CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2014 Trial Examination

FORM VI

MATHEMATICS EXTENSION 1

Friday 8th August 2014

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 120 boys

Examiner

DS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1

Which expression is equivalent to $\cos 2x$?

(A) $\sin^2 x - \cos^2 x$

(B) $2\sin^2 x - 1$

(C) $2\sin^2 x + 1$

(D) $2\cos^2 x - 1$

QUESTION TWO

1

A polynomial of degree four is divided by a polynomial of degree two.
What is the maximum possible degree of the remainder?

(A) 3

(B) 2

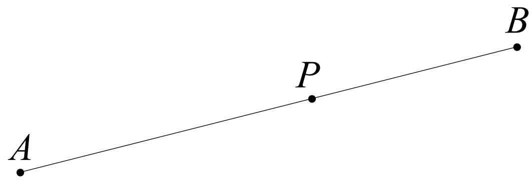
(C) 1

(D) 0

Exam continues next page ...

QUESTION THREE

1



In the diagram above the point P divides the interval AB in the ratio $3:2$. In what ratio does the point A divide the interval BP?

(A) $-5:3$

(B) $-5:2$

(C) $-3:5$

(D) $-2:5$

QUESTION FOUR

1

What is the exact value of $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$?

(A) $-\frac{2\pi}{3}$

(B) $-\frac{\pi}{3}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

QUESTION FIVE

1

Which function is a primitive of $\dfrac{1}{1+4x^2}$?

(A) $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}x\right)$

(B) $\frac{1}{4}\tan^{-1}\left(\frac{1}{2}x\right)$

(C) $\frac{1}{2}\tan^{-1}(2x)$

(D) $\frac{1}{4}\tan^{-1}(2x)$

Exam continues overleaf ...

QUESTION SIX

1

Which expression is equal to ${}^{n}\mathrm{C}_2$?

(A) $\frac{n}{2}$

(B) $\frac{n^2-n}{2}$

(C) $\frac{n^2+n}{2}$

(D) n

QUESTION SEVEN

1

The velocity v of a particle moving in a straight line is governed by the equation $v = x - 2$, where x is its displacement. The particle started at $x = 5$. What is the displacement function of the particle?

(A) $x = 5e^t$

(B) $x = 2 + \frac{1}{3}e^t$

(C) $x = 2 + e^t$

(D) $x = 2 + 3e^t$

QUESTION EIGHT

1

A particle is moving in simple harmonic motion about the origin according to the equation $x = 2\cos nt$, where x metres is its displacement after t seconds. It passes through the origin with speed $\sqrt{2}\,\mathrm{m/s}$. What is the value of n?

(A) $\dfrac{1}{\sqrt{2}}$

(B) $\sqrt{2}$

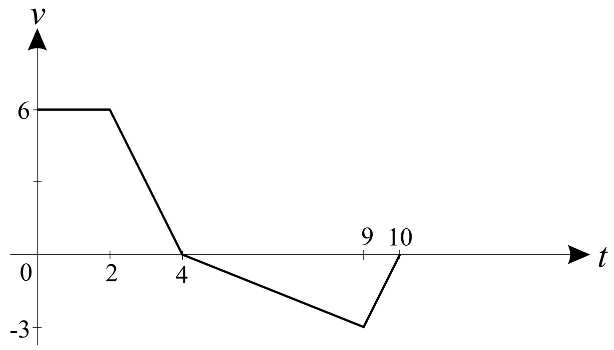
(C) $-\sqrt{2}$

(D) $\dfrac{\pi}{4}$

Exam continues next page ...

QUESTION NINE

1



The diagram above shows the velocity–time graph of an object that moves over a 10 second time interval. For what percentage of the time is the speed of the object decreasing?

(A) 30%

(B) 60%

(C) 70%

(D) It cannot be determined from the graph.

QUESTION TEN

1

How many solutions does the equation $2x + 3\pi \sin x = 0$ have in the domain $0 \leq x \leq 2\pi$?

(A) 1

(B) 2

(C) 3

(D) 4

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequation $\dfrac{2}{x} < 3$. **2**

(b) (i) Sketch the curve $y = \sin^{-1} x$. **1**

(ii) What is the gradient of the curve at $x = 0$? **1**

(c) Solve the equation $\sin 2x = \sin x$ for $-\pi \le x \le \pi$. **3**

(d) A curve is defined parametrically by the equations **2**

$$x = 1 - t$$
$$y = t^2.$$

Find the gradient of the tangent to the curve at the point where $t = -3$.

(e) By using the substitution $u = \sin x$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^5 x \cos x \, dx$. **3**

(f) A spherical balloon, with volume given by the formula $V = \frac{4}{3}\pi r^3$, is being filled with air at the constant rate of $200\,\text{cm}^3/\text{s}$. At what rate is its radius r increasing at the instant when it is $7\,\text{cm}$? Give your answer correct to three significant figures. **3**

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) The cubic equation $x^3 + 5x^2 + cx + d = 0$ has three real roots -3, 7 and α.

(i) Use the sum of the roots to find α. **1**

(ii) Find the values of c and d. **2**

(b) Find the coefficient of x^3 in the expansion of $\left(3x^2 - \dfrac{2}{x}\right)^9$. **3**

(c) (i) Write the expression $\sqrt{2}\sin x - \sqrt{6}\cos x$ in the form $A\sin(x - \theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$. **2**

(ii) Hence write down the maximum value of $\sqrt{2}\sin x - \sqrt{6}\cos x$, and find the smallest positive value of x for which this maximum value occurs. **2**

(d) Let $P(x) = x^3 + 3x - 7$.

(i) Show that the equation $P(x) = 0$ has a root between 1 and 2. **1**

(ii) Use two applications of Newton's method with initial approximation $x_1 = 1$ to approximate this root. Give your answer correct to two decimal places. **2**

(e) Suppose that θ is the acute angle between the lines $y = kx$ and $(k+1)y = kx$, where $k + 1 > 0$ and $k \neq 0$.

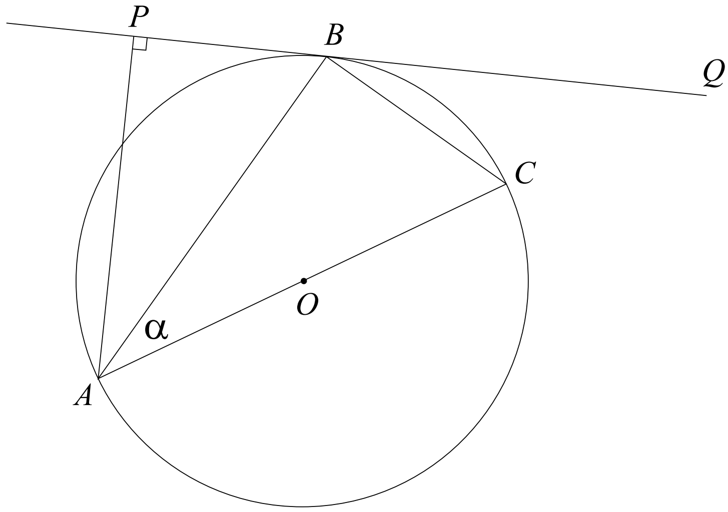
(i) Find an expression for $\tan\theta$ in simplest form. **1**

(ii) Explain why $\theta < 45°$. **1**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) **3**



The diagram above shows the points A, B and C lying on a circle, of which AC is a diameter. The line AP is perpendicular to the tangent at B.
Let $\angle BAC = \alpha$.

Prove that BA bisects $\angle PAC$.

(b) A particle is moving in simple harmonic motion. Its acceleration is defined by the equation $\ddot{x} = -9x$. Whenever the particle is 4 cm from the origin its speed is 6 cm/s. Find the amplitude of the motion. **2**

(c) Consider the quadratic polynomial $Q(x) = (x+h)^2 + k$, for some constants h and k. Find the values of h and k given that $x+2$ is a factor of $Q(x)$ and 16 is the remainder when $Q(x)$ is divided by x. **3**

(d) Prove by mathematical induction that for all positive integer values of n, **3**

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2\,(n+1) = \tfrac{1}{12}n(n+1)(n+2)(3n+1).$$

Exam continues next page ...

QUESTION THIRTEEN (Continued)

(e) A jug of cold water at $W°$ C, where $W > 0$, is taken out of a refrigerator. The air temperature in the room is $2W°$ C. The rate at which the water warms is proportional to the difference between the temperature of the surrounding air and the temperature of the water. Thus $\dfrac{dT}{dt} = k(2W - T)$, where $T°$ C is the temperature of the water after t minutes.

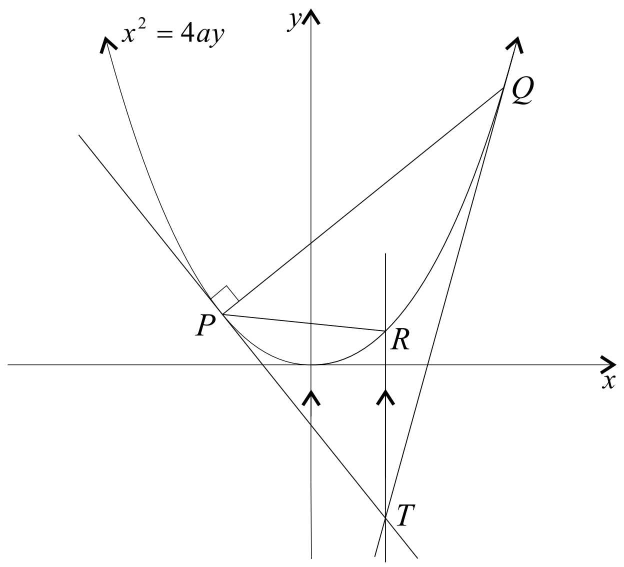
(i) Show that $T = 2W - We^{-kt}$ satisfies the differential equation. **1**

(ii) If the temperature of the water has increased by 50% after 20 minutes, find the value of k. **2**

(iii) Find the percentage increase in the temperature of the water 45 minutes after the water is taken out of the refrigerator. Give your answer correct to the nearest whole percent. **1**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above the normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ meets the parabola again at $Q(2aq, aq^2)$. You may assume that the normal at P has equation $x + py = 2ap + ap^3$.

(i) Show that $p^2 + pq + 2 = 0$. **2**

(ii) Given that the tangents at P and Q intersect at the point $T\,(a(p+q), apq)$, and the line through T parallel to the axis of the parabola meets the parabola at $R(2ar, ar^2)$, prove that PR is a focal chord. (That is, prove that $pr = -1$.) **2**

QUESTION FOURTEEN (Continued)

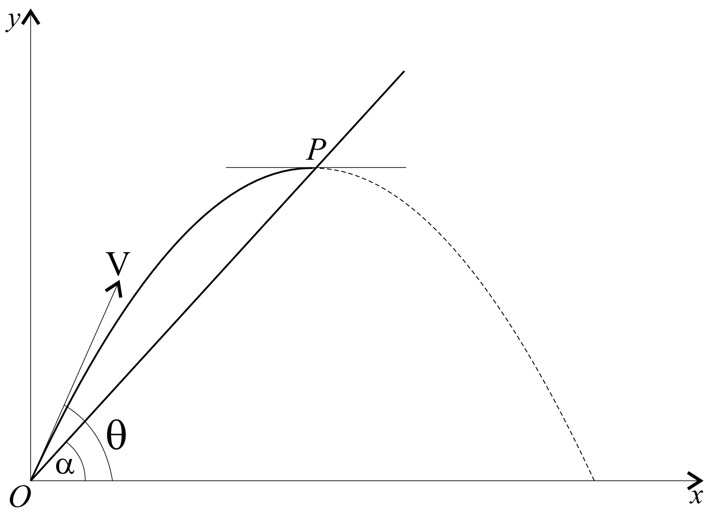
(b) (i) By considering the expansion of $(1+x)^n$, show that **1**

$$\binom{n}{1}+\binom{n}{2}x+\binom{n}{3}x^2+\cdots+\binom{n}{n}x^{n-1}=\frac{(1+x)^n-1}{x}.$$

(ii) By applying integration to the identity in part (i), with the substitution $u=1+x$ on the right-hand-side, show that **3**

$$\binom{n}{1}-\tfrac{1}{2}\binom{n}{2}+\tfrac{1}{3}\binom{n}{3}-\cdots+\tfrac{(-1)^{n-1}}{n}\binom{n}{n}=1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{n}.$$

(c)



In the diagram above the point O is the foot of a plane inclined at an angle α to the horizontal. A particle is projected with speed V from O at an angle of elevation θ to the horizontal, where $\theta > \alpha$. It strikes the inclined plane at P, which is the vertex of the parabolic path of the particle. You may assume that this parabolic path has parametric equations $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$.

(i) Show that $\tan\theta = 2\tan\alpha$. **3**

(ii) Show that the distance OP is given by $\dfrac{2V^2\sec\alpha\tan\alpha}{g(1+4\tan^2\alpha)}$. **4**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\,\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ \ x > 0$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Extension 1 Trial HSC 2014

Multiple Choice (One mark each.)

(1) $\cos 2x = 2\cos^2 x - 1$ Ⓓ

(2) degree one Ⓒ

(3) $-5:3$ Ⓐ

(4) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos\frac{\pi}{3}\right)$

$= \frac{\pi}{3}$ Ⓒ

(5) $\int \frac{1}{1+4x^2}\,dx$

$= \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}\,dx$

$= \frac{1}{4}\cdot 2\tan^{-1}2x + c$

$= \frac{1}{2}\tan^{-1}2x + c$ Ⓒ

(6) ${}^nC_2 = \frac{n!}{2!(n-2)!}$

$= \frac{n(n-1)}{2}$

$= \frac{n^2-n}{2}$ Ⓑ

(7) When $t=0$, $x=5$,

so only A or D are possible.

In D, $v = \frac{dx}{dt} = 3e^t = x-2$.

So it's Ⓓ.

(8) $x = 2\cos nt$

$v = -2n\sin nt$

Maximum speed $= 2n = \sqrt{2}$.

So $n = \frac{1}{\sqrt{2}}$. Ⓐ

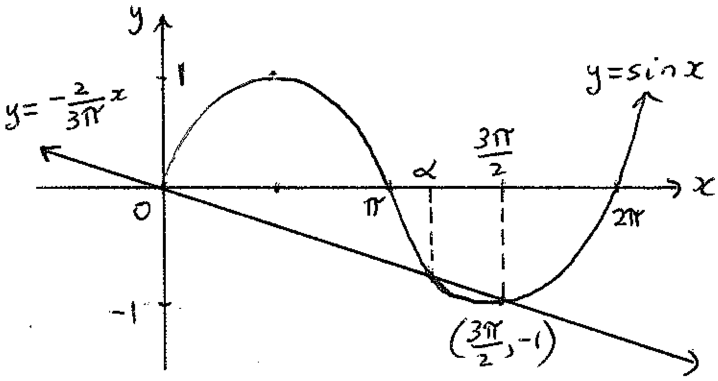
(9) Speed decreasing

for $2<t<4$ and $9<t<10$.

$\frac{3}{10} = 30\%$ Ⓐ

(10) $2x + 3\pi\sin x = 0$

$\sin x = -\frac{2}{3\pi}x$



Three solutions $x = 0, \alpha, \frac{3\pi}{2}$.

Ⓒ

Written Response

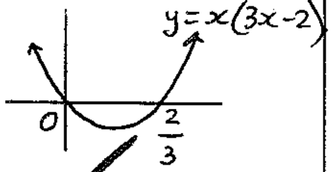
(11)(a) $\dfrac{2}{x} < 3$, $x \neq 0$

Multiply both sides by x^2.

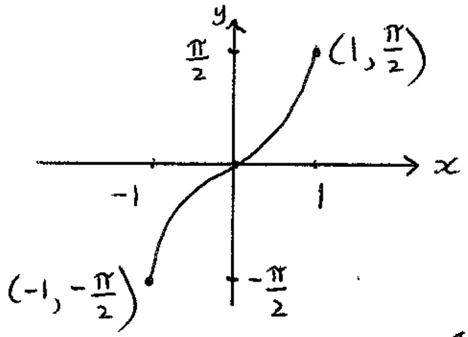
$2x < 3x^2$ ✓

$x(3x-2) > 0$

$x < 0$ or $x > \dfrac{2}{3}$ ✓



(b)(i)



✓

(ii) At $(0,0)$ $m = 1$. ✓

(c) $\sin 2x = \sin x$, $-\pi \le x \le \pi$

$2\sin x \cos x = \sin x$ ✓

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$ ✓

$x = -\pi, 0, \pi, -\frac{\pi}{3}, \frac{\pi}{3}$

$= 0, \pm\pi, \pm\frac{\pi}{3}$ ✓

(d) $\begin{cases} x = 1 - t \\ y = t^2 \end{cases}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-1}$$ ✓

At $t = -3$, $\dfrac{dy}{dx} = 6$. ✓

Alternatively eliminate t to get the Cartesian equation $y = (1-x)^2$.
Then $\dfrac{dy}{dx} = -2(1-x)$.

When $t = -3$, $x = 4$.

So $\dfrac{dy}{dx} = -2(-3) = 6$, as above.

(e) $u = \sin x$

$\dfrac{du}{dx} = \cos x$

$du = \cos x \, dx$

x	0	$\frac{\pi}{4}$
u	0	$\frac{1}{\sqrt{2}}$

✓

$$\int_0^{\frac{\pi}{4}} \sin^5 x \cos x \, dx = \int_0^{\frac{1}{\sqrt{2}}} u^5 \, du = \frac{1}{6}\left[u^6\right]_0^{\frac{1}{\sqrt{2}}}$$ ✓

$$= \frac{1}{6}\left(2^{-\frac{1}{2}}\right)^6 = \frac{1}{6} \cdot 2^{-3} = \frac{1}{48}$$ ✓

(f) Given $\dfrac{dV}{dt} = 200 \text{ cm}^3/\text{s}$.

$V = \frac{4}{3}\pi r^3$ so $\dfrac{dV}{dr} = 4\pi r^2$.

Find $\dfrac{dr}{dt}$ when $r = 7$cm. ✓

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$200 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{50}{\pi r^2}$$ ✓

When $r = 7$,

$$\frac{dr}{dt} = \frac{50}{49\pi} \text{ cm/s}$$

$\doteqdot 0.325$ cm/s ✓

(No penalty for incorrect rounding or incorrect number of figures.)

(12) (a) (i) $-3 + 7 + \alpha = -5$

$\alpha = -9$ ✓

(ii) product of roots $= -d$,

so $-3 \times 7 \times -9 = -d$,

so $d = -189$. ✓

Sum of products of pairs $= c$,

so $-21 + 27 - 63 = c$

so $c = -57$. ✓

(b) General term $= {}^9C_r \cdot (3x^2)^{9-r} \cdot \left(\frac{-2}{x}\right)^r$

$= {}^9C_r \cdot 3^{9-r} \cdot (-2)^r \cdot x^{18-3r}$ ✓

Let $18 - 3r = 3$.
Then $3r = 15$
$r = 5$. ✓

So the coefficient of x^3 is ✓

${}^9C_5 \cdot 3^4 \cdot (-2)^5 = -326\,592$.

(c) (i) $A\sin(x-\theta)$

$= A\sin x\cos\theta - A\cos x\sin\theta$

$= (A\cos\theta)\sin x - (A\sin\theta)\cos x$

Equating the coefficients of $\sin x$ and $\cos x$, we get

$A\cos\theta = \sqrt{2}$ (1)
$A\sin\theta = \sqrt{6}$ (2) ✓

Squaring and adding,

$A^2(\cos^2\theta + \sin^2\theta) = 2 + 6$

$A = 2\sqrt{2}$ $(A > 0)$

Dividing (2) by (1),

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$ $\left(0 < \theta < \frac{\pi}{2}\right)$ ✓

So $\sqrt{2}\sin x - \sqrt{6}\cos x = 2\sqrt{2}\sin\left(x - \frac{\pi}{3}\right)$

(ii) So the maximum value is $2\sqrt{2}$, ✓
since $-1 \leq \sin\left(x - \frac{\pi}{3}\right) \leq 1$.

(continued)
(ii) The maximum value first occurs when

$x - \frac{\pi}{3} = \frac{\pi}{2}$

$x = \frac{5\pi}{6}$. ✓

(d) (i) $P(1) = -3$ and $P(2) = 7$,
so $P(x) = 0$ has a root between 1 and 2. ✓

(ii) $P'(x) = 3x^2 + 3$

$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$

$= 1 - \frac{P(1)}{P'(1)}$

$= 1 - \frac{-3}{6}$

$= 1{\cdot}5$ ✓

$x_3 = 1{\cdot}5 - \frac{P(1{\cdot}5)}{P'(1{\cdot}5)}$

$= 1{\cdot}5 - \frac{1{\cdot}5^3 + 3(1{\cdot}5) - 7}{3(1{\cdot}5)^2 + 3}$

$= 1{\cdot}410256\ldots$

$= 1{\cdot}41$ to 2 decimal places. ✓

(e) (i) $m_1 = k$ and $m_2 = \frac{k}{k+1}$.

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ (θ is acute)

$= \left|\frac{k - \frac{k}{k+1}}{1 + \frac{k^2}{k+1}}\right|$

$= \left|\frac{k^2 + k - k}{k + 1 + k^2}\right|$

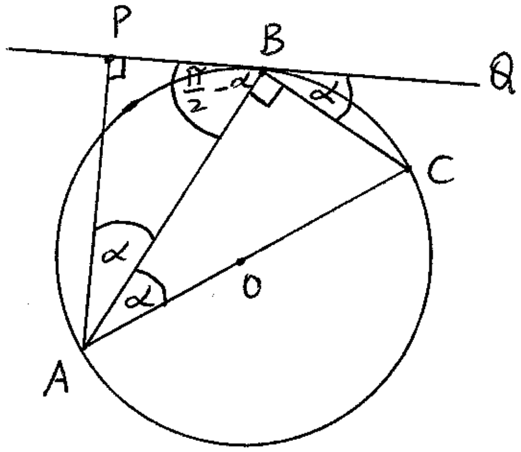
$= \frac{k^2}{k^2 + k + 1}$ ✓ (since $k + 1 > 0$)

(ii) $\frac{k^2}{k^2+k+1} < \frac{k^2}{k^2}$ (since $k + 1 > 0$)

so $\tan\theta < 1$ (θ is acute)

so $\theta < 45°$. ✓

(13)(a)



$\angle ABC = \frac{\pi}{2}$ (angle in semicircle)

and $\angle CBQ = \alpha$ (alternate segment theorem)

so $\angle ABP = \frac{\pi}{2} - \alpha$ (adjacent angles on a line)

so $\angle BAP = \alpha$ (angle sum of ΔBAP)

so BA bisects $\angle PAC$.

(b) $\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = -9x$

so $\frac{1}{2}v^2 = \int -9x \, dx$

$= -\frac{9}{2}x^2 + c$.

When $|x| = 4$, $|v| = 6$.

So $18 = -72 + c$

$c = 90$.

So $\frac{1}{2}v^2 = 90 - \frac{9}{2}x^2$

$v^2 = 180 - 9x^2$.

When $v = 0$,

$x^2 = 20$

$x = \pm 2\sqrt{5}$.

So the amplitude is $2\sqrt{5}$ cm.

OR use $v^2 = n^2(a^2 - x^2)$.

(c) $x+2$ is a factor

so $Q(-2) = 0$

so $(-2+h)^2 + k = 0$

so $4 - 4h + h^2 + k = 0$ ①

The remainder when dividing by x is 16

so $Q(0) = 16$

so $h^2 + k = 16$ ②

② − ① : $4h - 4 = 16$

$h = 5$

Substitute into ② :

$25 + k = 16$

$k = -9$

So $h = 5$ and $k = -9$.

(13)(d) When $n=1$, $LHS = 1^2 \times 2$

$= 2$

and $RHS = \frac{1}{12}(1)(2)(3)(4)$

$= 2$.

So the result is true for $n=1$.

Suppose that the result is true for the positive integer $n=k$.

i.e. suppose that $\sum_{n=1}^{k} n^2(n+1) = \frac{1}{12}k(k+1)(k+2)(3k+1)$. (*)

Prove that the result is true for $n=k+1$.

i.e. prove that $\sum_{n=1}^{k+1} n^2(n+1) = \frac{1}{12}(k+1)(k+2)(k+3)(3k+4)$.

$$LHS = \sum_{n=1}^{k} n^2(n+1) + (k+1)^2(k+2)$$

$$= \frac{1}{12}k(k+1)(k+2)(3k+1) + (k+1)^2(k+2) \quad \text{(using (*))}$$

$$= \frac{1}{12}(k+1)(k+2)\left(k(3k+1) + 12(k+1)\right)$$

$$= \frac{1}{12}(k+1)(k+2)(3k^2+13k+12)$$

$$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+4)$$

$$= RHS$$

So the result is true for $n=k+1$ if it is true for $n=k$.

But the result is true for $n=1$.

So, by induction, it is true for all positive integer values of n.

(e)(i) $T = 2W - We^{-kt}$

$$\frac{dT}{dt} = kWe^{-kt}$$

$= k(2W-T)$, as required.

(ii) When $t=20$, $T = 150\%$ of W

$= 1.5W$.

So $1.5W = 2W - We^{-20k}$

$$e^{-20k} = 0.5$$

$$e^{20k} = 2$$

$$20k = \ln 2$$

$$k = \frac{1}{20}\ln 2$$

(iii) When $t=45$,

$$T = 2W - We^{-45k}$$

$$= W(2-e^{-45k})$$

$$= (1.789\ldots)W$$

So the temperature has increased by about 79%.

(14)(a)(i) The coordinates $Q(2aq, aq^2)$ satisfy $x + py = 2ap + ap^3$,

$$\text{so } 2aq + apq^2 = 2ap + ap^3$$

$$2q + pq^2 = 2p + p^3$$

$$2p - 2q + p^3 - pq^2 = 0$$

$$2(p-q) + p(p-q)(p+q) = 0$$

$$(p-q)(2 + p^2 + pq) = 0$$

$p \neq q$, as P and Q are distinct,

so $p^2 + pq + 2 = 0$.

(ii) The vertical line TR has equation $x = a(p+q)$ and meets the parabola at $R(2ar, ar^2)$.

So $a(p+q) = 2ar$

$$p + q = 2r$$

Multiply both sides by p:

$$p^2 + pq = 2pr$$

But from (i), $p^2 + pq = -2$,

so $-2 = 2pr$

so $pr = -1$.

(b)(i) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$

Subtract 1 from both sides and then divide by x:

$$\frac{(1+x)^n - 1}{x} = \binom{n}{1} + \binom{n}{2}x + \binom{n}{3}x^2 + \dots + \binom{n}{n}x^{n-1}$$

(ii) $\displaystyle\int \frac{(1+x)^n - 1}{x}\,dx = \int \frac{u^n - 1}{u - 1}\,du$ (where $u = 1 + x$, so that $dx = du$)

$$= \int (1 + u + u^2 + \dots + u^{n-1})\,du$$

$$= u + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 + \dots + \tfrac{1}{n}u^n + c_1$$

$$= (1+x) + \tfrac{1}{2}(1+x)^2 + \tfrac{1}{3}(1+x)^3 + \dots + \tfrac{1}{n}(1+x)^n + c_1$$

So integrating both sides of the identity in (a) gives

$$\binom{n}{1}x + \tfrac{1}{2}\,{}^nC_2x^2 + \tfrac{1}{3}\binom{n}{3}x^3 + \dots + \tfrac{1}{n}\binom{n}{n}x^n + C = (1+x) + \tfrac{1}{2}(1+x)^2 + \dots + \tfrac{1}{n}(1+x)^n$$

Let $x = 0$, then $C = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Let $x = -1$, then

$$-\binom{n}{1} + \tfrac{1}{2}\binom{n}{2} - \tfrac{1}{3}\binom{n}{3} + \dots + \frac{(-1)^n}{n}\binom{n}{n} + \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{n}\right) = 0$$

so $\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \dots + \frac{(-1)^{n-1}}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

(A better approach is to integrate both sides of (i) from -1 to 0.)

(c)(i) $\dot{y} = V\sin\theta - gt$

At P, $\dot{y} = 0$, so $t = \dfrac{V\sin\theta}{g}$. ✓

So $x = \dfrac{V^2\sin\theta\cos\theta}{g}$ and $y = \dfrac{V^2\sin^2\theta}{g} - \dfrac{1}{2}\cdot g\cdot\dfrac{V^2\sin^2\theta}{g^2}$

$= \dfrac{V^2\sin^2\theta}{2g}$. ✓

Substitute these expressions into $y = x\tan\alpha$, the equation of OP:

$$\frac{V^2\sin^2\theta}{2g} = \frac{V^2\sin\theta\cos\theta}{g}\cdot\tan\alpha$$

so $\dfrac{\sin\theta}{2} = \cos\theta\tan\alpha$

so $\tan\theta = 2\tan\alpha$. ✓

(ii) By Pythagoras, $OP^2 = (x\text{-value of } P)^2 + (y\text{-value of } P)^2$.

$$= \frac{V^4\sin^2\theta\cos^2\theta}{g^2} + \frac{V^4\sin^4\theta}{4g^2}$$

$$= \frac{V^4}{4g^2}\left(4\sin^2\theta\cos^2\theta + \sin^4\theta\right) \quad ✓$$

$$= \frac{V^4}{4g^2\sec^4\theta}\left(\frac{4\sin^2\cos^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}\right)$$

$$= \frac{V^4}{4g^2}\cdot\frac{4\tan^2\theta + \tan^4\theta}{(1+\tan^2\theta)^2} \quad ✓$$

BUT $\tan\theta = 2\tan\alpha$,

so $$OP^2 = \frac{V^4}{4g^2}\cdot\frac{16\tan^2\alpha + 16\tan^4\alpha}{(1+4\tan^2\alpha)^2} \quad ✓$$

$$= \frac{4V^4}{g^2}\cdot\frac{\tan^2\alpha(1+\tan^2\alpha)}{(1+4\tan^2\alpha)^2}$$

$$= \frac{4V^4}{g^2}\cdot\frac{\sec^2\alpha\tan^2\alpha}{(1+4\tan^2\alpha)^2}$$

So $$OP = \frac{2V^2\sec\alpha\tan\alpha}{g(1+4\tan^2\alpha)}. \quad ✓$$