Student's name ________________________ **Student's number** __________ **Teacher's name** ____________________

PRESBYTERIAN LADIES' COLLEGE SYDNEY 1888

2014
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using blue or black pen Black is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 70

Section I: Pages 3-6
10 marks

- Attempt questions 1-10, using the answer sheet on page 13.
- Allow about 15 minutes for this section

Section II: Pages 7-10
60 marks

- Attempt questions 11-14, using the booklets provided.
- Allow about 1 hours 45 minutes for this section

Question	1-10	11	12	13	14	Total	%
Marks	/10	/15	/15	/15	/15	/70	

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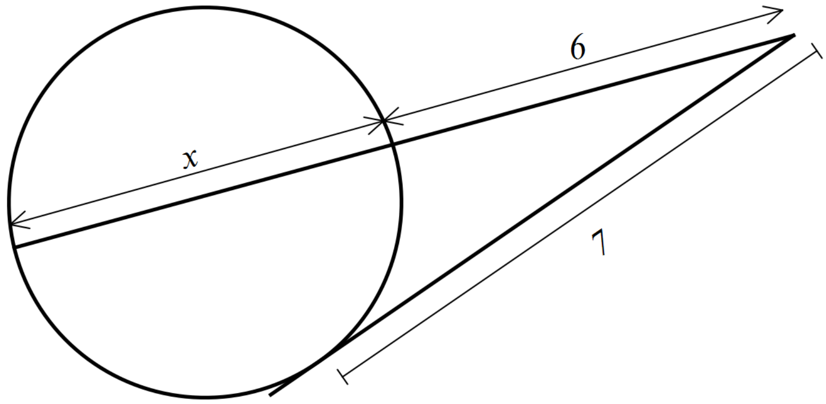
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

1.



What is the value of x?

(A) 1

(B) $1\frac{1}{6}$

(C) $2\frac{1}{6}$

(D) $8\frac{1}{6}$

2. What is the solution of $\frac{5}{1-x} < 3$?

(A) $x < -\frac{2}{3}, x > 1$

(B) $x < -\frac{2}{3}$

(C) $x > 1$

(D) $-\frac{2}{3} < x < 1$

3. What is the value of

$$\lim_{x \to 0}\left(\frac{2x}{\sin 5x}\right)?$$

(A) $\frac{2}{\sin 5}$

(B) $\frac{2}{5}$

(C) $\frac{5}{2}$

(D) $\sin 3x$

4. What are the co-ordinates of the point which divides the interval joining $A(3, -2)$ and $B(-5, 4)$ **externally** in the ratio of 5:3?

(A) $\left(0, \frac{1}{4}\right)$

(B) $(15, -11)$

(C) $(-17, 13)$

(D) $\left(-2, \frac{3}{2}\right)$

5. The inverse of the function $f(x) = e^{2x-1}$ is?

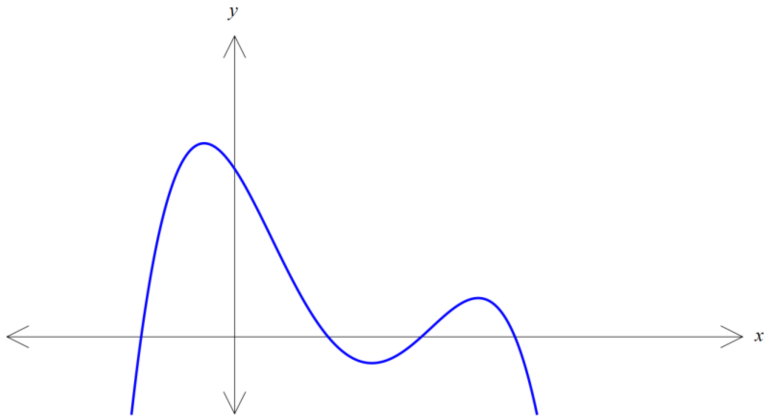
(A) $f^{-1}(x) = -e^{2x-1}$

(B) $f^{-1}(x) = \frac{e^{x+1}}{2}$

(C) $f^{-1}(x) = -\log_e(2x+1)$

(D) $f^{-1}(x) = \log_e \sqrt{x} + \frac{1}{2}$

6. The graph below shows a polynomial function, $y = P(x)$.



Which of the following could be the equation of *P(x)*?

(A) $P(x)=(x+1)(x+2)(x+3)(x-1)$

(B) $P(x)=-(x+1)(x+2)(x+3)(x-1)$

(C) $P(x)=(x+1)(x-1)(x-2)(x-3)$

(D) $P(x)=-(x+1)(x-1)(x-2)(x-3)$

7. The co-efficient of x^2 in the expansion $(2x-3)^5$ is?

(A) -1080
(B) -540
(C) 540
(D) 1080

8. Using $u = \cos x$,

$\int_0^{\frac{\pi}{3}} sin^3 x\, cos^4 x\, dx$ can be expressed in terms of *u* as

(A) $\int_0^{\frac{\pi}{3}} u^6 - u^4 du$

(B) $\int_{\frac{1}{2}}^{1} u^6 - u^4 du$

(C) $\int_{\frac{1}{2}}^{1} u^4 - u^6 du$

(D) $\int_0^{\frac{\sqrt{3}}{2}} u^4 - u^6 du$

9. A particle is moving along the x-axis, initially moving to the left from the origin. Its velocity and acceleration are given by

$v^2 = 2\log_e(3+\cos x)$ and

$\ddot{x} = \dfrac{-\sin x}{3+\cos x}$.

Which of the following describes the subsequent motion?

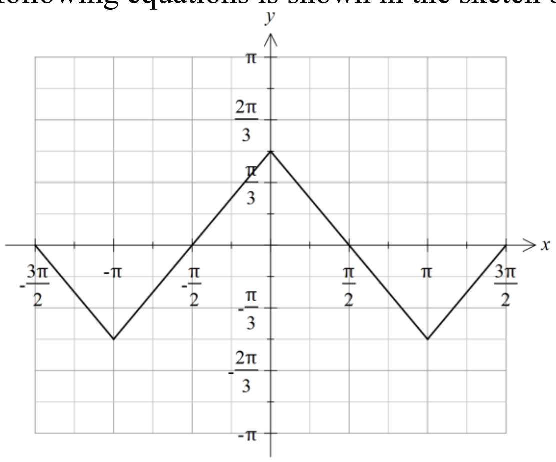
(A) Moves only to the left, alternately speeding up and slowing down, without becoming stationary.

(B) Moves only to the left, alternately slowing to a stop and speeding up.

(C) Slowing to a stop, then heading to the right forever.

(D) Oscillates between two points.

10. Which of the following equations is shown in the sketch below?



(A) $y = \cos^{-1}(\sin x)$

(B) $y = \sin^{-1}(\cos x)$

(C) $y = \sin^{-1}(x) + \sin(x)$

(D) $y = \cos^{-1}(x) + \cos(x)$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Find the acute angle between the lines $x-2y+3=0$ and $y=3x-1$ at their point of intersection. **2**

b) Find $\int \frac{1}{25+9x^2}\,dx$. **2**

c) Find $\frac{d}{dx} sin^{-1}\left(2x^3\right)$ **2**

d) The polynomial $P(x)=x^3-3x^2+kx+12$ has 3 roots. It is known that two of the roots are of equal magnitude but opposite in sign. What is the value of k? **3**

e) Explain why Newton's method does not work for the root of the equation $x^3-3x+6=0$ if the initial approximation is chosen to be $x=1$. Use mathematics to support your answer. **2**

f) If $cot^2\theta - cot\theta = 1$, where $0<\theta<\frac{\pi}{2}$,

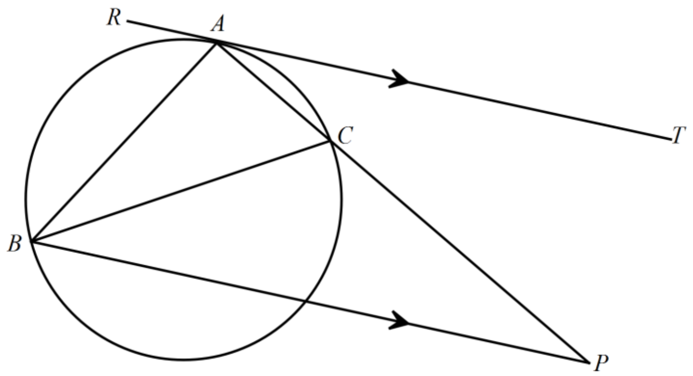
(i) Show that $cot\theta = \frac{1+\sqrt{5}}{2}$. **1**

(ii) Hence, show that the exact value of $cot\,2\theta = \frac{1}{2}$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) AT is a tangent and is parallel to BP. Prove that $\angle ABP = \angle ACB$. **3**



b) A roast duck is taken out of the oven once it is cooked. A thermometer records the temperature of the duck to be 75^0C. The roast duck is then allowed to cool in a room with a constant temperature of 23^0C.

(i) Show that $T = 23 + Ae^{-kt}$ satisfies the differential equation **1**

$$\frac{dT}{dt} = -k(T-23) \text{ where}$$

T is the temperature of the duck in degrees Celsius, ^0C,
t is the time in minutes and
k is a constant.

(ii) Show that $A = 52$. **1**

(iii) Find the value of k (in exact form) if after 5 minutes the duck's temperature is 65^0C. **2**

(iv) Bacteria start to develop rapidly in the duck after 8 minutes. What will be the duck's temperature when the bacteria start to develop? Answer to the nearest degree. **1**

c) Using the substitution $u = e^x$, find $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$ **2**

d) (i) Find the domain and range of the function $f(x) = \sin^{-1}(2x)$. **1**

(ii) Sketch the graph of the function $f(x) = \sin^{-1}(2x)$. **1**

(iii) The region bounded by the graph $f(x) = \sin^{-1}(2x)$ and the x-axis between $x = 0$ and $x = \frac{1}{2}$ is rotated about the y-axis to form a solid. Find the exact volume of the solid. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) The speed v m/s of a particle moving in a straight line is given by $v^2 = 84 + 16x - 4x^2$ where the displacement of the particle relative to a fixed point is x cm.

(i) Find an expression for the particle's acceleration in terms of x. **2**

(ii) Hence show that the particle is moving in simple harmonic motion. **1**

(iii) Find the period, amplitude and centre of motion. **2**

b) (i) The monic polynomial, $P(x)$, has a root at $x = 3$, a double root at $x = -1$ and is of degree 4. If the polynomial passes through the point (1,0), find the equation of the polynomial $P(x)$. **2**

(ii) The polynomial $Q(x)$ has equation $Q(x) = x^2 + 1$. **2**

Show that $\dfrac{P(x)}{Q(x)}$ has a remainder of $4x + 8$.

c) A balloon has the shape of a right circular cylinder of radius r and length twice the radius, with a hemisphere at each end of radius r. The balloon is being filled at the rate of $10\,cm^3/s$. Find the rate of change of r when $r = 8$ centimetres **2**

d) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are the ends of a focal chord on the parabola $x^2 = 4ay$.

(i) Show that PQ has equation $(p+q)x - 2y - 2apq = 0$. **1**

(ii) Show that $pq = -1$ if PQ is a focal chord. **1**

(iii) Show that the equation of the tangent at P is $y = px - ap^2$. **1**

(iv) Hence find the locus of the point of intersection of the tangents at the ends of the focal chord. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

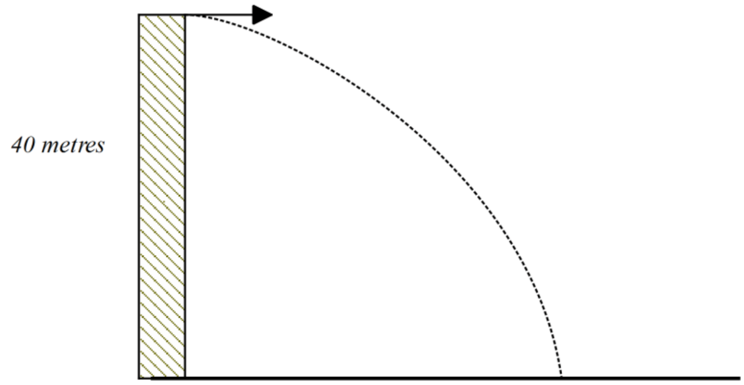
a) Prove by Mathematical Induction that **3**

$\sum_{r=1}^{n} log_e\left(\frac{r+1}{r}\right) = log_e(n+1)$ for all positive integers, n.

b) Find the general solutions for $2\cos x = \sqrt{3}\cot x$. **3**

c) An object is projected horizontally from the top edge of a vertical cliff 40 metres above sea level with a velocity of $40m/s$.

Take $g = 10m/s^2$.



(i) Using the top edge of the cliff as the origin, prove that the parametric equations of the path of the object are: **2**

$x = 40t$ $\qquad\qquad$ $y = -5t^2 + 40$

(ii) Calculate when and where the object hits the water. **1**

(iii) Find the velocity and angle of the object the instant it hits the water. **2**

d) By considering $(1-x)^n\left(1+\frac{1}{x}\right)^n$, or otherwise, express **4**

$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \ldots + (-1)^n\binom{n}{n}\binom{n}{n-2}$

in simplest form.

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

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Mathematics Extension 1:

Multiple Choice Answer Sheet

Student Number__________

Completely fill the response oval representing the most correct answer.

1.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
2.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
3.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
4.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
5.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
6.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
7.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
8.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
9.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
10.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭

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Mathematics Extension 1:
Multiple Choice Answer Sheet

Student Number ANSWERS

Completely fill the response oval representing the most correct answer.

1.	**A** ○	**B** ○	**C** ●	**D** ○
2.	**A** ●	**B** ○	**C** ○	**D** ○
3.	**A** ○	**B** ●	**C** ○	**D** ○
4.	**A** ○	**B** ○	**C** ●	**D** ○
5.	**A** ○	**B** ○	**C** ○	**D** ●
6.	**A** ○	**B** ○	**C** ○	**D** ●
7.	**A** ●	**B** ○	**C** ○	**D** ○
8.	**A** ○	**B** ○	**C** ●	**D** ○
9.	**A** ●	**B** ○	**C** ○	**D** ○
10.	**A** ○	**B** ●	**C** ○	**D** ○

PLC Sydney Maths Department

Solutions for exams and assessment tasks

Ver 1

Academic Year		Calendar Year	
Course		Name of task/exam	

Section I

1. $(x+6)6 = 7^2$

$6x + 36 = 49$

$6x = 13$

$x = \frac{13}{6}$

$x = 2\frac{1}{6}$

$\therefore C$

2. $\frac{5}{1-x} < 3$

Critical pts: $x = 1$

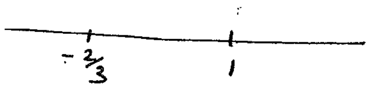
$\frac{5}{1-x} = 3$

$5 = 3(1-x)$

$5 = 3 - 3x$

$2 = -3x$

$x = -\frac{2}{3}$



Check $x = 0$

$\frac{5}{1} \nless 3$ false

$\therefore x < -\frac{2}{3}, \; x > 1$

$\therefore A$

3. $\lim_{x \to 0} \frac{2x}{\sin 5x} = \frac{2}{5}$

$\therefore B$

4. $A(3, -2) \quad B(-5, 4)$

$-5 : 3$ (m : n)

$\therefore \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$

$= \left(\frac{-5(-5) + 3(3)}{-2}, \frac{-5(4) + 3(-2)}{-2} \right)$

$= \left(\frac{25+9}{-2}, \frac{-20-6}{-2} \right)$

$= (-17, 13)$

$\therefore C$

5. $f: \quad f(x) = e^{2x-1}$

$f^{-1}: \quad x = e^{2y-1}$

$\ln x = (2y-1)$

$\ln x + 1 = 2y$

$y = \frac{1}{2}(\ln x + 1)$

$\therefore f^{-1}(x) = \frac{1}{2}\ln x + \frac{1}{2}$

$= \ln x^{\frac{1}{2}} + \frac{1}{2}$

$= \ln\sqrt{x} + \frac{1}{2}$

$\therefore D$

Academic Year		Calendar Year	
Course		Name of task/exam	

6 D

7 $(2x-3)^5$

$T_{k+1} = {}^nC_k (a)^{n-k}(b)^k$

$= {}^5C_k (2x)^{5-k}(-3)^k$

$= {}^5C_k \, 2^{5-k} (-3)^k x^{5-k}$

for co-efficient of x^2

$\therefore 5-k=2$

$k=3$.

$\therefore$ coeff $= {}^5C_3 \, 2^2 (-3)^3$

$= 10 \times 4 \times -27$

$= -1080$

$\therefore$ A

8 $\int_0^{\pi/3} \sin^3 x \cos^4 x \, dx$

$u = \cos x$

when $x=0$ $u=1$

when $x=\frac{\pi}{3}$ $u=\frac{1}{2}$.

$\therefore$ integral goes from 1 to $\frac{1}{2}$

or $-$ integral goes from $\frac{1}{2}$ to 1

$\therefore u = \cos x$

$du = -\sin x \, dx$

$\sin x = \sqrt{1-\cos^2 x}$

$= \sqrt{1-u^2}$

$\int_0^{\frac{\pi}{3}} \sin^2 x \sin x \cos^4 x \, dx$

$= \int_1^{\frac{1}{2}} (-)(1-u^2)u^4 \, du$

$= \int_{\frac{1}{2}}^1 (1-u^2)u^4 \, du$

$= \int_{\frac{1}{2}}^1 (u^4 - u^6) \, du$

$\therefore$ C

9 $v^2 = 2\ln(3+\cos x)$

for stationary $v=0$

$\therefore 0 = 2\ln(3+\cos x)$

$0 = \ln(3+\cos x)$

$e^0 = 3+\cos x$

$1 = 3+\cos x$

$-2 = \cos x$

no sol^n

$\therefore$ does not stop

$\therefore$ A

10 $\therefore$ B

Academic Year		Calendar Year	
Course		Name of task/exam	

Q11

a) $m_1 = \frac{1}{2}$

$m_2 = 3$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{1}{2} - 3}{1 + \frac{3}{2}}\right|$$

$$= \left|\frac{-\frac{5}{2}}{\frac{5}{2}}\right|$$

$$= |-1|$$

$\tan\theta = 1$

$\therefore \theta = \pi/4$

b) $$\int \frac{1}{25+9x^2}\,dx$$

$$= \frac{1}{15}\tan^{-1}\frac{3x}{5} + c$$

c) $$\frac{1}{\sqrt{1-4x^6}} \times 6x^2$$

$$= \frac{6x^2}{\sqrt{1-4x^6}}$$

d) $P(x) = x^3 - 3x^2 + kx + 12$

Let roots be $\alpha, -\alpha, \beta$

sum of roots 1 at a time

$\beta = 3$

sum of roots 2 at a time:

$-\alpha^2 + \alpha\beta - \alpha\beta = k$

$-\alpha^2 = k$

$-(\alpha^2) = k$

product of roots:

$-\alpha^2\beta = -12$

$\alpha^2\beta = 12$

$\alpha^2(3) = 12$

$\alpha^2 = 4.$

$\therefore \alpha = \pm 2.$

$\therefore k = -(4)$

$k = -4$

e) $P(x) = x^3 - 3x + 6$

$P'(x) = 3x^2 - 3$

at $x = 1$

$P'(1) = 3 - 3$

$= 0$

$\therefore$ at $x=1$ there is a stationary point.

Since Newton's method finds where the tangent at a point crosses the x-axis, this would not work for the tangent at $x=1$. It wouldn't cross the x-axis as it would be a horizontal tangent.

$\therefore$ Newton's method would not work at $x=1$.

Academic Year		Calendar Year	
Course		Name of task/exam	

f i $\cot^2\theta - \cot\theta = 1$

$\cot^2\theta - \cot\theta - 1 = 0$

$\cot\theta = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{1+4}}{2}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

since $0 < \theta < \frac{\pi}{2}$

in the first quadrant.

$\therefore$ all trig ratios are positive.

$\therefore \cot\theta = \dfrac{1+\sqrt{5}}{2}$

ii RTS $\cot 2\theta = \frac{1}{2}$

LHS $= \cot 2\theta$

$= \dfrac{1}{\tan 2\theta}$

$= \dfrac{1}{\frac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{1-\tan^2\theta}{2\tan\theta}$

as $\cot\theta = \dfrac{1+\sqrt{5}}{2}$

$\tan\theta = \dfrac{2}{1+\sqrt{5}}$

$\left(\text{since } \tan\theta = \dfrac{1}{\cot\theta}\right)$

$\therefore \tan\theta = \dfrac{2}{1+\sqrt{5}}$

LHS $= \dfrac{1 - \frac{4}{(1+\sqrt{5})^2}}{2\left(\frac{2}{1+\sqrt{5}}\right)}$

$= \dfrac{(1+\sqrt{5})^2 - 4}{(1+\sqrt{5})^2} \div \dfrac{4}{(1+\sqrt{5})}$

$= \dfrac{1+2\sqrt{5}+5-4}{(1+\sqrt{5})^2} \times \dfrac{(1+\sqrt{5})}{4}$

$= \dfrac{2+2\sqrt{5}}{4(1+\sqrt{5})}$

$= \dfrac{2(1+\sqrt{5})}{4(1+\sqrt{5})}$

$= \dfrac{2}{4}$

$= \dfrac{1}{2}$

$=$ RHS

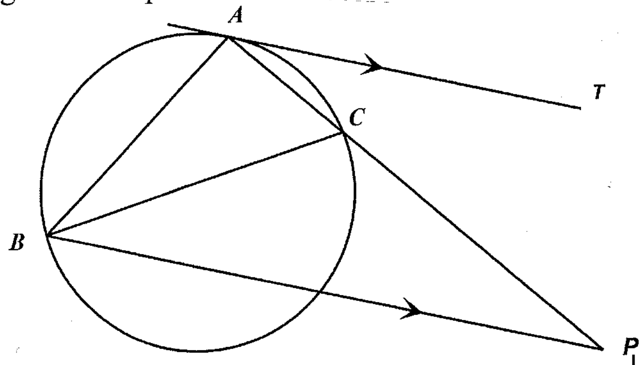
$\therefore$ shown

Academic Year		Calendar Year	
Course		Name of task/exam	

Q12

a.

AT is a tangent and is parallel to *BP*. Prove $\angle ABP = \angle ACB$.



Let $\angle ABC = x$

$\therefore \angle TAC = x$ (angle between a tangent and a chord equals the angle in the alternate segment).

$\angle TAC = \angle CPB = x$ (alternate angles equal $AT \parallel BP$).

Let $\angle CBP = y$.

$\therefore \angle ABP = x + y$ (adjacent angles)

$\angle ACB = x + y$ (exterior angle of ΔCBP equals sum of 2 interior opposite angles).

$\therefore \angle ABP = \angle ACB$.

b. i $T = 23 + Ae^{-kt}$ .... ①

$\frac{dT}{dt} = -kAe^{-kt}$

from ① $Ae^{-kt} = T - 23$

$\therefore \frac{dT}{dt} = -k(T-23)$

ii $T = 23 + Ae^{-kt}$

when $t = 0$, $T = 75$

$75 = 23 + Ae^{0}$

$A = 75 - 23$

$= 52$

iii $t = 5, T = 65$

$65 = 23 + 52e^{-k \times 5}$

$42 = 52e^{-5k}$

$\frac{42}{52} = e^{-5k}$

$\ln\left(\frac{42}{52}\right) = -5k$

$k = -\frac{1}{5}\ln\left(\frac{42}{52}\right)$.

iv $T = ?$ $t = 8$

$T = 23 + 52e^{-\left[-\frac{1}{5}\ln\left(\frac{42}{52}\right)\right] \times 8}$

$T = 60°$

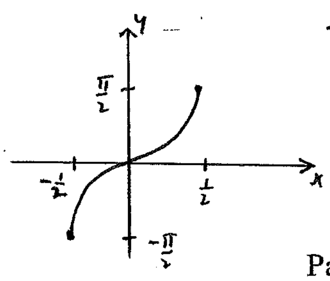
c $\int \frac{e^x \, dx}{\sqrt{1 - e^{2x}}}$ $u = e^x$, $du = e^x \, dx$.

$= \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u + C$

$= \sin^{-1} e^x + C$

d ii

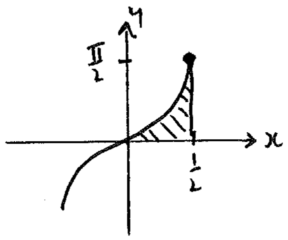


i D: $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

R: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

iii)



V = cylinder − V curve y-axis.

$V = \pi\left(\frac{1}{2}\right)^2\frac{\pi}{2} - \int_0^{\pi/2}\pi x^2\,dy$

$= \frac{\pi^2}{8} - \pi\int_0^{\frac{\pi}{2}} x^2\,dy$

Now $y = \sin^{-1}2x$

$\sin y = 2x$

$x = \frac{1}{2}\sin y$

$\therefore V = \frac{\pi^2}{8} - \pi\int_0^{\pi/2}\left(\frac{1}{2}\sin y\right)^2 dy$

$= \frac{\pi^2}{8} - \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\sin^2 y\,dy$

$= \frac{\pi^2}{8} - \frac{\pi}{4}\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos 2y\right)dy$

$\cos 2y = 1 - 2\sin^2 y$

$2\sin^2 y = 1 - \cos 2y$

$\sin^2 y = \frac{1}{2} - \frac{1}{2}\cos 2y.$

$= \frac{\pi^2}{8} - \frac{\pi}{4}\left[\frac{1}{2}y - \frac{\sin 2y}{4}\right]_0^{\pi/2}$

S C −S −C ↑

$= \frac{\pi^2}{8} - \frac{\pi}{4}\left[\left(\frac{\pi}{4} - 0\right) - (0 - 0)\right]$

$= \frac{\pi^2}{8} - \frac{\pi^2}{16}$

$= \frac{\pi^2}{16}$ units^3

Q13

a i $v^2 = 84 + 16x - 4x^2$

$\frac{1}{2}v^2 = 42 + 8x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(42 + 8x - 2x^2\right)$

$a = 8 - 4x$

or $\ddot{x} = 8 - 4x$

ii $\ddot{x} = -4(x - 2)$

which is of the form

$\ddot{x} = -n^2(x - h)$

∴ SHM

iii ∴ Centre is at $x = 2$.

when $v = 0$

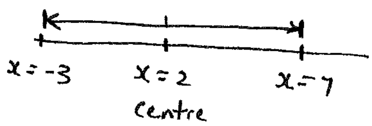
$84 + 16x - 4x^2 = 0$

$21 + 4x - x^2 = 0$

$x^2 - 4x - 21 = 0$

$(x + 3)(x - 7) = 0$

$\therefore x = -3,\ x = 7.$



∴ amplitude = 5

period $= \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

∴ Period = π sec.

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b i $P(x) = (x-3)(x+1)^2(x-h)$

(1, 0) satisfies

$0 = (1-3)(1+1)^2(1-h)$

$0 = (-2)(4)(1-h)$

$1 - h = 0$

$h = 1.$

$\therefore P(x) = (x-3)(x-1)(x+1)^2$

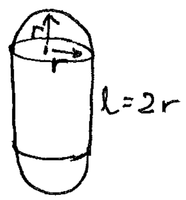
ii $P(x) = x^4 - 2x^3 - 4x^2 + 2x + 3$

$$\frac{P(x)}{Q(x)} = \frac{x^4 - 2x^3 - 4x^2 + 2x + 3}{x^2+1}$$

$$\begin{array}{r} x^2 - 2x - 5 \\ x^2+1 \,\big)\, x^4 - 2x^3 - 4x^2 + 2x + 3 \\ x^4 \qquad\quad + x^2 \qquad\qquad \\ \hline -2x^3 - 5x^2 + 2x + 3 \\ -2x^3 \qquad\quad - 2x \qquad \\ \hline -5x^2 + 4x + 3 \\ -5x^2 \qquad\ - 5 \\ \hline 4x + 8 \end{array}$$

$\therefore$ remainder is $4x + 8$

c/



V = cylinder + 2 hemi-spheres

$V = \pi r^2 h + \frac{4}{3}\pi r^3$

$V = \pi r^2(2r) + \frac{4}{3}\pi r^3$

$V = 2\pi r^3 + \frac{4}{3}\pi r^3$

$V = 3\frac{1}{3}\pi r^3$

$V = \frac{10\pi r^3}{3}$

$\frac{dV}{dr} = 10\pi r^2$

$\frac{dV}{dt} = 10\ \text{cm}^3/\text{s}$

$\therefore \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\cancel{10}\pi r^2} \times \cancel{10}$

$= \frac{1}{\pi r^2}$

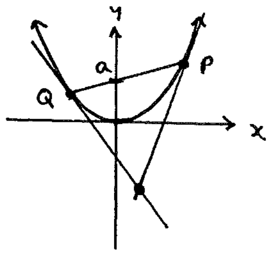
when $r = 8$

$\frac{dr}{dt} = \frac{1}{64\pi}$ cm/sec.

d $x^2 = 4ay$

$P(2ap, ap^2)$

$Q(2aq, aq^2)$



i $m = \frac{aq^2 - ap^2}{2aq - 2ap}$

$= \frac{\cancel{a}\cancel{(q-p)}(q+p)}{2\cancel{a}\cancel{(q-p)}}$

$= \frac{p+q}{2}.$

eqn $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = (p+q)x - (p+q)(2ap)$

$2y - \cancel{2ap^2} = (p+q)x - \cancel{2ap^2} - 2apq$

$(p+q)x - 2y - 2apq = 0$

ii If focal chord (0, a) satisfies

$0 - 2a - 2apq = 0$

$\cancel{2a} = -\cancel{2a}pq$

$-1 = pq$

$\therefore pq = -1$

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iii) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

at $P(2ap, ap^2)$

$m_{tang} = \frac{2ap}{2a} = p$

$\therefore$ eqn

$y - ap^2 = p(x - 2ap)$

similarly eqn tangent at Q is

$y - aq^2 = q(x - 2aq)$.

Solving simultaneously:

$p(x - 2ap) + ap^2 = q(x - 2aq) + aq^2$

$px - 2ap^2 + ap^2 = qx - 2aq^2 + aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$, $p \neq q$

$x = a(p+q)$

$y = px - 2ap^2 + ap^2$

$= p[a(p+q)] - ap^2$

$= ap^2 + apq - ap^2$

$= apq$.

$\therefore x = a(p+q)$ $\quad y = apq$

since $pq = -1$

$y = -a$

$\therefore$ locus is $y = -a$

Q14

a) RTP

$$\sum_{r=1}^{n} \ln\left(\frac{r+1}{r}\right) = \ln(n+1)$$

Step 1: Prove true for $n=1$

LHS $= \ln 2$

RHS $= \ln(1+1) = \ln 2$.

$\therefore$ true for $n = 1$

Step 2: Assume true for $n=k$

i.e.

$\ln 2 + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{k+1}{k}\right) = \ln(k+1)$

Step 3: Prove true for $n = k+1$

i.e. $\ln 2 + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{k+1}{k}\right) + \ln\left(\frac{k+2}{k+1}\right) = \ln(k+2)$

LHS $= \underbrace{\ln 2 + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{k+1}{k}\right)} + \ln\left(\frac{k+2}{k+1}\right)$

$= \ln(k+1) + \ln\left(\frac{k+2}{k+1}\right)$ by assumption

$= \ln(k+1) + \ln(k+2) - \ln(k+1)$

$= \ln(k+2)$

$=$ RHS

$\therefore$ prove by M.I. for all positive integers n.

b) $2\cos x = \sqrt{3}\cot x$

$2\cos x = \sqrt{3}\dfrac{\cos x}{\sin x}$

$2\cos x - \dfrac{\sqrt{3}\cos x}{\sin x} = 0$

$\cos x\left(2 - \dfrac{\sqrt{3}}{\sin x}\right) = 0$

$\therefore \cos x = 0$ $\quad 2 = \dfrac{\sqrt{3}}{\sin x}$

$\sin x = \dfrac{\sqrt{3}}{2}$

$\therefore x = k\pi + \dfrac{\pi}{2}$, k integer

and

$x = k\pi + (-1)^k\dfrac{\pi}{3}$, k integer

c) i $\ddot{y} = -g$

$\dot{y} = \int -g\,dt$

$\dot{y} = -gt + c_1$

when $t = 0$ $\dot{y} = 0 \Rightarrow c_1 = 0$

$\dot{y} = -gt$

$y = \int -gt\,dt$

$y = -\dfrac{gt^2}{2} + c_2$

when $t = 0$ $y = 40$

$c_2 = 40$

$\therefore y = -\frac{1}{2}gt^2 + 40$

when $g = 10$

$y = -5t^2 + 40$

$\ddot{x} = 0$

$\dot{x} = \int 0\,dt$

$\dot{x} = c_3$

when $t = 0$ $\quad \dot{x} = 40\cos 0$

$= 40$

$\therefore c_3 = 40$

$\dot{x} = 40$

$x = \int 40\,dt$

$x = 40t + c_4$

when $t = 0$ $x = 0 \Rightarrow c_4 = 0$

$\therefore x = 40t$

ii object hits when $y = 0$

$\therefore 0 = -5t^2 + 40$

$5t^2 = 40$

$t^2 = 8$

$t = 2\sqrt{2}$ $\quad (t > 0)$

and $x = 40 \times 2\sqrt{2}$

$= 80\sqrt{2}$ m.

iii $v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{40^2 + (-10(\sqrt{8}))^2}$

$= 48.99$ m/s

$\tan\theta = \dfrac{\dot{y}}{\dot{x}}$

$\tan\theta = \dfrac{10\sqrt{8}}{40}$

$\theta = 35°16'$

d $(1-x)^n\left(1+\frac{1}{x}\right)^n$

$= \left[(1-x)\left(1+\frac{1}{x}\right)\right]^n$

$= \left[1+\frac{1}{x}-x-1\right]^n$

$= \left[\frac{1}{x}-x\right]^n$

$= {}^nC_0\left(\frac{1}{x}\right)^n(-x)^0 + {}^nC_1\left(\frac{1}{x}\right)^{n-1}(-x) + {}^nC_2\left(\frac{1}{x}\right)^{n-2}(-x)^2 + \ldots$

Consider $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^n\, {}^nC_n x^n$

$\left(1+\frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1 x^{-1} + {}^nC_2 x^{-2} + {}^nC_3 x^{3} + \ldots + {}^nC_n x^{-n}$

$(1-x)^n\left(1+\frac{1}{x}\right)^n = {}^nC_0\left({}^nC_0 + {}^nC_1 x^{-1} + \ldots\right) - {}^nC_1\left({}^nC_0 + {}^nC_1 + \ldots\right) + \ldots$

Coeffs of x^2 term:

$${}^nC_2\,{}^nC_0 - {}^nC_3\,{}^nC_1 + {}^nC_4\,{}^nC_2 - {}^nC_5\,{}^nC_3 + \ldots + (-1)^n\,{}^nC_n\,{}^nC_{n-2}$$

Coeff of x^2 term in $\left(\frac{1}{x}-x\right)^n$

$T_{k+1} = {}^nC_k\left(\frac{1}{x}\right)^{n-k}(-x)^k$

$= {}^nC_k\, x^{k-n}(-1)^k x^k$

$= {}^nC_k\, x^{2k-n}(-1)^k$

if x^2 term $\quad 2k-n=2$

$2k = 2+n$

$k = \frac{2+n}{2}$

$\therefore$ n must be even.

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$\therefore$ for even n,

$$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \dots + (-1)^n \binom{n}{n}\binom{n}{n-2} = \binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}}$$

for odd n,

$$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \dots + (-1)^n \binom{n}{n}\binom{n}{n-2} = 0.$$