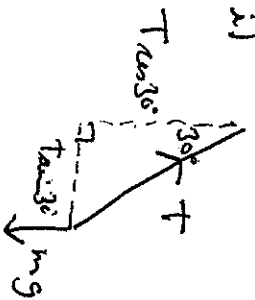
$$\text{(b) } \frac{1}{2} \int \frac{dx}{\sqrt{\frac{1}{4} + x^2}} = \frac{1}{2} \log \left(x + \sqrt{\frac{1}{4} + x^2} \right) + c$$

$$\text{(c) } \text{let } u = x \quad u' = e^{-x}$$

$$u' = 1 \quad v = -e^{-x}$$

$$\begin{aligned} \int_0^1 xe^{-x} dx &= \left[-xe^{-x} \right]_0^1 + \int_0^1 e^{-x} dx \\ &= -e^{-1} + 0 + \left[-e^{-x} \right]_0^1 \\ &= -e^{-1} - e^{-1} + 1 \\ &= 1 - 2e^{-1} \end{aligned}$$

(d) i)



$$T \cos 60 = mg$$

$$5 \cos 30^\circ = m \times 10$$

$$m = \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4}$$

ii)

$$T = m \lambda \omega^2$$

$$5 = \frac{\sqrt{3}}{4} \times \frac{3}{2} \times \omega^2$$

$$\frac{4 \times 5}{3\sqrt{3}} = \omega^2$$

$$\omega = \sqrt{\frac{4 \times 5}{3\sqrt{3}}}$$

$$P_{\text{avg}} = \frac{2\pi}{\sqrt{\frac{4 \times 5}{3\sqrt{3}}}}$$

$$= 2.2 \text{ c/s } \approx 1.8 \text{ s}$$

Question 13:

a)

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta} d\theta = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2+2t}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} 2(t+1)^{-2} dt$$

$$= \left[\frac{2}{t+1} \right]_0^{\frac{1}{\sqrt{3}}}$$

$$= 2 - \frac{2\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{2+2\sqrt{3}-2\sqrt{3}}{1+\sqrt{3}}$$

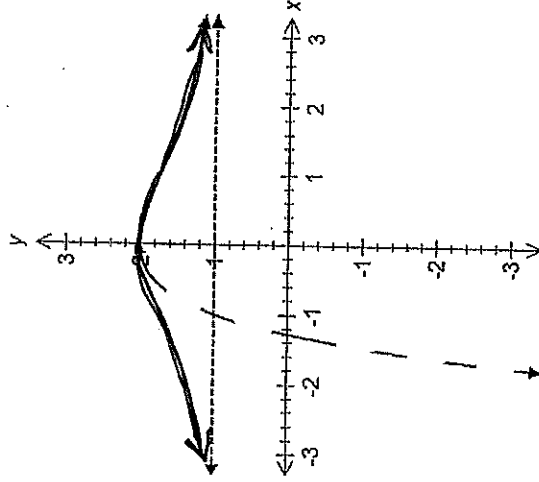
$$= \frac{2}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{2(1-\sqrt{3})}{-2}$$

$$= \sqrt{3}-1$$

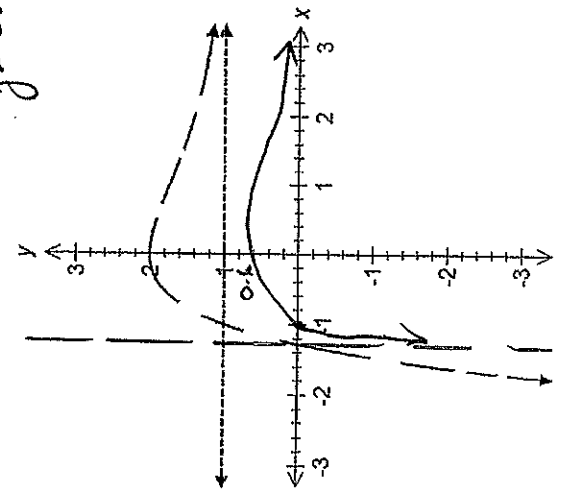
b)

i) $y = f(|x|)$

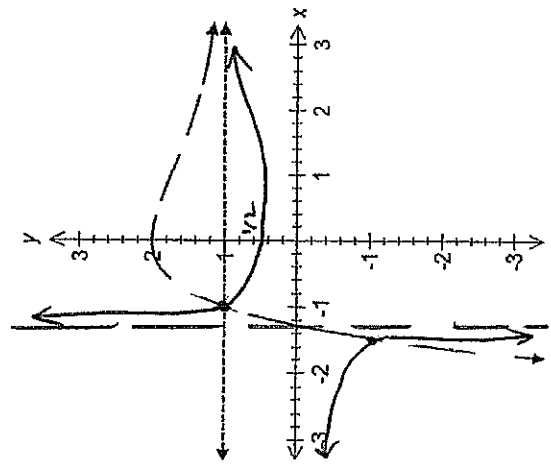


$y = \ln 2 \approx 0.6$

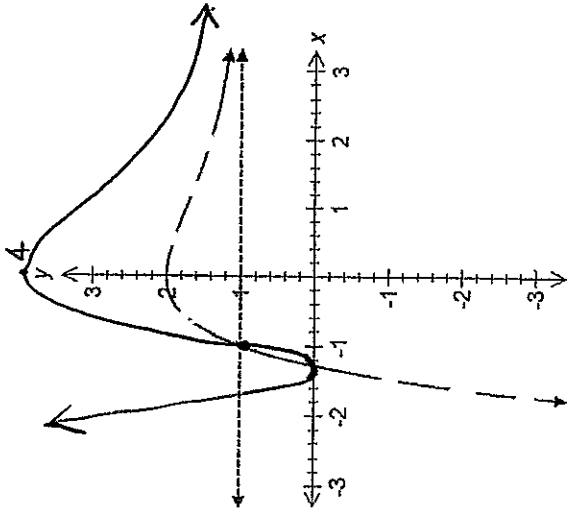
iv) $y = \ln[f(x)]$

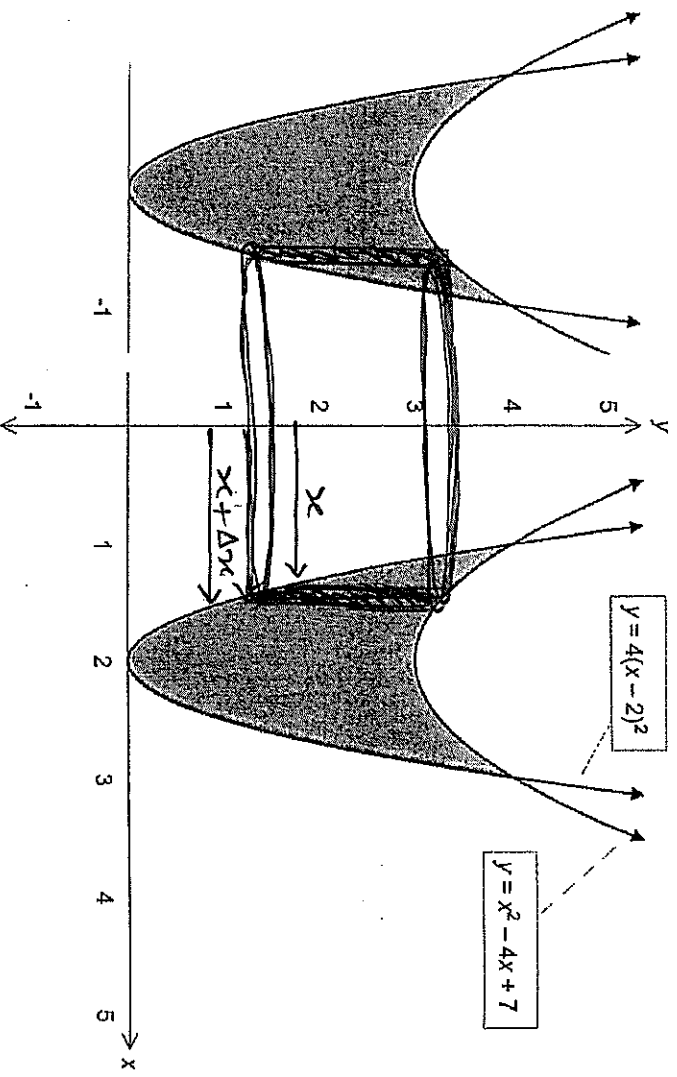


ii) $y = \frac{1}{f(x)}$



iii) $y = [f(x)]^2$





Area of base of cylindrical shell: $\Delta A(x) = \pi(R^2 - r^2)$

$$\Delta A(x) = \pi((x + \Delta x)^2 - x^2)$$

$\Delta A(x) = 2\pi x \Delta x$, (since Δx^2 is very small).

Volume of shell:

$$\Delta V(x) = \Delta A(x) \times \text{height}$$

$$\Delta V(x) = \Delta A(x) \times (x^2 - 4x + 7 - 4(x-2)^2)$$

$$\Delta V(x) = \Delta A(x) \times (-3x^2 + 12x - 9)$$

$$\Delta V(x) = 2\pi x \times 3(-x^2 + 4x - 3) \Delta x$$

Volume of solid:

$$V = \lim_{\Delta x \rightarrow 0} \sum_{x=1}^3 6\pi(-x^3 + 4x^2 - 3x) \Delta x$$

$$= 6\pi \left[\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2} \right]_1^3$$

$$= 6\pi \left(\left(-\frac{81}{4} + \frac{108}{3} - \frac{27}{2} \right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2} \right) \right)$$

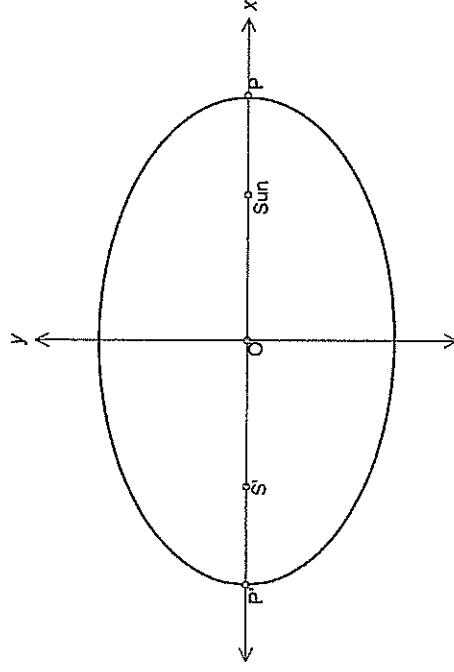
$$= 6\pi \left(\frac{9}{4} - \left(-\frac{5}{12} \right) \right)$$

$$= 6\pi \times \frac{8}{3}$$

$$\therefore V = 16\pi \text{ units}^3$$

Question 14:

a)



i) Earth is closest to the Sun at P.

$$\begin{aligned}
 PS &= PO - ae \\
 &= 1.486 \times 10^8 - (1.486 \times 10^8 \times 0.017) \\
 &= 1.486 \times 10^8 - 2526200 \\
 &= 146073800 \text{ km} \\
 &= 1.460738 \times 10^8 \text{ km}
 \end{aligned}$$

ii) Earth is furthest from the Sun at P'.

$$\begin{aligned}
 P'S &= PO + ae \\
 &= 1.486 \times 10^8 + (1.486 \times 10^8 \times 0.017) \\
 &= 1.017 \times 1.486 \times 10^8 \\
 &= 15112 \ 6200 \text{ km} \\
 &= 1.511262 \times 10^8 \text{ km}
 \end{aligned}$$

b)

Let

$$P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$$

$$P'(x) = 4x^3 - 15x^2 - 18x$$

$$P''(x) = 12x^2 - 30x - 18$$

For a root of multiplicity 3, $P''(x) = 0$.

That is:

$$0 = 12x^2 - 30x - 18$$

$$0 = 2x^2 - 5x - 3$$

$$0 = (x-3)(2x+1)$$

$$\therefore x = 3 \text{ or } x = -\frac{1}{2}$$

Testing roots: $P(3) = 0$

$\therefore x = 3$ is a triple root.

Consider the sum of the roots of $P(x)$:

$$3 + 3 + 3 + \alpha = 5$$

$$\alpha = -4$$

$$\therefore P(x) = (x-3)^3(x+4) \text{ and}$$

$P(x)$ has roots 3, 3, 3 and -4.

c)

$$i) P(x) = x^3 + ax^2 + bx + 6$$

Since all the coefficients are real, then by the conjugate root theorem both $(1-i)$ and $(1+i)$ are factors of $P(x)$.

Product of roots of $P(x)$:

$$(1-i)(1+i)\alpha = -6$$

$$(1-i^2)\alpha = -6$$

$$2\alpha = -6$$

$$\alpha = -3$$

Now:

$$P(-3) = 0$$

$$0 = (-3)^3 + a(-3)^2 + b(-3) + 6$$

$$0 = -27 + 9a - 3b + 6$$

$$0 = -21 + 9a - 3b \dots\dots\dots(1)$$

Sum of roots:

$$(1-i) + (1-i) - 3 = -a$$

$$2 - 3 = -a$$

$$a = 1 \dots \text{sub into (1)}$$

$$0 = -21 + 9(1) - 3b$$

$$0 = -12 - 3b$$

$$3b = -12$$

$$b = -4$$

ii)

Over the complex field:

$$P(x) = (x-1+i)(x-1-i)(x+3)$$

iii)

Over the real field:

$$P(x) = ((x-1)^2 - i^2)(x+3)$$

$$= (x^2 - 2x + 1 + 1)(x+3)$$

$$= (x^2 - 2x + 2)(x+3)$$

d)

Let

$$P(x) = 3x^5 + 20x^3 + 45x - c$$

$$P'(x) = 15x^4 + 60x^2 + 45$$

For stationary points $P'(x) = 0$.

$$0 = 15x^4 + 60x^2 + 45$$

$$0 = x^4 + 4x^2 + 3$$

$$0 = (x^2 + 3)(x^2 + 1)$$

No real solutions

No turning points.

For points of inflexion $P''(x) = 0$.

$$P''(x) = 60x^3 + 120x$$

$$0 = x(x^2 + 2)$$

\therefore at $x = 0$ ($y = c$) there is a point of inflexion.

Hence there must be only one real root and two pairs of conjugate complex roots of $P(x)$.

Sum of 'other' roots is given as -7.

Consider the sum of the roots of $P(x)$:

$$-7 + \alpha = 0$$

$$\alpha = 7 \text{ (is the real root)}$$

Now $P(7) = 0$ and by substitution $c = 57596$.

$$15 (a) (i) 16 = 25(e^2 - 1)$$

$$\frac{16}{25} = e^2 - 1$$

$$\frac{41}{25} = e^2$$

$$e = \frac{\sqrt{41}}{5}$$

$$(ii) (\pm 5 \times \frac{\sqrt{41}}{5}, 0)$$

$$= (\pm \sqrt{41}, 0)$$

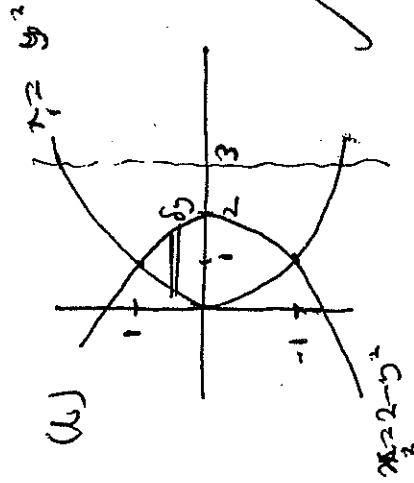
$$(iii) y = \pm \frac{5}{\sqrt{41}}$$

$$= \pm \frac{25}{\sqrt{41}}$$

$$(iv) y = \pm \frac{4}{5}x$$

$$(v) \frac{1 \times x}{25} - \frac{2 \times y}{16} = 1$$

$$\frac{x}{25} - \frac{y}{8} = 1$$



$$V_{\text{shaded}} = \pi(x_1^2 - x_2^2)$$

$$= \pi \int_{-1}^1 (3 - x_1)^2 - (3 - x_2)^2 dy$$

$$= \pi \int_{-1}^1 (9 - 6x_1 + x_1^2 - 9 + 6x_2 - x_2^2) dy$$

$$= \pi \int_{-1}^1 (6x_2 - 6x_1 + x_1^2 - x_2^2) dy$$

$$V_{\text{total}} = \lim_{dy \rightarrow 0} \sum_{y=-1}^1 \pi(6x_2 - 6x_1 + x_1^2 - x_2^2) dy$$

$$= \pi \int_{-1}^1 (6x_2 - 6x_1 + x_1^2 - x_2^2) dy$$

$$= \pi \int_{-1}^1 (12 - 6y^2 - 6y^2 + y^2 - 9 + 4y^2 - y^2) dy$$

$$(vii) V = \pi L$$

$$= A'B' \cdot B'C' \cdot \delta x$$

$$A'B' = mx + k$$

$$O = mx + k$$

$$k = 0$$

$$\therefore A'B' = mx$$

$$B'C' = mh$$

$$m = \frac{a}{h}$$

$$\therefore A'B' = \frac{a}{h}x$$

$$\therefore B'C' = \frac{b-c}{h}x + c$$

$$\therefore V = \left(\frac{a}{h}x\right) \left(\frac{b-c}{h}x + c\right) \delta x$$

$$(viii) V = \lim_{\delta x \rightarrow 0} \sum_{x=0}^h \left(\frac{a}{h}x\right) \left(c + \frac{b-c}{h}x\right) \delta x$$

$$= \frac{a}{h} \int_0^h x \left(c + \frac{b-c}{h}x\right) dx$$

$$= \frac{a}{h} \int_0^h \left(cx + \frac{b-c}{h}x^2\right) dx$$

$$= \frac{a}{h} \left[\frac{cx^2}{2} + \frac{b-c}{2h}x^3 \right]_0^h$$

$$= \frac{a}{h} \left(\frac{ch^2}{2} + \frac{b-c}{2h}h^3 \right)$$

$$= \frac{ach}{2} + \frac{abr - ac^2h}{3}$$

$$= \frac{ach}{6} + \frac{abh}{3}$$

$$= \frac{abh}{6} (c + 2a)$$

G.E.D.

$$= \pi \int_1^1 (8 + dy^2) dy$$

$$= 16\pi \int_0^1 (1 + y^2) dy$$

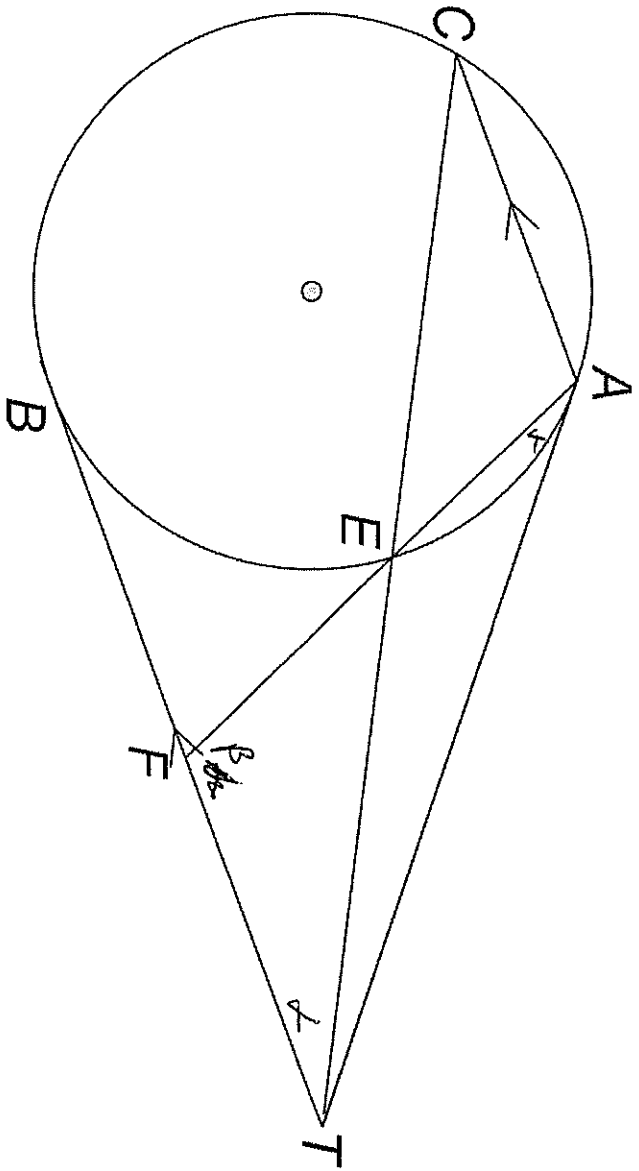
$$= 16\pi \left[y + \frac{y^3}{3} \right]_0^1$$

$$= 16\pi \left(1 + \frac{1}{3}\right)$$

$$= \frac{32\pi}{3}$$

Question 16

a)



i) In $\triangle TFA$ and $\triangle ETF$

$$\angle TAF = \angle ACE = \alpha \quad (\angle \text{ in alt segment})$$

$$\angle ACE = \angle ETF = \alpha \quad (\text{alt } \angle \text{ s } AC \parallel BT)$$

$$\therefore \angle TAF = \angle ETF$$

$$\angle EFT = \angle TFA \quad (\text{common})$$

$\therefore \triangle TFA \sim \triangle ETF$ (equiangular)

$$\frac{TF}{EF} = \frac{AF}{TF} \quad (\text{corresp sides of sim } \Delta \text{ s in same ratio})$$

$$TF^2 = AF \times EF$$

$$\text{iii) } BF^2 = AF \times EF \quad (\text{tangent}^2 = \text{prod. of intercepts})$$

$$\therefore TF^2 = BF^2$$

$$TF = BF$$

\therefore AE extended bisects BT.

$$v) i) \quad x = \frac{1}{1+t^4}$$

$$\frac{dx}{dt} = \frac{-4t^3}{(1+t^4)^2}$$

$$y = \frac{t}{1+t^4}$$

$$\frac{dy}{dt} = \frac{5t^4(1+t^4) - t^5(4t^3)}{(1+t^4)^2}$$

$$= \frac{5t^4 + 8t^8}{(1+t^4)^2}$$

$$= \frac{t^4(5+t^4)}{(1+t^4)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{t^4(5+t^4)}{(1+t^4)^2} \cdot \frac{(1+t^4)^2}{-4t^3}$$

$$= \frac{t(5+t^4)}{-4}$$

$$y - \frac{t^5}{1+t^4} = \frac{t(5+t^4)}{-4} \left[x - \frac{1}{1+t^4} \right]$$

$$4y + x(5t+t^5) = \frac{4t^5}{1+t^4} + \frac{5t+t^5}{1+t^4}$$

$$= \frac{5t(1+t^4)}{(1+t^4)}$$

ii) When $x=0$

$$4y = 5t$$

$$y = \frac{5t}{4}$$

$$\therefore OY = \frac{5t}{4}$$

When $y=0$

$$x = \frac{5t^5}{5t+t^5}$$

$$= \frac{5}{5+t^4}$$

$$\therefore OX = \frac{5}{5+t^4}$$

$$\therefore A = \frac{1}{2} \times \frac{5t}{4} \times \frac{5}{5+t^4}$$

$$= \frac{25t}{8(5+t^4)}$$

$$\frac{dA}{dt} = \frac{25 \times 8(5+t^4) + 32t^3(25t)}{64(5+t^4)^2}$$

For a max area $\frac{dA}{dt} = 0$

$$200(5+t^4) - 800t^4 = 0$$

$$1000 + 200t^4 - 800t^4 = 0$$

$$t^4 = \frac{5}{3}$$

$$t = \left(\frac{5}{3}\right)^{1/4}$$

$$\therefore \text{Max } A = \frac{1}{2} \times \frac{5}{4} \left(\frac{5}{3}\right)^{1/4} \times \frac{5}{5 + \frac{5}{3}}$$

$$= \frac{5}{8} \left(\frac{5}{3}\right)^{1/4} \times \frac{3}{4}$$

$$= \frac{15}{32} \left(\frac{5}{3}\right)^{1/4}$$

c) Show true for $n=1$

$$\text{LHS} = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\text{RHS} = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left(\frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}\right)$$

$$= \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \text{LHS}$$

\therefore true for $n=1$

Assume true for $n=k$

$$\tan^{-1}\left(\frac{1}{2 \times 1^2}\right) + \dots + \tan^{-1}\left(\frac{1}{2k^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right)$$

Prove true for $n=k+1$

R.T.P

$$\tan^{-1}\left(\frac{1}{2 \times 1^2}\right) + \dots + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+3}\right)$$

$$\text{LHS} = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) \quad [\text{using assumption}]$$

$$= \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right) - \tan^{-1}\left(\frac{1}{2k+1}\right)$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{1}{2(k+1)^2} - \frac{1}{2k+1}\right] \div \left\{1 + \frac{1}{2(k+1)^2} \times \frac{1}{2k+1}\right\}$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{2k+1 - 2(k+1)^2}{2(2k+1)(k+1)^2}\right] \div \frac{2(k+1)^2(2k+1) + 1}{2(2k+1)(k+1)^2}$$

$$= \frac{\pi}{4} + \tan^{-1}\left[\frac{2k+1 - 2(k+1)^2}{2(k+1)^2(2k+1) + 1}\right]$$

$$= \frac{\pi}{4} + \tan^{-1} \left[\frac{2k+1 - 2k^2 - 4k - 2}{(2k^2 + 4k + 2)(2k+1) + 1} \right]$$

$$= \frac{\pi}{4} + \tan^{-1} \left[\frac{-2k^2 - 2k - 1}{4k^3 + 8k^2 + 4k + 2k^2 + 4k + 2 + 1} \right]$$

$$= \frac{\pi}{4} + \tan^{-1} \left[\frac{-(2k^2 + k + 1)}{(2k+3)(2k^2 + k + 1)} \right]$$

$$= \frac{\pi}{4} + \tan^{-1} \left[\frac{-1}{2k+3} \right]$$

$$= \frac{\pi}{4} - \tan^{-1} \left[\frac{1}{2k+3} \right]$$

= RHS

∴ Proven true for $n = k+1$.

∴ Proposition is true for all $n \geq 1$ by induction.

