2014 Extension 2 Solutions

1) $a(x^2+1) + x(bx+c) = 5x^2 - x + 1$

$\underline{x = i}$

$-b + ic = -4 - i$

$\therefore \underline{b = 4,\ c = -1}$ (D)

2) If $2 - i$ is a root then so is $2 + i$

$\alpha + \beta = 4$
$\alpha\beta = 5$

$\therefore \underline{z^2 - 4z + 5 \text{ is a factor}}$ (A)

3) $9x^2 + 16y^2 = 25$

$\dfrac{9x^2}{25} + \dfrac{16y^2}{25} = 1$

$$e^2 = \frac{a^2 - b^2}{a^2}$$
$$= \left(\frac{25}{9} - \frac{25}{16}\right) \times \frac{9}{25}$$
$$= \left(\frac{1}{9} - \frac{1}{16}\right) \times 9$$
$$= 1 - \frac{9}{16}$$
$$= \frac{7}{16}$$

$\therefore \underline{e = \dfrac{\sqrt{7}}{4}}$ (B)

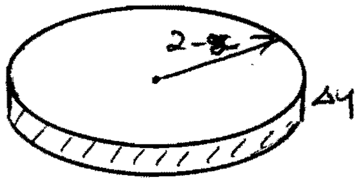
4) $$(\bar{z})^{-1} = \frac{z}{|z|^2}$$
$$= \frac{2}{4}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= \underline{\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$ (C)

5) $y^2 = x^2 - 2x$

$(x-1)^2 - y^2 = 1$

$\therefore \underline{\text{hyperbola centre at } (1,0)}$ (C)

6)



$$A(y) = \pi\left(2 - \frac{y^2}{8}\right)^2$$
$$\Delta V = \pi\left(2 - \frac{y^2}{8}\right)^2 \Delta y$$
$$V = \lim_{\Delta y \to 0} \sum_{y=-4}^{4} \pi\left(2 - \frac{y^2}{8}\right)^2 \Delta y$$
$$\underline{V = 2\pi \int_0^4 \left(2 - \frac{y^2}{8}\right)^2 dy}$$ (D)

7) $$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$$
$$= \int \left[\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right] dx$$
$$= \int (\sec^2 x + \sec x \tan x)\,dx$$
$$= \underline{\tan x + \sec x + c}$$ (B)

8) $\arg u = \arg z + \arg w \Rightarrow u = zw$

$\overrightarrow{Ou} = \overrightarrow{Oz} + \overrightarrow{wO} \Rightarrow u = z - w$ (B)

9) $x = 1,\ v = 2,\ a = 4$

A: $v = 2\sin(x-1) + 2$
$= 2\sin 0 + 2$
$= 2$ ✓

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$\frac{1}{2}v^2 = \frac{1}{2} \times 4(\sin(x-1)+1)^2$
$= 2(\sin^2(x-1) + 2\sin(x-1) + 1)$
$a = 2[2\sin(x-1)\cos(x-1) + 2\cos(x-1)]$
$= 2(0+2)$
$= 4$ ✓ $\therefore$ (A)

B: $v = 2 + 4\log x$
$= 2 + 4\log(1)$
$= 2$ ✓

C: $v^2 = 4(x^2 - 2)$
$= 4(1-2)$
$= -4$ ✗

D: $v = x^2 + 2x + 4$
$= (x+2)^2$
$= (1+2)^2$
$= 9$ ✗

10) $\int_0^a f(a-x)\,dx$ $\quad u = a - x,\ du = -dx$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

$\int_0^a f(x-a)\,dx$ $\quad u = x - a,\ du = dx$

$= \int_{-a}^0 f(u)\,du$

$= \int_{-a}^0 f(x)\,dx$

$\therefore \int_0^a \left(f(x-a) + f(a-x)\right)dx$

$= \int_0^a f(x)\,dx + \int_{-a}^0 f(x)\,dx$

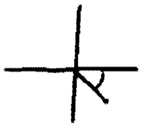
$= \int_{-a}^a f(x)\,dx$ (D)

Question 11

a) $z + w = -2 - 2i + 3 + i$

$= 1 - i$

$= \underline{\underline{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}}$



(ii) $\frac{z}{w} = \frac{-2-2i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{-6+2i-6i-2}{10}$

$= \underline{\underline{-\frac{4}{5} - \frac{2}{5}i}}$

b) $\int_0^{\frac{1}{2}} (3x-1)\cos(\pi x)\,dx$

$= 3\int_0^{\frac{1}{2}} x\cos\pi x\,dx - \int_0^{\frac{1}{2}} \cos\pi x\,dx$

$u = x \qquad v = \frac{1}{\pi}\sin\pi x$

$du = dx \qquad dv = \cos\pi x\,dx$

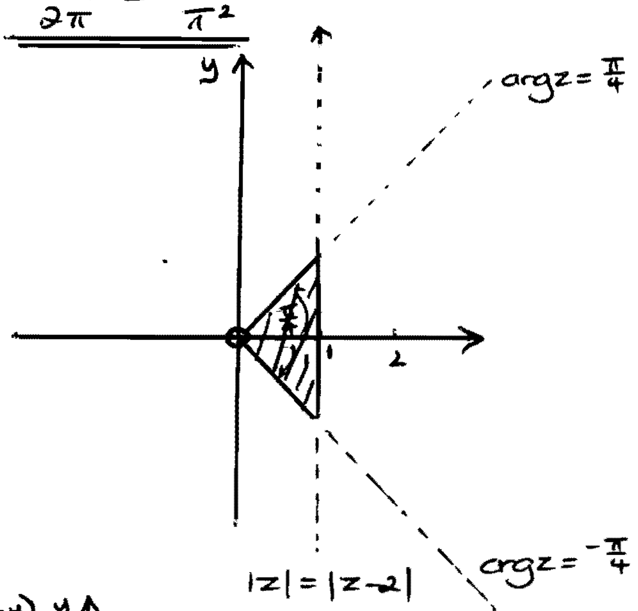
$= \left[\frac{3x}{\pi}\sin\pi x - \frac{1}{\pi}\sin\pi x\right]_0^{\frac{1}{2}} - \frac{3}{\pi}\int_0^{\frac{1}{2}} \sin\pi x\,dx$

$= \frac{3}{2\pi} - \frac{1}{\pi} + \frac{3}{\pi^2}\left[\cos\pi x\right]_0^{\frac{1}{2}}$

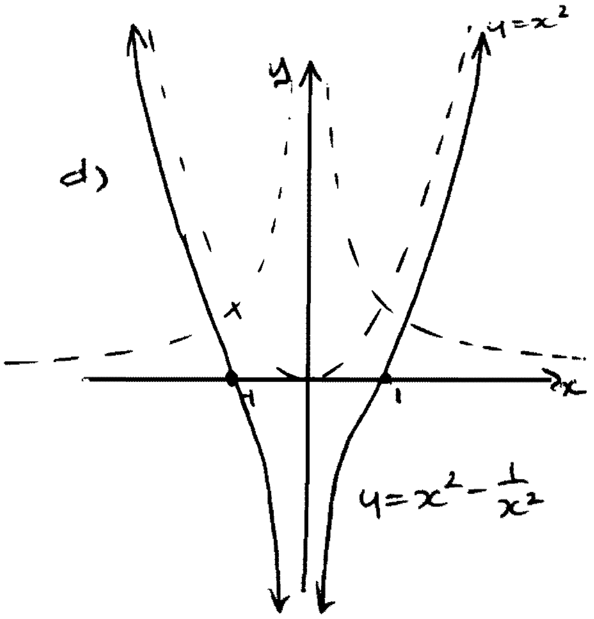
$= \frac{1}{2\pi} + \frac{3}{\pi^2}(0 - 1)$

$= \underline{\underline{\frac{1}{2\pi} - \frac{3}{\pi^2}}}$

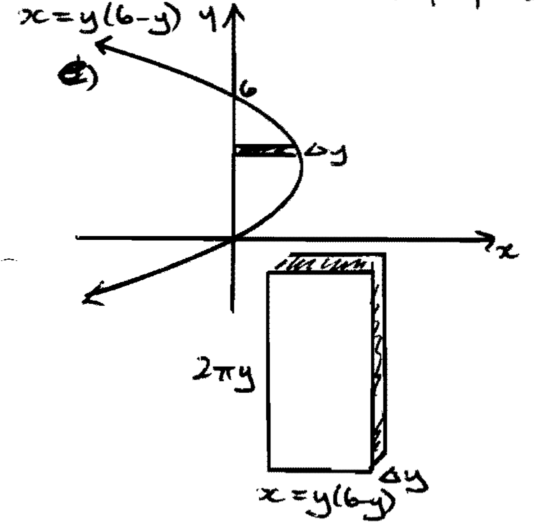
c)



d)



e)



$A(y) = 2\pi y^2(6-y)$

$\Delta V = (12\pi y^2 - 2\pi y^3)\,\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{6} (12\pi y^2 - 2\pi y^3)\,\Delta y$

$= 2\pi\int_0^6 (6y^2 - y^3)\,dy$

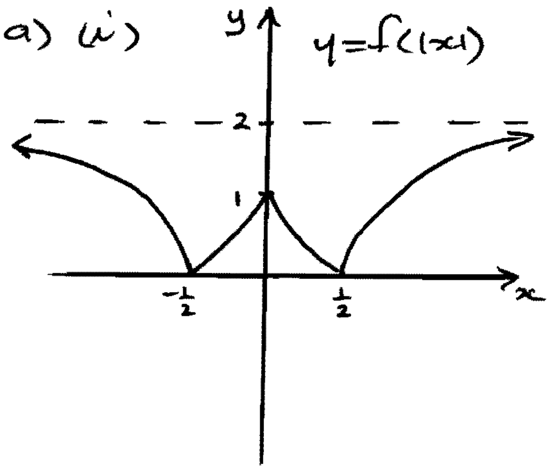
$= 2\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$

$= 2\pi\left(2(6)^3 - \frac{1}{4}(6)^4\right)$

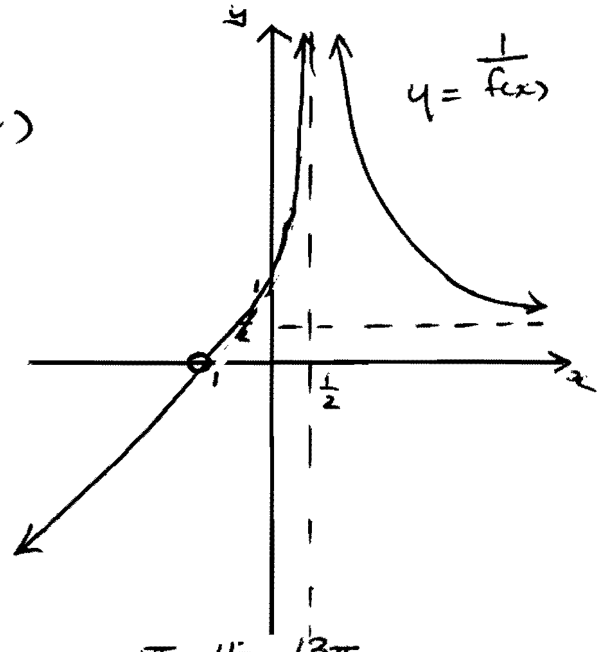
$= \underline{\underline{216\pi \text{ units}^3}}$

Question 12

a) (i)



(ii)



b) $x^3 - 3x = \sqrt{3}$

$x = 2\cos\theta$

$\therefore 8\cos^3\theta - 6\cos\theta = \sqrt{3}$

$2\cos 3\theta = \sqrt{3}$

$\cos 3\theta = \frac{\sqrt{3}}{2}$

(ii) $3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$

$\therefore x = 2\cos\frac{\pi}{18}, 2\cos\frac{11\pi}{18}, 2\cos\frac{13\pi}{18}$

c) $x^2 - y^2 = 5$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

$xy = 6$

$y = \frac{6}{x}$

$\frac{dy}{dx} = -\frac{6}{x^2}$

at $P(x_0, y_0)$ the slope of the tangents are $\frac{x_0}{y_0}$ and $-\frac{6}{x_0^2}$

$m_1 \times m_2 = \frac{x_0}{y_0} \times -\frac{6}{x_0^2}$

$= \frac{-6}{x_0 y_0}$

however as P lies on $xy=6$, $x_0y_0 = 6$

$\therefore m_1 \times m_2 = -\frac{6}{6}$

$= -1$

Thus the tangents are perpendicular

d) $I_0 = \int_0^1 \frac{dx}{x^2+1}$

$= \left[\tan^{-1}x\right]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$

(ii) $I_n + I_{n-1} = \int_0^1 \frac{x^{2n} + x^{2n-2}}{x^2+1}\,dx$

$= \int_0^1 \frac{x^{2n-2}(x^2+1)}{x^2+1}\,dx$

$= \int_0^1 x^{2n-2}\,dx$

$= \left[\frac{x^{2n-1}}{2n-1}\right]_0^1$

$= \frac{1}{2n-1} - 0$

$= \frac{1}{2n-1}$

(iii) $\int_0^1 \frac{x^4}{x^2+1}\,dx = I_2$

$= \frac{1}{3} - I_1$

$= \frac{1}{3} - (1 - I_0)$

$= \frac{1}{3} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{4} - \frac{2}{3}$

Question 13

a) $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{3\sin x - 4\cos x + 5}$$

$t = \tan\frac{x}{2}$

$dx = \frac{2dt}{1+t^2}$

$$= \int_{\frac{1}{\sqrt3}}^{1} \frac{2\,dt}{6t - 4 + 4t^2 + 5 + 5t^2}$$

$$= \int_{\frac{1}{\sqrt3}}^{1} \frac{2\,dt}{9t^2 + 6t + 1}$$

$$= 2\int_{\frac{1}{\sqrt3}}^{1} (3t+1)^{-2}\,dt$$

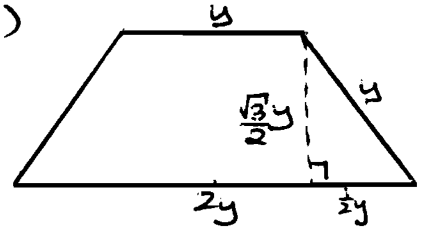
$$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt3}}^{1}$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt3 + 1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{\sqrt3 - 1}{2}\right)$$

$$= -\frac{2}{3} \times \frac{3 - 2\sqrt3}{4}$$

$$= \frac{2\sqrt3 - 3}{6}$$

b) 

$$A(x) = \frac{\sqrt3}{4}y\left(2y + y\right)$$

$$= \frac{3\sqrt3}{4}y^2$$

$$= \frac{3\sqrt3}{4}x^4$$

$$\Delta V = \frac{3\sqrt3}{4}x^4\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} \frac{3\sqrt3}{4}x^4\,\Delta x$$

$$= \frac{3\sqrt3}{4}\int_0^2 x^4\,dx$$

$$= \frac{3\sqrt3}{20}\left[x^5\right]_0^2$$

$$= \frac{3\sqrt3}{20}(32 - 0)$$

$$= \frac{24\sqrt3}{5} \text{ units}^3$$

c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2(t^2+1)^2}{4a^2t^2} - \frac{b^2(t^2-1)^2}{4b^2t^2}$

$= \frac{(t^2+1)^2 - (t^2-1)^2}{4t^2}$

$= \frac{t^4+2t^2+1-t^4+2t^2-1}{4t^2}$

$= \frac{4t^2}{4t^2}$

$= 1$ $\quad \therefore$ M lies on the hyperbola

(ii) $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{b^2x}{a^2y}$

at M, $\frac{dy}{dx} = \frac{ab^2(t^2+1)}{2t} \times \frac{2t}{a^2b(t^2-1)}$

$= \frac{b(t^2+1)}{a(t^2-1)}$

$\therefore m_{tangent} = \frac{b(t^2+1)}{a(t^2-1)}$

$m_{PQ} = \frac{bt + \frac{b}{t}}{at - \frac{a}{t}}$

$= \frac{bt^2+b}{at^2-a}$

$= \frac{b(t^2+1)}{a(t^2-1)} = m_{tangent}$

as M lies on PQ, PQ is the tangent at M.

(iii) $OP \times OQ = \sqrt{a^2t^2+b^2t^2} \times \sqrt{\frac{a^2}{t^2} + \frac{b^2}{t^2}}$

$= \sqrt{a^2+b^2} \times \sqrt{a^2+b^2}$

$= a^2+b^2$ $\qquad (\because e^2 = \frac{a^2+b^2}{a^2})$

$= a^2e^2$

$= OS^2$

(iv) $at = ae$

$\therefore t = e$

$m_{MS} = \frac{\frac{b(e^2-1)}{2e}}{\frac{a(e^2+1)}{2e} - ae}$

$= \frac{b(e^2-1)}{a(e^2+1-2e^2)}$

$= \frac{b(e^2-1)}{a(1-e^2)}$

$= -\frac{b}{a}$

$\therefore$ MS is parallel to the asymptote $y = \frac{-b}{a}x$

Question 14

a) $P(x) = x^5 - 10x^2 + 15x - 6$

$P'(x) = 5x^4 - 20x + 15$

$P''(x) = 20x^3 - 20$

$P(1) = P'(1) = P''(1) = 0$ $\therefore$ $x = 1$ is a triple root

(ii) $P(x) = (x-1)^3(x^2 + 3x + 6)$

$\therefore \quad x = \dfrac{-3 \pm \sqrt{-15}}{2}$

complex roots are $-\dfrac{3}{2} \pm \dfrac{\sqrt{15}}{2} i$

b) $x = a\cos\theta \qquad y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta \qquad \dfrac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{b\cos\theta}{-a\sin\theta} \qquad \therefore\ m_N = \frac{a\sin\theta}{b\cos\theta}$$

$$m_{op} = \frac{b\sin\theta}{a\cos\theta}$$

$$\tan\phi = \left| \frac{\frac{a\sin\theta}{b\cos\theta} - \frac{b\sin\theta}{a\cos\theta}}{1 + \frac{ab\sin^2\theta}{ab\cos^2\theta}} \right|$$

$$= \left| \frac{\frac{a\sin\theta\cos\theta}{b} - \frac{b\sin\theta\cos\theta}{a}}{\cos^2\theta + \sin^2\theta} \right|$$

$$= \left| \frac{a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta}{ab} \right|$$

$$= \left(\frac{a^2 - b^2}{ab}\right) |\sin\theta\cos\theta|$$

If P is as illustrated then θ is acute $\Rightarrow \sin\theta > 0, \cos\theta > 0$

$$\tan\phi = \left(\frac{a^2 - b^2}{ab}\right)\sin\theta\cos\theta$$

(ii) $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ which is a maximum

when $2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

as $\dfrac{a^2 - b^2}{ab}$ is constant then

$\tan\phi$ is a maximum when $\theta = \frac{\pi}{4}$

c) $\overset{Kv^2}{\longleftarrow}\bullet\overset{F}{\longrightarrow}$ $\qquad m\ddot{x} = F - Kv^2$

terminal velocity occurs when $\ddot{x} = 0$

$$\text{ie} \quad 0 = F - K(300)^2$$

$$K = \frac{F}{(300)^2}$$

$$\therefore \; m\ddot{x} = F - \frac{F}{(300)^2}v^2$$

$$= \underline{\underline{F\left(1 - \left(\frac{v}{300}\right)^2\right)}}$$

(ii)

$$\ddot{x} = \frac{F}{m}\left(1 - \frac{v^2}{300^2}\right)$$

$$\frac{dv}{dt} = \frac{F}{m}\left(\frac{300^2 - v^2}{300^2}\right)$$

$$\int_0^T dt = \frac{300^2 m}{F}\int_0^{200} \frac{dv}{300^2 - v^2}$$

$$T = \frac{300^2 m}{F} \times \frac{1}{600}\left[\ln\left(\frac{300+v}{300-v}\right)\right]_0^{200}$$

$$= \frac{150m}{F}\ln\left(\frac{500}{100}\right)$$

$$= \underline{\underline{\frac{150m}{F}\ln 5 \text{ hours}}}$$

Question 15

a) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$

$1 = a^2+b^2+c^2+2(ab+bc+ac)$

but $c \geqslant b \geqslant a$

$b \geqslant a \qquad c \geqslant a \qquad c \geqslant b$

$\therefore ab \geqslant a^2 \qquad ac \geqslant a^2 \qquad bc \geqslant b^2$

Thus $1 \geqslant a^2+b^2+c^2+2(a^2+b^2+a^2)$

$= 5a^2+3b^2+c^2$

$\therefore 5a^2+3b^2+c^2 \leq 1$

b) $(1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n$

$= (\sqrt{2})^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$

$(1-i)^n = \overline{(1+i)^n}$

$= (\sqrt{2})^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$

$\therefore (1+i)^n+(1-i)^n = 2(\sqrt{2})^n\cos\frac{n\pi}{4}$

(ii) $(1+x)^n+(1-x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$

$= \sum_{k=0}^{n}\binom{n}{k}x^k\left(1+(-1)^k\right)$

(as n is even)

$= 2\binom{n}{0}+2\binom{n}{2}x^2+2\binom{n}{4}x^4+2\binom{n}{6}x^6+\dots+2\binom{n}{n}x^n$

$= 2\left[\binom{n}{0}+\binom{n}{2}x^2+\binom{n}{4}x^4+\binom{n}{6}x^6+\dots+\binom{n}{n}x^n\right]$

let $x=i$

$(1+i)^n+(1-i)^n = 2\left[\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\dots+\binom{n}{n}\right]$

(as n is a multiple of 4
$i^n = (i^4)^k$
$= 1^k = 1$

$(\sqrt{2})^n\cos\frac{n\pi}{4} = \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\dots+\binom{n}{n}$

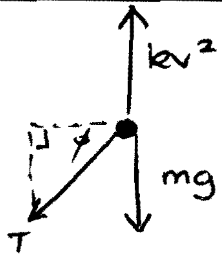
let $n=4k$, where k is an integer

ie n is divisible by 4

$$\cos\frac{n\pi}{4} = \cos\frac{4k\pi}{4}$$
$$= \cos k\pi$$
$$= (\cos\pi)^k$$
$$= (-1)^k$$
$$= (-1)^{\frac{n}{4}}$$

$$\therefore \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots + \binom{n}{n} = (\sqrt{2})^n (-1)^{\frac{n}{4}}$$

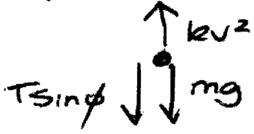
c)



horizontal $F = \frac{mv^2}{r}$

$$T\cos\phi = \frac{mv^2}{r}$$

vertical $F = 0$



$$T\sin\phi + mg - kv^2 = 0$$
$$T\sin\phi = kv^2 - mg$$

$$\frac{\sin\phi}{\cos\phi} = \frac{kv^2 - mg}{\frac{mv^2}{r}}$$
$$= \frac{r(kv^2 - mg)}{mv^2}$$

but $\cos\phi = \frac{r}{l} \Rightarrow r = l\cos\phi$

$$\frac{\sin\phi}{\cos\phi} = \frac{l\cos\phi(kv^2 - mg)}{mv^2}$$
$$\frac{\sin\phi}{\cos^2\phi} = \frac{lk}{m} - \frac{lg}{v^2}$$

(ii) $$\frac{\sin\phi}{\cos^2\phi} < \frac{lk}{m}$$
$$\sin\phi < \frac{lk}{m}(1 - \sin^2\phi)$$
$$lk\sin^2\phi + m\sin\phi - lk < 0$$
$$\frac{-m - \sqrt{m^2 + 4l^2k^2}}{2lk} < \sin\phi < \frac{-m + \sqrt{m^2 + 4l^2k^2}}{2lk}$$

Thus it is true that

$$\sin\phi < \frac{\sqrt{m^2 + 4l^2k^2}}{2lk}$$

(iii) $\frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right) = \frac{(\cos^2\phi)(\cos\phi) - (\sin\phi)(-2\cos\phi\sin\phi)}{\cos^4\phi}$

$= \frac{\cos^2\phi + 2\sin^2\phi}{\cos^3\phi}$

$\cos^3\phi > 0$ for $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$\therefore \frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right) > 0$, the function is increasing

(iv) as v increases, $\frac{lg}{v^2}$ decreases

$\therefore \frac{\sin\phi}{\cos^2\phi} \rightarrow \frac{lk}{m}$

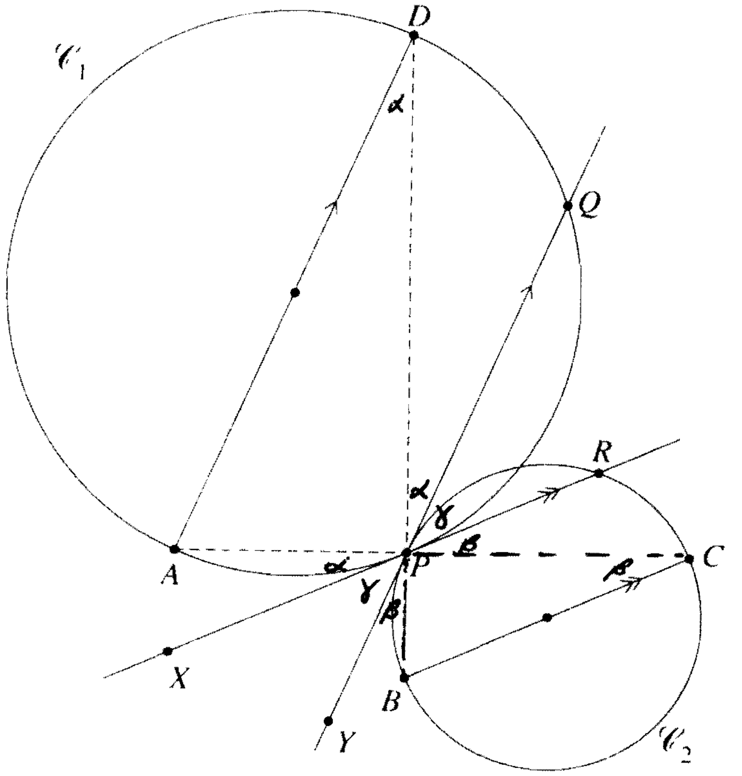
but $\frac{\sin\phi}{\cos^2\phi}$ is an increasing function $< \frac{lk}{m}$

so as ϕ increases $\frac{\sin\phi}{\cos^2\theta} \rightarrow \frac{lk}{m}$

ie as ϕ increases, v increases.

Question 16

a)



a) (i) $\angle APX = \angle ADP$ (alternate segment theorem)
$\angle ADP = \angle DPQ$ (alternate $\angle$'s =, $AD \parallel PQ$)

$\therefore \angle APX = \angle DPQ$

(ii) $\angle APX = \angle DPQ = \alpha$ (proven in (i))
$\angle BPY = \angle RPC = \beta$ (by similar method)
$\angle XPY = \angle QPR = \gamma$ (vertically opposite $\angle$'s =)
$\angle DPA = \angle BPC = 90°$ ($\angle$ in semicircle)

$\angle APX + \angle APD + \angle DPQ + \angle QPR + \angle RPC + \angle CPB + \angle BPY + \angle XPY = 360$ (revolution)
$\alpha + 90 + \alpha + \gamma + \beta + 90 + \beta + \gamma = 360$
$2\alpha + 2\beta + 2\gamma = 180$
$\alpha + \beta + \gamma = 90$

$\angle APC = \angle APD + \angle DPQ + \angle QPR + \angle RPC$ (common $\angle$)
$\angle APC = 90 + \alpha + \gamma + \beta$
$\angle APC = 180$

$\therefore$ A, P, C are collinear

(iii) $\angle DPQ = \angle BPY$ (vertically opposite $\angle$'s)
$\alpha = \beta$

$\angle BPY = \angle PCB$ (alternate segment theorem)

$\therefore \angle PCB = \alpha$

Thus $\angle PCB = \angle ADP = \alpha$

ABCD is a cyclic quadrilateral
as $\angle$'s in same segment are =.

b) $1-x^2+x^4-x^6+\ldots+(-1)^{n-1}x^{2n-2}$

$a=1$, $r=-x^2$, $n=n$

$$S_n=\frac{1(1-(-x^2)^n)}{1+x^2}$$

if n is odd
$S_n=\dfrac{1+x^{2n}}{1+x^2}$

if n is even
$S_n=\dfrac{1-x^{2n}}{1+x^2}$

Thus

$$\frac{1-x^{2n}}{1+x^2}\le S_n\le\frac{1+x^{2n}}{1+x^2}$$

$$\frac{-1-x^{2n}}{1+x^2}\le -S_n\le\frac{-1+x^{2n}}{1+x^2}$$

$$\frac{-x^{2n}}{1+x^2}\le\frac{1}{1+x^2}-S_n\le\frac{x^{2n}}{1+x^2}$$

$1+x^2\geqslant 1$

$\therefore \dfrac{1}{1+x^2}\le 1$

$\dfrac{-x^{2n}}{1+x^2}\geqslant -x^{2n}$ and $\dfrac{x^{2n}}{1+x^2}\le x^{2n}$

$$\therefore -x^{2n}\le\frac{1}{1+x^2}-(1-x^2+x^4-x^6+\ldots+(-1)^{n-1}x^{2n-2})\le x^{2n}$$

(ii)

$$-\int_0^1 x^{2n}\,dx\le\int_0^1\left[\frac{1}{1+x^2}-(1-x^2+x^4-x^6+\ldots+(-1)^{n-1}x^{2n-2})\right]dx\le\int_0^1 x^{2n}\,dx$$

$$-\left[\frac{x^{2n+1}}{2n+1}\right]_0^1\le\left[\tan^{-1}x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right)\right]_0^1\le\left[\frac{x^{2n+1}}{2n+1}\right]_0^1$$

$$-\frac{1}{2n+1}\le\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^{n-1}\frac{1}{2n-1}\right)\le\frac{1}{2n+1}$$

(iii)

$$\lim_{n\to\infty}\frac{-1}{2n+1}\le\lim_{n\to\infty}\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^{n-1}\frac{1}{2n-1}\right)\le\lim_{n\to\infty}\frac{1}{2n+1}$$

$$0\le\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)\le 0$$

$$\therefore\ \frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)=0$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

c) $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$

$$= \int \frac{1+\ln x}{(1+\ln x)^2}\,dx - \int \frac{dx}{(1+\ln x)^2}$$

$$= \int \frac{dx}{1+\ln x} - \int \frac{dx}{(1+\ln x)^2} \qquad u = \frac{1}{1+\ln x} \quad v = x$$

$$du = \frac{-dx}{x(1+\ln x)^2} \quad dv = dx$$

$$= \frac{x}{1+\ln x} + \int \frac{dx}{(1+\ln x)^2} - \int \frac{dx}{(1+\ln x)^2}$$

$$= \frac{x}{1+\ln x} + c$$