3f) $x^n - 1$ is divisible by $x - 1$

$\underline{n=1}$

$x^1 - 1$

$= x - 1$ Hence true for $n=1$

Assume true for $n=k$ where k is a integer

$\underline{ie}$ $x^k - 1 = (x-1)P(x)$ where $P(x)$ is a polynomial

Prove true for $n=k+1$

$x^{k+1} - 1 = (x-1)Q(x)$ where $Q(x)$ is a polynomial

Proof:

$$x^{k+1} - 1$$

$$= x \cdot x^k - 1$$

$$= x\left[(x-1)P(x) + 1\right] - 1$$

$$= x(x-1)P(x) + x - 1$$

$$= (x-1)(xP(x) + 1)$$

$$= (x-1)Q(x)$$ where $Q(x) = xP(x) + 1$ which is a polynomial

4a (i) n^3+2n is divisible by 12 for even n.

$\underline{n=2}$

$2^3+2(2)$

$=12$ Hence true for $n=2$

Assume true for $n=k$ where k is an even intiger

$\underline{\text{ie}}$ $k^3+2k=12P$, P is an integer

Prove true for $n=k+2$

$\underline{\text{ie}}$ $(k+2)^3+2(k+2)=12Q$, Q is an integer

Proof: $(k+2)^3 + 2(k+2)$

$= k^3 + 6k^2 + 12k + 8 + 2k + 4$

$= k^3 + 2k + 6k^2 + 12k + 12$

$= 12P + 12\left(\frac{1}{2}k^2 + k + 1\right)$

$= 12\left(P + \frac{1}{2}k^2 + k + 1\right)$

$= 12Q$ $\quad Q = P + \frac{1}{2}k^2 + k + 1$ which is an integer as k^2 is even

6b) $2^n > 3n^2$, $n \geq 8$

$n=8$

LHS $2^8 = 256$ RHS $= 3\times(8)^2$
$= 192$

LHS > RHS

Hence true for $n=8$

Assume true for $n=k$, where k is a positive integer

$2^k > 3k^2$, $k \geq 8$

Prove true for $n=k+1$

ie $2^{k+1} > 3(k+1)^2$

Proof

$$
\begin{aligned}
2^{k+1} &= 2\times 2^{k} \\
&> 6k^{2} \\
&= 3k^{2}+3k^{2} \\
&\geq 3k^{2}+24k \qquad (\because k\geq 8) \\
&= 3k^{2}+6k+18k \\
&\geq 3k^{2}+6k+144 \qquad (\because k\geq 8) \\
&> 3k^{2}+6k+3 \\
&= 3(k+1)^{2}
\end{aligned}
$$

$$\therefore\ 2^{k+1} > 3(k+1)^{2}$$

9 a) $T_n = 2T_{n-1} + 1$, $T_1 = 5$

Prove $T_n = 6 \times 2^{n-1} - 1$

<u>n=1</u>

$T_1 = 6 \times 2^0 - 1$

$= 5$ hence true for n=1

Assume true for n=k, where k is positive integer

i.e. $T_k = 6 \times 2^{k-1} - 1$

Prove true for n=k+1

i.e. $T_{k+1} = 6 \times 2^k - 1$

Proof:

$$T_{k+1} = 2 \times T_k + 1$$

$$= 2 \times \left(6 \times 2^{k-1} - 1\right) + 1$$

$$= 12 \times 2^{k-1} - 1$$

$$= 6 \times 2 \times 2^{k-1} - 1$$

$$= 6 \times 2^{k} - 1$$

Hence the result is true for $n = k+1$ if it is true for $n = k$

etc

10/ sum of angles of an n-sided polygon is $(n-2)180$

$\underline{n=3}$

LHS = $\angle$ sum of $\triangle = 180$ RHS = $(3-2)180$

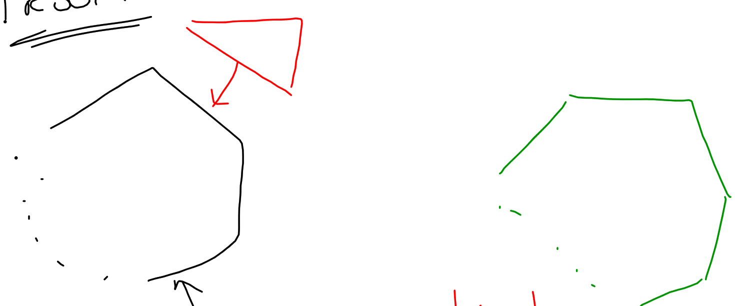
$= 180$

Hence true for $n=3$.

Assume $\angle$ sum of k sided polygon is $(k-2)180$

Prove $\angle$ sum of $(k+1)$ sided polygon is $(k-1)180$.

Proof:



k sided polygon + triangle = $k+1$ sided polygon.

$\angle$ sum $(k+1)$ sided polygon = $\angle$sum k sided polygon + $\angle$ sum $\triangle$

$= (k-2)180 + 180$

$= \underline{(k-1)180}$

15a) Prove $\sqrt{n+1} - \sqrt{n} > \frac{1}{2\sqrt{n+1}}$

$$\frac{\sqrt{n+1} - \sqrt{n}}{1} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$> \frac{1}{2\sqrt{n+1}} \qquad \left(\because \sqrt{n+1} > \sqrt{n}\right)$$

15 a) $\underline{\sqrt{n+1} - \sqrt{n} > \dfrac{1}{2\sqrt{n+1}}}$. $\Rightarrow \sqrt{k} < \sqrt{k+1} - \dfrac{1}{2\sqrt{k+1}}$

b) $\underline{\text{Prove}}$: $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \sqrt{n}$, $n \geq 7$

$\underline{n=7}$

LHS $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

$= 2.59\ldots$

RHS $= \sqrt{7}$

$= 2.65\ldots.$

LHS < RHS

Hence true for $n=7$

Assume true for $n=k$, where k integer ≥ 7

ie $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} < \sqrt{k}$

Prove true for $n=k+1$

$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k+1} < \sqrt{k+1}$

$<$

Proof

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}$$

$$< \sqrt{k}+\frac{1}{k+1}$$

$$< \sqrt{k+1}-\frac{1}{2\sqrt{k+1}}+\frac{1}{k+1}$$

$$= \sqrt{k+1}-\frac{\sqrt{k+1}-2}{2(k+1)}$$

$$\leq \sqrt{k+1}-\frac{\sqrt{8}-2}{2(k+1)} \qquad k\geq 7$$

$$< \sqrt{k+1}$$

17 $T_{n+2} = 3T_{n+1} - 2T_n$

$T_1 = 5,\ T_2 = 7$

Prove $T_n = 3 + 2^n$

$\underline{n=1}$

$T_1 = 3 + 2^1$

$= 5$

$\underline{n=2}$

$T_2 = 3 + 2^2$

$= 7$

Hence true for $n = 1$ and 2

Assume true for $n=k$ and $k+1$, where k is an integer

ie $T_k = 3+2^k$ $\qquad T_{k+1} = 3+2^{k+1}$

Prove true for $n=k+2$

ie $T_{k+2} = 3+2^{k+2}$

Proof

$$
\begin{aligned}
T_{k+2} &= 3T_{k+1} - 2T_k \\
&= 3(3+2^{k+1}) - 2(3+2^k) \\
&= 3 + 3\times2^{k+1} - 2\times2^k \\
&= 3 + 2\times2^{k+1} \\
&= \overline{3+2^{k+2}}
\end{aligned}
$$

Hence true for $n=k+2$, if true for both $n=k$ and $n=k+1$

Since true for $n=1$ and $n=2$ true for all positive integer by induction.