VIRILE AGITUR

Number: ____________________

Teacher: ____________________

2014

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen only
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

Teachers:
Mr Bradford
Mr Vuletich
Mrs Dempsey
Ms Yun

Total marks – 70

Section I: Pages 1-3

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II: Pages 4-8

60 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Write your Board of Studies Student Number and your teacher's name on the front cover of each writing booklet

This paper MUST NOT be removed from the examination room

Number of Students in Course: 80

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Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Find

$$\lim_{x \to 0} \frac{\sin 7x}{5x}$$

(A) 0

(B) $\frac{5}{7}$

(C) 1

(D) $\frac{7}{5}$

2 The point P divides the interval from $A(-2, 2)$ to $B(8, -3)$ internally in the ratio 3:2.

What is the x-coordinate of P?

(A) 4

(B) 2

(C) 0

(D) -1

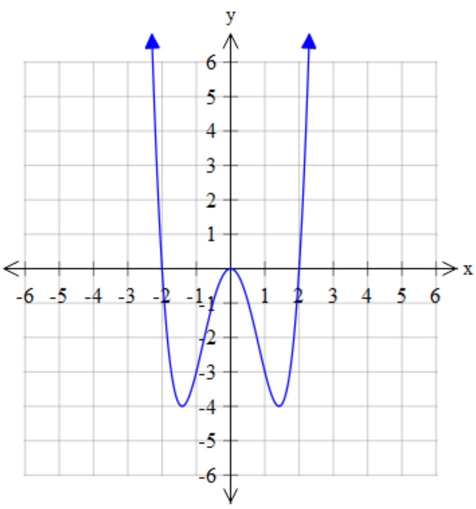
3 Which function best describes the polynomial $y = P(x)$?

(A) $y = x^2(4 - x^2)$

(B) $y = x^3(x-2)(x+2)$

(C) $y = x^2(x^2 - 4)$

(D) $y = x^3(2-x)(x+2)$



4 Which of the following is the domain of the function $y = \sin^{-1} 2x$?

(A) $-2 \le x \le 2$

(B) $0 \le x \le 2\pi$

(C) $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

(D) $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

5 In the expression $(1+x)(1+x)^9$, the coefficient of x^5 is:

(A) 126

(B) 210

(C) 252

(D) 504

6 The points *A*, *B* and *C* lie on a circle centred at O. *AB* is parallel to *CD* and $\angle ABO = 42^\circ$.

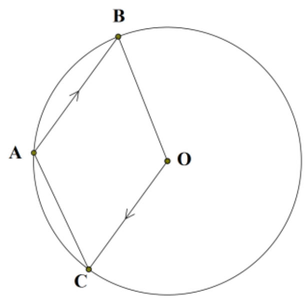
What is the size of $\angle CAB$?

(A) 42°

(B) 82°

(C) 138°

(D) 111°



7 A family of eight is seated randomly around a circular table.

What is the probability that the two youngest members of the family sit together?

(A) $\dfrac{6!2!}{7!}$

(B) $\dfrac{6!}{7!2!}$

(C) $\dfrac{6!2!}{8!}$

(D) $\dfrac{6!}{8!2!}$

8 When the polynomial *P(x)* is divided by $(x+1)(x-3)$, the remainder is $2x+7$.

What is the remainder when *P(x)* is divided by $x-3$?

(A) 1

(B) 7

(C) 9

(D) 13

9 The function $f(x)=3x-x^3$ has a relative maximum at (1, 2) and a relative minimum at $(-1,-2)$. What is the largest domain containing the origin for which *f(x)* has an inverse function $f^{-1}(x)$.

(A) $-1\le x\le 1$

(B) $-1<x<1$

(C) $-2\le x\le 2$

(D) $-2<x<2$

10 Which expression represents the ratio of the sum of *2n* terms to the sum of *n* terms of any Geometric Progression?

(A) 2 : 1

(B) $r^n:1$

(C) $(r^n+1):1$

(D) $(r^n-1):1$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the values of k such that $(x-2)$ is a factor of the polynomial $P(x)=x^3-2x^2+kx+k^2$. **2**

(b) Find the value of $\sum_{n=2}^{5} {}^{n}c_2$. **2**

(c) Solve the equation $\frac{4}{2x-1}<1$. **3**

(d) Find the angle that $y=\tan^{-1}2x$ makes with the <u>y-axis</u> at the origin. Give your answer to the nearest degree. **3**

(e) Use the substitution $u=\ln x$ to evaluate, **3**

$$\int_e^{e^2}\frac{1}{x\ln x}\,dx.$$

(f) The probability of snow falling in the Snowy Mountains on any one of the thirty-one days in August is 0.2. Find the probability that August has exactly 10 days in which snow falls. Give your answer as a percentage to the nearest whole per cent. **2**

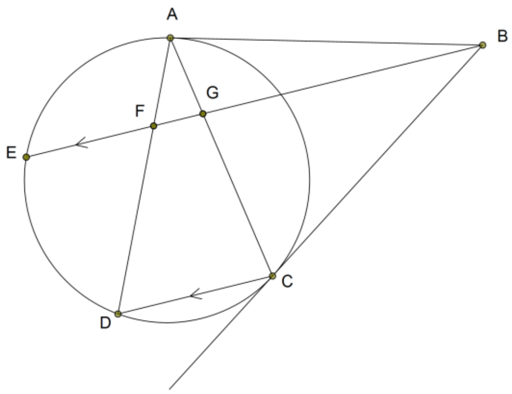
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $f(x) = 2\sin x - 10x + 5$ has a root between 0.6 and 0.7. **1**

(ii) Use one step of Newton's method and starting with x=0.6, find a better approximation. Answer to 2 decimal places. **2**

(b) In the diagram *EB* is parallel to *DC*. Tangents from *B* meet the circle at *A* and *C*.



Copy or trace the diagram into your writing booklet.

(i) Prove that $\angle BCA = \angle BFA$.
3

(ii) Prove *ABCF* is a cyclic quadrilateral. **1**

(c) Use mathematical induction to prove that $2^{3n} - 3^n$ is divisible by 5 for $n \geq 1$. **3**

(d) Consider the function $f(x) = \cos^{-1} 2x$.

(i) Sketch the function $f(x)$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ **2**

(ii) Find the exact volume formed when $y = f(x)$ is rotated about the y-axis from $0 \leq y \leq \pi$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In how many ways can a committee of 3 men and 4 woman be selected from a group of 8 men and 10 women? **1**

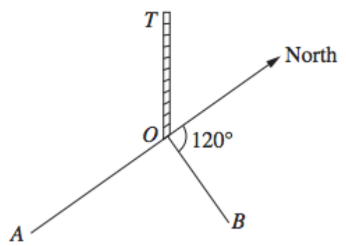
(b) It is known that two of the roots of the equation $2x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other. Find the value of *k*. **2**

(c) A salad, which is initially at a temperature of $25\,^\circ\text{C}$, is placed in a refrigerator that has a constant temperature of $3\,^\circ\text{C}$. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, *T* of the salad. That is, *T* satisfies the equation $\frac{dT}{dt} = -k(T-3)$ where *t* is the number of minutes after the salad is placed in the refrigerator.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. **1**

(ii) The temperature of the salad is $11\,^\circ\text{C}$ after 10 minutes. **3**
Find the temperature of the salad after 15 minutes.

(d) From a point *A* due south of a tower, the angle of elevation of the top of the tower *T*, is 23°. From another point *B*, on a bearing of 120° from the tower, the angle of elevation of *T* is 32°. The distance *AB* is 200 metres.

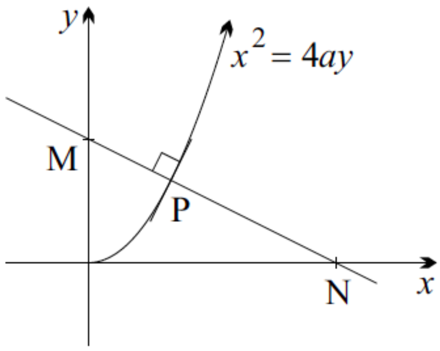


(i) Copy the diagram into your writing booklet, adding the given information to your diagram. **1**

(ii) Hence find the height of the tower. **3**

Question 13 continues on page 7

(e) A normal is drawn to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$, where $p > 0$. The normal intersects the x-axis at N and the y-axis at M as shown in the diagram below.



(i) Find the equation of the normal PN. **2**

(ii) It is known that the point P divides NM in the ratio 3:2. Find the value of the parameter p. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line so that its acceleration is given by

$\frac{dv}{dt} = x - 1$, where v is its velocity and x is its displacement from the origin.

Initially, the particle is at the origin and has velocity $v = 1$.

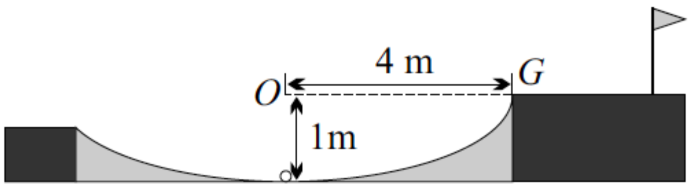
(i) Show that $v^2 = (x-1)^2$ **2**

(ii) By finding an expression for $\frac{dt}{dx}$, or otherwise, find x as a function of t. **2**

(b) The diagram below shows a golf ball in the middle of a 1m deep bunker and 4m from the edge of the green at G. The ball is hit with an initial speed of 12 m/s at an angle of elevation α. By taking the origin at 0, the horizontal and vertical equations of motion are:

$$x = 12t\cos\alpha \text{ and } y = -5t^2 + 12t\sin\alpha - 1.$$

(Do NOT prove these equations)



(i) Find the maximum height the ball reaches above G when $\alpha = 30°$. **2**

(ii) Find the range of values that α may take so that the ball lands on the green at or beyond G. Give your answer correct to the nearest $5°$. **3**

(c) (i) Use mathematical induction to show that $\ln(n!) > n$ for all positive integers $n \geq 6$. **3**

(ii) Hence show that $\frac{1}{n!} < \frac{1}{e^n}$ for all positive integers $n \geq 6$. **1**

(iii) Hence show that, **2**

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots < \frac{103}{60} + \frac{1}{e^5(e-1)}$$

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \textbf{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

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① $\lim_{x \to 0} \frac{\sin 7x}{5x} = \lim_{x \to 0} \frac{\sin 7x}{7x} \times \frac{7}{5}$

$= \frac{7}{5}$ (D)

② A$(-2, 2)$ B$(8, -3)$

3 : 2

$x = \frac{-4 + 24}{5}$

$= 4$ (A)

③ $y = x^2(x+2)(x-2)$

$= x^2(x^2-4)$ (C)

④ D: $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$ (D)

⑤ x^5: $(1)\,{}^9C_5 x^5 + (x)\,{}^9C_4 x^4$

$= (126 + 126)x^5$

$= 252x^5$ (C)

⑥ $\angle ABO = 42$

$\angle ABO + \angle BOC = 180$ (cointerior $\angle$'s, AB∥CO)

$\angle BOC = 138$

$\therefore \angle BOC_{reflex} = 222$

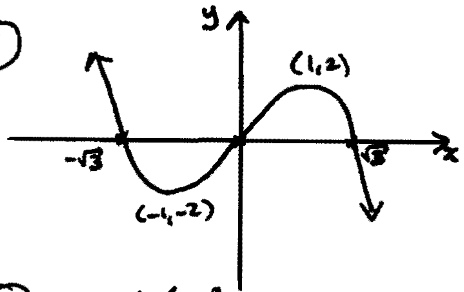
$\angle BOC_{reflex} = 2\angle BAC$ ($\angle$ at centre twice $\angle$ at circumference on same arc)

$\angle BAC = 111$ (D)

⑦ $P(\text{youngest together}) = \frac{2!6!}{7!}$ (A)

⑧ $P(x) = (x+1)(x-3)Q(x) + 2x + 7$

$P(3) = 0 + 6 + 7 = 13$ (D)

⑨ 

D: $-1 \le x \le 1$ (A)

⑩ $\frac{a(r^{2n}-1)}{r-1} \times \frac{r-1}{a(r^n-1)}$

$= \frac{r^{2n}-1}{r^n-1}$

$= \frac{(r^n+1)(r^n-1)}{(r^n-1)}$ $\therefore$ ratio $r^n+1 : 1$ (C)

$= r^n + 1$

Question 11

a) $P(2) = 0$

$2^3 - 2(2)^2 + 2k + k^2 = 0$

$k^2 + 2k = 0$

$k(k+2) = 0$

$\underline{\underline{k = 0 \text{ or } k = -2}}$

b) $\sum_{n=2}^{5} {}^nC_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$

$= 1 + 3 + 6 + 10$

$= \underline{\underline{20}}$

c) $\frac{4}{2x-1} < 1$

$x \neq \frac{1}{2}$

$\frac{4}{2x-1} = 1$

$4 = 2x - 1$

$2x = 5$

$x = \frac{5}{2}$

$\underline{\underline{x < \frac{1}{2} \text{ or } x > \frac{5}{2}}}$

d) $y = \tan^{-1} 2x$

$\frac{dy}{dx} = \frac{2}{4x^2 + 1}$

at $(0,0)$; $\frac{dy}{dx} = 2$

If α is angle with y-axis

$\tan\alpha = \frac{1}{2}$

$\alpha = 26 \cdot 5650\ldots$

$\underline{\underline{\alpha = 27^\circ}}$

e) $\int_e^{e^2} \frac{dx}{x \ln x} = \int_1^2 \frac{du}{u}$

$u = \ln x$ $\quad$ $x = e,\ u = 1$

$du = \frac{dx}{x}$ $\quad$ $x = e^2,\ u = 2$

$= [\ln u]_1^2$

$= \underline{\underline{\ln 2}}$

f) X = # days snow falls in August.

$P(X = 10) = {}^{31}C_{10} (0.8)^{21} (0.2)^{10}$

$= 0.041889\ldots$

$= \underline{\underline{4\%}}$

Question 12

a) $f(x) = 2\sin x - 10x + 5$

$f(0.6) = 0.129\ldots > 0$
$f(0.7) = -0.711\ldots < 0$

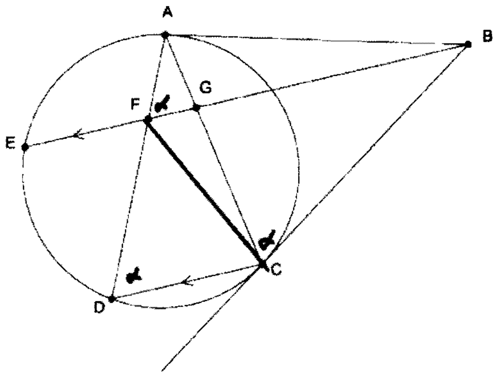
$\therefore$ as $f(x)$ is continuous, a root exists between 0.6 and 0.7

(ii) $f'(x) = 2\cos x - 10$

$$x_1 = 0.6 - \frac{2\sin 0.6 - 10(0.6) + 5}{2\cos 0.6 - 10}$$

$= 0.6154844719\ldots$
$= \underline{0.62}$ (to 2dp)

b)



$\angle BCA = \angle ADC$ (alternate segment thm)

$\angle AFG = \angle ADC$ (corresponding $\angle$'s, $FG \parallel DC$)

$\therefore \angle BCA = \angle BFA$

(ii) $\therefore ABCF$ is a cyclic quadrilateral

$\angle$'s in same segment are =

c) $2^{3n} - 3^n$ is divisible by 5

Prove for $n=1$

$2^3 - 3^1 = 5$ which is divisible by 5

Hence result is true for $n=1$

Assume the result is true for $n=k$

ie $2^{3k} - 3^k = 5P$, where P is an integer

Prove true for $n=k+1$

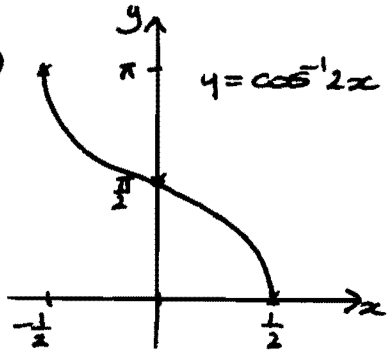
ie $2^{3k+3} - 3^{k+1} = 5Q$, where Q is an integer.

Proof $2^{3k+3} - 3^{k+1} = 8 \times 2^{3k} - 3^{k+1}$
$= 8(5P + 3^k) - 3 \times 3^k$
$= 40P + 5 \times 3^k$
$= 5(8P + 3^k)$
$= 5Q$ where $Q = 8P + 3^k$ which is an integer

Hence the result is true for $n=k+1$ if it is true for $n=k$

Since the result is true for $n=1$ then it is true for all integral values of n by induction.

d) (i)



(ii) $V = \pi \int x^2 \, dy$

$= 2\pi \int_0^{\frac{\pi}{2}} \frac{1}{4} \cos^2 y \, dy$

$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} (1 + \cos 2y) \, dy$

$= \frac{\pi}{4} \left[y + \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4} \left(\frac{\pi}{2} + 0 - 0 \right)$

$= \frac{\pi^2}{8}$ units^3

$y = \cos^{-1} 2x$

$2x = \cos y$

$x = \frac{1}{2} \cos y$

Question 13

a) Ways $= {}^{8}C_3 \times {}^{10}C_4$

$= 56 \times 210$

$= \underline{11760}$

b) let the roots be α and $\frac{1}{\alpha}$ and β

$(\alpha)(\frac{1}{\alpha})(\beta) = -3$

$\beta = -3$

$P(-3) = 0$

$2(-3)^3 + (-3)^2 + k(-3) + 6 = 0$

$-54 + 9 - 3k + 6 = 0$

$3k = 39$

$\underline{k = 13}$

c) $T = 3 + Ae^{-kt} \Rightarrow Ae^{-kt} = (T-3)$

$\frac{dT}{dt} = -kAe^{-kt}$

$= \underline{-k(T-3)}$

(ii) when $t=0$, $T=25$

$\therefore A = 22$

when $t = 10$, $T = 11$

$11 = 3 + 22e^{-10k}$

$e^{-10k} = \frac{8}{22}$

$-10k = \ln\frac{4}{11}$

$k = \frac{1}{10}\ln\frac{11}{4}$

when $t = 15$

$T = 3 + 22e^{-15k}$

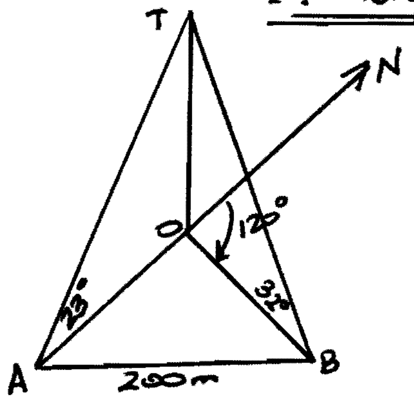
$= 3 + 22e^{-\frac{15}{10}\ln\frac{11}{4}}$

$= 3 + 22\left(\frac{11}{4}\right)^{-\frac{3}{2}}$

$= 7.824181513\ldots$

$\therefore$ temperature is $8°C$ after 15 min.

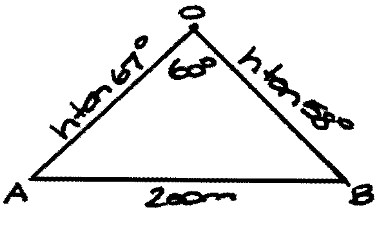
d)



$\frac{OA}{h} = \tan 67°$

$OA = h\tan 67°$

$OB = h\tan 58°$



$200^2 = h^2\tan^2 67 + h^2\tan^2 58 - 2h\tan 67\tan 58\cos 60$

$h = \dfrac{200}{\sqrt{\tan^2 67 + \tan^2 58 - 2\tan 67\tan 58\cos 60}}$

$= 20.16669369\ldots$

$= \underline{20\,m}$

c) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

when $x = 2ap$, $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

$\therefore$ slope of normal is $-\frac{1}{p}$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

(ii) M is $(0, ap^2 + 2a)$ N is $(ap^3 + 2ap, 0)$

3 : 2

$2ap = \frac{0 + 3ap^3 + 6ap}{5}$

$10ap = 3ap^3 + 6ap$

$3ap^3 - 4ap = 0$

$ap(3p^2 - 4) = 0$

$p = 0$ or $p^2 = \frac{4}{3}$

$p = \frac{2}{\sqrt{3}}$, $p > 0$

$ap^2 = \frac{2ap^2 + 4a + 0}{5}$

$5ap^2 = 2ap^2 + 4a$

$3ap^2 - 4a = 0$

$a(3p^2 - 4) = 0$

$p^2 = \frac{4}{3}$

$\therefore p = \frac{2}{\sqrt{3}}$

Question 14

a) (i) $v^2 = (x-1)^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}(x-1)^2\right)$
$= x-1$

when $x=0$, $v^2 = 1^2$
$v = \pm 1$

$\therefore v^2 = (x-1)^2$ satisfies the given conditions

(ii) $\frac{dx}{dt} = -(x-1)$

NOTE! $v = -(x-1)$ in order satisfy inital conditions of $x=0, v=1$

$\int_0^t dt = -\int_0^x \frac{dx}{x-1}$

$t = \left[\ln(1-x)\right]_0^x$

$t = \ln(1-x)$

$1-x = e^t$

$x = 1-e^t$

b) (i) $y = -5t^2 + 12t\sin\alpha - 1$

$\alpha = 30° \Rightarrow y = -5t^2 + 6t - 1$

$y_{max} = \frac{-\Delta}{4a}$

$= \frac{-(6^2 - 4\times5\times1)}{-20}$

$= \frac{16}{20} = \frac{4}{5}$

$\therefore$ maximum height above G is $\frac{4}{5}$ m

(ii) to reach edge of the green $x=4, y=0$

$4 = 12t\cos\alpha$

$t = \frac{1}{3\cos\alpha}$

$0 = \frac{-5}{9\cos^2\alpha} + \frac{4\sin\alpha}{\cos\alpha} - 1$

$0 = -5\sec^2\alpha + 36\tan\alpha - 9$

$-5\tan^2\alpha - 5 + 36\tan\alpha - 9 = 0$

$5\tan^2\alpha - 36\tan\alpha + 14 = 0$

$\tan\alpha = \frac{36 \pm \sqrt{1016}}{10}$

$\alpha = 22.4°$ or $81.6°$

$\therefore$ ball must be hit at angle α $22° \leq \alpha \leq 82°$

c) $\ln(n!) > n$ for $n \geq 6$

Prove for $n=6$

LHS $= \ln(6!)$ RHS $= 6$

$= 6.579\ldots$ $\therefore$ LHS > RHS

Hence result is true for $n=6$

Assume true for $n=k$, $k \geq 6$

ie $\ln(k!) > k$

Prove true for $n=k+1$

ie $\ln[(k+1)!] > k+1$

Proof

$$\ln[(k+1)!] = \ln(k!) + \ln(k+1)$$

$$> k + \ln(k+1)$$

$$\geq k + \ln 7 \qquad (\because k \geq 6)$$

$$> k+1$$

Hence the result is true for $n=k+1$ if it is also true for $n=k$

Since the result is true for $n=6$, then the result is true for all integral values of $n \geq 6$ by induction.

(ii) $\ln(n!) > n$

$$n! > e^n$$

$$\frac{1}{n!} < \frac{1}{e^n}$$

(iii) $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

$$= \frac{103}{60} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

$$< \frac{103}{60} + \frac{1}{e^6} + \frac{1}{e^7} + \frac{1}{e^8} + \ldots$$

$$= \frac{103}{60} + \frac{\frac{1}{e^6}}{1 - \frac{1}{e}}$$

$$= \frac{103}{60} + \frac{1}{e^6 - e^5}$$

$$= \frac{103}{60} + \frac{1}{e^5(e-1)}$$