Student's name	Student's number	Teacher's name
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2014
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time - 5 minutes
- Working time - 3 hours
- Write using blue or black pen
  Black is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 100

Section I: Pages 3-6
10 marks
- Attempt questions 1-10, using the answer sheet on page 19.
- Allow about 15 minutes for this section

Section II: Pages 7-16
90 marks
- Attempt questions 11-16, using the booklets provided.
- Allow about 2 hours 45 minutes for this section

Multiple Choice	11	12	13	14	15	16	Total
							%

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1. If $z=\left(1-i\sqrt{3}\right)^{2014}$ what is $Arg\,z$?

(A) $-\dfrac{2014\pi}{3}$

(B) $-\dfrac{2014\pi}{6}$

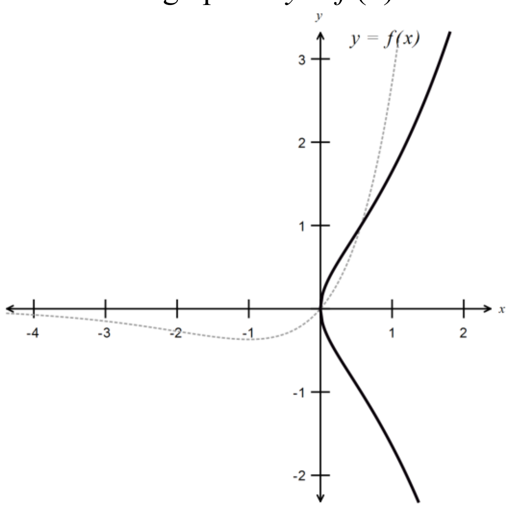
(C) $\dfrac{2014\pi}{3}$

(D) $\dfrac{2014\pi}{6}$

2. What does the equation $x^2+2y^2-24=0$ represent?

(A) Parabola
(B) Hyperbola
(C) Ellipse
(D) None of these

3. Which of the following transformations best describes the graph below? The graph of $y=f(x)$ is shown on the same diagram.



(A) $|y|=f(x)$

(B) $y^2=f(x)$

(C) $y=|f(x)|$

(D) $y=\left[f(x)\right]^2$

4. Which expression is equal to $\int \frac{dx}{\sqrt{4x^2+1}}$?

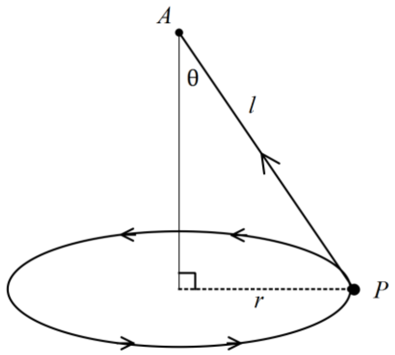
(A) $\sin^{-1} 2x + c$

(B) $\log_e\left(2x+\sqrt{4x^2+1}\right)+c$

(C) $\frac{1}{2}\log_e\left(x+\sqrt{x^2+\frac{1}{4}}\right)+c$

(D) $\frac{1}{4x}\sqrt{4x^2+1}+c$

5. A particle, P, of mass m kilograms, is suspended from a fixed point by a string of length, l metres with acceleration due to gravity, $g\ ms^{-2}$. P is moving with uniform circular motion about a horizontal circle with velocity $\omega\ rads/second$ and radius r. The forces acting on the particle are the gravitational force and the tension force T along the string. **1**



Which of the following expressions are the correct horizontal and vertical components of the force acting on P?

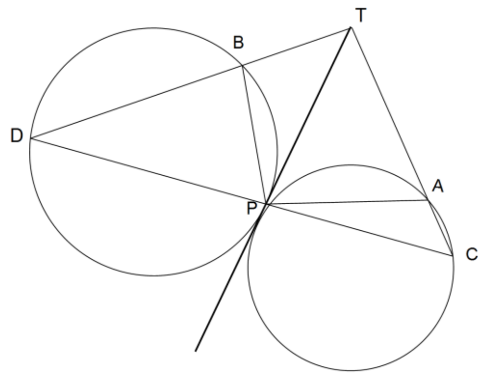
(A) $T\,sin\,\theta = mg$
$T\,cos\,\theta = mr\omega$

(B) $T\,cos\,\theta = mg$
$T\,sin\,\theta = mr\omega$

(C) $T\,sin\,\theta - mg = 0$
$T\,cos\,\theta = mr\omega^2$

(D) $T\,cos\,\theta - mg = 0$
$T\,sin\,\theta = mr\omega^2$

6. If TP is a common tangent to the circles in the diagram below, which line has an error in proving that *ATBP* is a cyclic quadrilateral?



(A) $\angle TPA = \angle TPB$ (common tangent bisects $\angle APB$)

(B) $\angle TPA = \angle PCA$ (angle between the tangent and the chord is equal to the angle in the alternate segment)

(C) $\angle TPB = \angle PDB$ (angle between the tangent and the chord is equal to the angle in the alternate segment)

(D) $\angle DTC = 180 - \angle TDC - \angle TCD$ (angle sum of a triangle)
$\therefore \angle APB + \angle DTC = 180$
Opposite angles in a cyclic quadrilateral are supplementary
$\therefore ATBP$ is a cyclic quadrilateral

7. A particle is moving in a circular path of radius *r*, with a constant angular speed of ω. The normal component of the acceleration is:

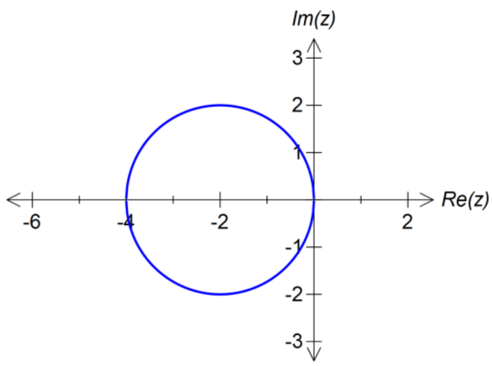
(A) ω

(B) $r\omega$

(C) $r\omega^2$

(D) $(r\omega)^2$

8.



Which one of the following is the equation of the circle in the diagram above?

(A) $(z+2)(\overline{z}+2)=4$

(B) $(z-2)(\overline{z}+2)=4$

(C) $(z-2)(\overline{z}-2)=4$

(D) $(z+2i)(\overline{z}-2i)=4$

9. The roots of $x^3+5x+3=0$ are α, β and γ. Which one of the following polynomials has roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$?

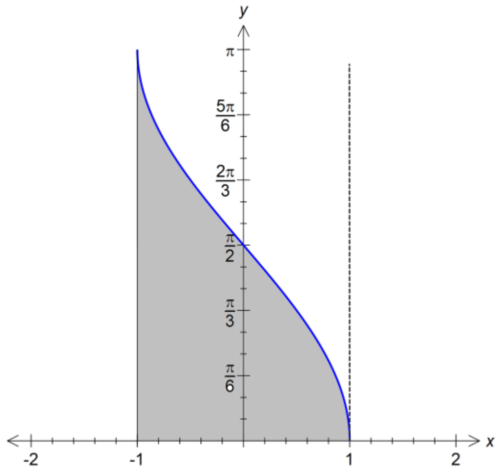
(A) $x^3-5x^2-9=0$

(B) $x^3+5x^2+9=0$

(C) $x^3-125x-375=0$

(D) $x^3+125x-+375=0$

10. In the diagram, the shaded region is bounded by the x-axis, the line $x=-1$ and the curve $y=\cos^{-1}x$.



Find the volume of the solid formed when this region is rotated about $x=1$.

(A) $\dfrac{3+\pi^2}{2}$

(B) $\dfrac{3}{2}$

(C) $\dfrac{5\pi^2}{2}$

(D) None of the above

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) (i) Show that $tan^3 x = sec^2 x\, tan\, x - tan\, x$. **1**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} tan^3 x\, dx$ **2**

b) If $\omega = \frac{1 - i\sqrt{3}}{2}$

(i) Show that $\omega^3 = -1$. **2**

(ii) Hence calculate ω^{16} **1**

c) (i) Find $\sqrt{5 - 12i}$ in $x + iy$ form. **2**

(ii) Hence, or otherwise, solve the equation $z^2 + 4z - 1 + 12i = 0$ **2**

d) Consider the equation $z^3 - z^2 - 2z - 12 = 0$. Given that $z = 2cis\left(\frac{2\pi}{3}\right)$ is a root of the equation, factorise fully over the

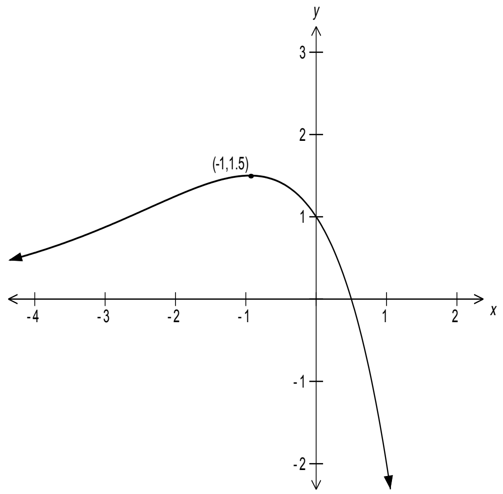
(i) real field **2**

(ii) complex field **1**

Question 11 continued over page

Question 11 continued

e) The following diagram shows the graph of $y = f(x)$.



On your answer sheet, draw separate one-third page sketches of the graphs of the following

(i) $y = -f(x)$ **1**

(ii) $y = \sqrt{f(x)}$ **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) The equation of the ellipse, E, is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

The point P is on the ellipse with co-ordinates $\left(x_1, y_1\right)$.

(i) Find the eccentricity of the ellipse. **1**

(ii) Find the co-ordinates of the foci and the equations of the directrices of the ellipse. **2**

(iii) Show that the equation of the tangent at P is $\frac{x_1 x}{25}+\frac{y_1 y}{9}=1$. **2**

(iv) Let the tangent at P meet a directrix at a point J. Show that $\angle PSJ$ is a right angle where S is the corresponding focus. **3**

b) Consider $f(x)=sin\,x+cos\,x$

(i) Find A and B such that $sin\,x+cos\,x=A\,sin(x+B)$ **2**

(ii) Sketch $f(x)=sin\,x+cos\,x$ for $-2\pi\le x\le 2\pi$. **2**

(iii) Hence, or otherwise, sketch $y=\frac{1}{f(x)}$ for $-2\pi\le x\le 2\pi$. **1**

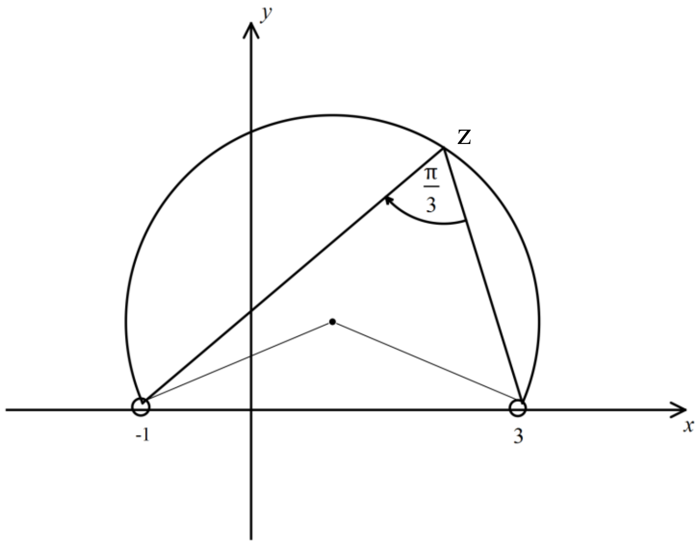
(iv) Sketch $y=\frac{f(x)}{x}$ **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) The diagram shows the locus of a point z in the complex plane such that **3**

$arg(z-3) - arg(z+1) = \frac{\pi}{3}$.



This locus is part of a circle. The angle between the lines from -1 to z and from 3 to z is $\frac{\pi}{3}$, as shown.

Find the centre and radius of the circle.

b) Find $\int \frac{x^2+2x}{(x-2)(x^2+4)}dx$ **3**

c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{dx}{cos^2 x + 2\,sin^2 x}$ using $t = tan\,x$ **4**

d) (i) Show that $\frac{1+sin\,\theta+i\,cos\,\theta}{1+sin\,\theta-i\,cos\,\theta} = sin\,\theta + i\,cos\,\theta$ **3**

(ii) Hence prove that **2**

$\left(\frac{1+sin\,\theta+i\,cos\,\theta}{1+sin\,\theta-i\,cos\,\theta}\right)^n = cos\left(\frac{n\pi}{2}-n\theta\right)+i\,sin\left(\frac{n\pi}{2}-n\theta\right)$, where n is a positive integer.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Find all the roots of the equation $16x^3 - 4x^2 - 8x + p = 0$ if two of the roots are equal. **3**

b) Prove by Mathematical Induction that **3**
$2^n > n^2$ for all integers $n \geq 5$

c) A particle of mass, *m* kilograms, is initially projected from the ground at an angle of θ, to the horizontal where $\theta = tan^{-1}\left(\frac{3}{4}\right)$ and an initial velocity of $25 m / s$.

The equations of motion for the particle are

$x = 25t\, cos\,\theta$
$y = -\frac{1}{2}gt^2 + 25t\, sin\,\theta$, where *g* is the acceleration due to gravity.
(DO NOT PROVE THESE RESULTS)

(i) Show that $x = 20t$ and $y = 15t - 5t^2$ if $g = 10 m / s^2$. **2**

(ii) If the particle hits a wall 40 metres from the point of projection

(α) find the height above the ground the particle hits. **1**

(β) show that the velocity of the particle, at the point of impact, is $\sqrt{425}\, m / s$. **2**

(iii) At impact, the particle is instantaneously at rest. It then falls vertically to the ground with a resistance force acting against the vertical motion equal to $0.01mv^2$ Newtons.

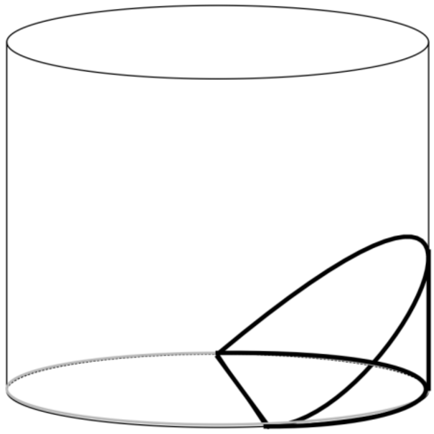
(α) Show that $a = 10 - 0.01v^2$, where *a* is the acceleration and *v* is the velocity of the particle. **1**

(β) Find the velocity on returning to the ground. Answer correct to 2 decimal places. **3**

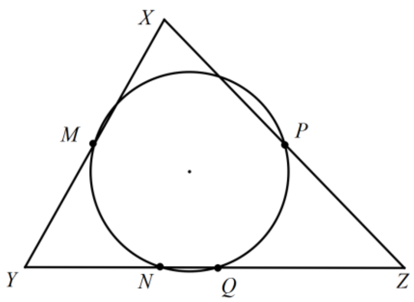
End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) A wedge is cut out of a right circular cylinder of radius 4 centimetres by two planes. One plane is perpendicular to the axis of the cylinder. The other plane intersects the first at an angle of 30^0, along a diameter of the cylinder. Find the volume of the wedge. **3**



b) In the acute-angled triangle *XYZ*, *M* is the midpoint of *XY*, *Q* is the midpoint of *YZ* and *P* is the midpoint of *ZX*. The circle through *M*, *Q* and *P* also cuts *YZ* at *N* as shown in the diagram.



(i) Prove *MPQY* is a parallelogram. **1**

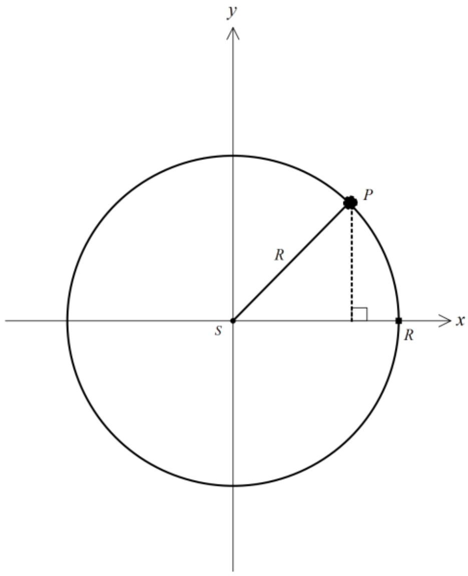
(ii) Prove $\angle MNY = \angle MPQ$. **1**

(iii) Prove that $XN \perp YZ$. **2**

Question 15 continued over page

Question 15 continued

c) A planet P of mass, m kilograms, moves in a circular orbit of radius R metres, around a star, S, in uniform circular motion. The position of the planet at time t seconds is given by the equations $x = R\cos\frac{2\pi t}{T}$ and $y = R\sin\frac{2\pi t}{T}$, where T is a constant.



(i) Show that $\ddot{x} = \frac{-4\pi^2}{T^2}x$ and $\ddot{y} = \frac{-4\pi^2}{T^2}y$ **2**

(ii) Show the acceleration of P is $\frac{-4\pi^2}{T^2}R$. **1**

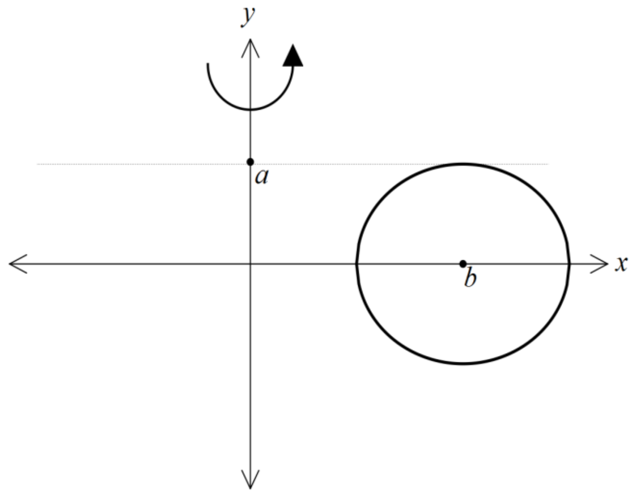
(iii) Find the force exerted by the star, S, on the planet, P. **1**

(iv) It is known that the magnitude of the gravitational force pulling the planet towards the star is given by $F = \frac{GMm}{R^2}$, where G is constant and M is the mass of the star, S, in kilograms. Show that the expression for T in terms of R, M and G is $T = 2\pi R\sqrt{\frac{R}{GM}}$. **2**

Question 15 continued over page

Question 15 continued

d) A donut shaped solid called a torus is formed by revolving $(x-b)^2+y^2=a^2$, $0<a<b$ about the y-axis. **2**



Express the volume of the torus as a definite integral in x. Do not evaluate this integral.

End of Question 15

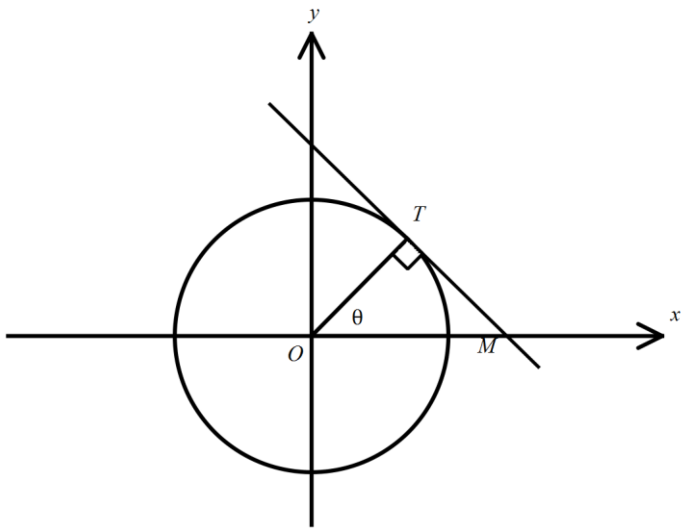
Question 16 (15 marks) Use a SEPARATE writing booklet.

a) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where $n \geq 2$,

(i) Show that $I_n = \frac{(n-1)}{n} I_{n-2}$ **3**

(ii) Hence or otherwise, evaluate $\int_0^2 (4 - x^2)^{\frac{5}{2}} \, dx$. **3**

b) The figure shows the circle $x^2 + y^2 = a^2$.



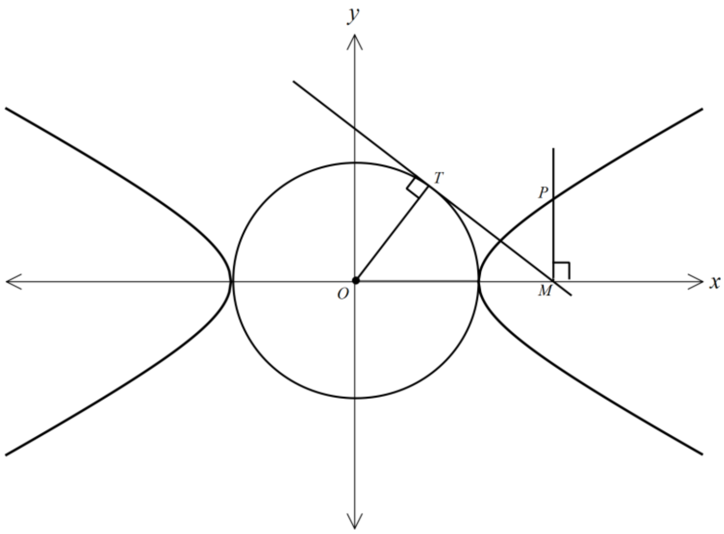
The point T lies on the circle. $\angle TOx = \theta$, where $0 \leq \theta \leq \frac{\pi}{2}$. The tangent to the circle at T meets the x-axis at M.

(i) Show that the co-ordinates of M are $(a \sec \theta, 0)$. **1**

Question 16 continued over page

Question 16 continued

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ where a, b>0 are shown on the diagram below:



MP is perpendicular to Ox and P is a point on the hyperbola in the first quadrant.

(ii) Show that the co-ordinates of P are $(a\sec\theta, b\tan\theta)$. **2**

(iii) If Q is another point on the hyperbola with co-ordinates $(a\sec\phi, b\tan\phi)$ where $\theta + \phi = \frac{\pi}{2}$ and $\theta \neq \frac{\pi}{4}$, show that the equation of the chord PQ is $y = \frac{b}{a}(\cos\theta + \sin\theta)x - b$. **3**

(iv) Show that every such chord passes through a fixed point and determine its co-ordinates. **1**

(v) State the equation of the asymptotes for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **1**

(vi) Show that as $\theta \to \frac{\pi}{2}$, the chord PQ approaches a line parallel to an asymptote. **1**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

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Extension 2 Mathematics
Multiple Choice Answer Sheet

Student Number__________

Completely fill the response oval representing the most correct answer.

1.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
2.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
3.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
4.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
5.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
6.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
7.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
8.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
9.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭
10.	**A** ⬭	**B** ⬭	**C** ⬭	**D** ⬭

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PLC Sydney Maths Department

Solutions for exams and assessment tasks

Ver 1

Academic Year	Yr 12	Calendar Year	2014
Course	Ext. 2	Name of task/exam	Trials

1. $z = (1 - i\sqrt{3})^{2014}$

$\text{Arg}(1 - i\sqrt{3}) = \tan^{-1} \frac{-\sqrt{3}}{1}$

$= -\frac{\pi}{3}$

$\therefore z = (1 - i\sqrt{3})^{2014}$

$\text{Arg}(1 - i\sqrt{3})^{2014} = 2014\,\text{Arg}(1 - i\sqrt{3})$

$= 2014\left(-\frac{\pi}{3}\right)$

$= -\frac{2014\pi}{3}$

$\therefore$ A

2. $x^2 + 2y^2 - 24 = 0$

$x^2 + 2y^2 = 24$

$\frac{x^2}{24} + \frac{y^2}{12} = 1$

$\therefore$ ellipse

$\therefore$ C

3. B

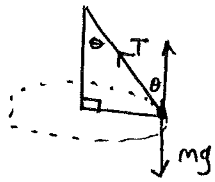
4. $\int \frac{dx}{\sqrt{4x^2 + 1}}$

$= \int \frac{dx}{2\sqrt{x^2 + (\frac{1}{2})^2}}$

$= \frac{1}{2} \ln\left(x + \sqrt{x^2 + \frac{1}{4}}\right) + c$

$\therefore$ C

5.



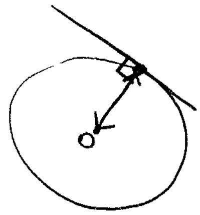
Horizontally: $T\sin\theta = mr\omega^2$

Vertically: $T\cos\theta = mg$

$\therefore$ D

6. A

7.



normal component is the F acting towards the centre of the circle

$\therefore a = r\omega^2$

$\therefore$ C

8. Circle in diagram is

$|z + 2| = 2$

$\therefore \sqrt{(x+2)^2 + y^2} = 2$

$(x+2)^2 + y^2 = 4.$

Consider $(z+2)(\bar{z}+2) = 4$

$4 = (x + iy + 2)(x - iy + 2)$

$4 = x^2 + y^2 + 2(x + iy) + 2(x - iy) + 4$

$0 = x^2 + y^2 + 4x$

$0 = (x+2)^2 + y^2 - 4$

$\therefore$ A

Academic Year		Calendar Year	
Course		Name of task/exam	

9. $x^3 + 5x + 3 = 0$

α, β, γ

$\alpha + \beta + \gamma = -\frac{b}{a} = 0$ ①

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 5$ ②

$\alpha\beta\gamma = -\frac{d}{a} = -3.$ ③

If roots $\alpha\beta, \beta\gamma, \alpha\gamma$

sum of roots 1 at a time:

$\alpha\beta + \beta\gamma + \alpha\gamma = 5$ from ②.

sum of roots 2 at a time:

$\alpha\beta\beta\gamma + \alpha\beta\alpha\gamma + \beta\gamma\alpha\gamma$

$= \alpha\beta\gamma(\beta + \alpha + \gamma)$

$= -3(0)$

$= 0$

product of roots

$\alpha\beta\beta\gamma\alpha\gamma = \alpha^2\beta^2\gamma^2$

$= (\alpha\beta\gamma)^2$

$= (-3)^2$

$= 9$

$\therefore$ polynomial is

$x^3 - (\alpha\beta + \beta\gamma + \alpha\gamma)x^2 + (\alpha\beta\gamma(\alpha+\beta+\gamma))x - \alpha^2\beta^2\gamma^2 = 0$

$x^3 - 5x^2 + 0x - 9 = 0$

$x^3 - 5x^2 - 9 = 0$

$\therefore$ A

10. $A(y) = \pi(R^2 - r^2)$

$= \pi(R - r)(R + r)$

$R = 2$

$r = 1 - x$

$\therefore A(y) = \pi(2 - 1 + x)(2 + 1 - x)$

$= \pi(1 + x)(3 - x)$

$V = \pi\int_0^\pi (1 + \cos y)(3 - \cos y)\,dy$

Note $y = \cos^{-1}x$

$\cos y = x.$

$\therefore V = \pi\int_0^\pi (3 + 2\cos y - \cos^2 y)\,dy$

$= \pi\left[[3y + 2\sin y]_0^\pi - \int_0^\pi \cos^2 y\,dy\right]$

$= \pi\left[(3\pi + 0) - 0 - \int_0^\pi (\frac{1}{2}\cos 2y + \frac{1}{2})\,dy\right]$

$= \pi\left[3\pi - [\frac{1}{2}y + \frac{1}{4}\sin 2y]_0^\pi\right]$

$= \pi\left[3\pi - (\frac{\pi}{2} - 0 - 0)\right]$

$= 3\pi^2 - \frac{\pi^2}{2}$

$= \frac{5\pi^2}{2}$

$\therefore$ C

Academic Year		Calendar Year	
Course		Name of task/exam	

Q11

a i RTS $\tan^3 x = \sec^2 x \tan x - \tan x$

$RHS = \tan x(\sec^2 x - 1)$

$= \tan x(\tan^2 x)$

$= \tan^3 x$

$= LHS$

ii $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x \tan x - \tan x) \, dx$

$= \left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= \left(\frac{1}{2} - 0\right) + \left[\ln(\cos x)\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2} + \left(\ln \frac{1}{\sqrt{2}} - \ln 1\right)$

$= \frac{1}{2} + \ln \frac{1}{\sqrt{2}}$

$= \frac{1}{2} + \ln 2^{-\frac{1}{2}}$

$= \frac{1}{2} - \frac{1}{2} \ln 2.$

b $\omega = \frac{1 - \sqrt{3}i}{2}$

i $\omega^3 = \left(\frac{1 - \sqrt{3}i}{2}\right)^3$

$= \left(\frac{1 - 2\sqrt{3}i - 3}{4}\right)\left(\frac{1 - \sqrt{3}i}{2}\right)$

$= \left(\frac{-2 - 2\sqrt{3}i}{4}\right)\left(\frac{1 - \sqrt{3}i}{2}\right)$

$\omega^3 = \frac{-2 + 2\sqrt{3}i - 2\sqrt{3}i - 6}{8}$

$= -\frac{8}{8}$

$= -1.$

$\therefore \omega^3 = -1$

ii $\omega^{16} = (\omega^3)^5 \times \omega$

$= (-1)^5 \omega$

$= -\omega$

OR $\frac{-1 + \sqrt{3}i}{2}$

c i $\sqrt{5 - 12i} = a + ib$

$5 - 12i = (a + ib)^2$

$5 - 12i = a^2 - b^2 + 2abi$

equating:

$5 = a^2 - b^2$

$-12 = 2ab$

$b = -\frac{6}{a}$

$\therefore 5 = a^2 - \left(-\frac{6}{a}\right)^2$

$5 = a^2 - \frac{36}{a^2}$

$5a^2 = a^4 - 36$

$(a^2 + 4)(a^2 - 9) = 0$

$a = \pm 3$, a real

$\therefore b = \mp 2$

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

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ii $z^2 + 4z - 1 + 12i = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-4 \pm \sqrt{16 - 4(-1+12i)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 + 4 - 48i}}{2}$$

$$= \frac{-4 \pm \sqrt{20 - 48i}}{2}$$

$$= \frac{-4 \pm 2\sqrt{5 - 12i}}{2}$$

$$= -2 \pm \sqrt{5 - 12i}$$

$$= -2 \pm (3 - 2i)$$

$$= -2 + 3 - 2i, \ -2 - 3 + 2i$$

$$= 1 - 2i, \ -5 + 2i$$

d $z^3 - z^2 - 2z - 12 = 0$

Since coefficients are real roots occur in conjugate pairs

$$\therefore 2\,\text{cis}\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -1 + \sqrt{3}i$$

$\therefore -1 - \sqrt{3}i$ is also a root.

$$\therefore \left(z - (-1 + \sqrt{3}i)\right)\left(z - (-1 - \sqrt{3}i)\right)$$

$$= z^2 - z(-1 - \sqrt{3}i) - z(-1 + \sqrt{3}i) + (-1 + \sqrt{3}i)(-1 - \sqrt{3}i)$$

$$= z^2 + z + \sqrt{3}iz + z - \sqrt{3}iz + 1 + 3$$

$$= z^2 + 2z + 4$$

$$\begin{array}{r} z + 3 \\ z^2 + 2z + 4 \enclose{longdiv}{z^3 - z^2 - 2z - 12} \\ \underline{z^3 + 2z^2 + 4z} \\ -3z^2 - 6z - 12 \\ \underline{-3z^2 - 6z - 12} \\ \end{array}$$

$\therefore$ i real field

$(z^2 + 2z + 4)(z - 3)$

ii complex field

$\left(z - (-1 + \sqrt{3}i)\right)\left(z - (-1 - \sqrt{3}i)\right)(z - 3)$

e see separate sheet.

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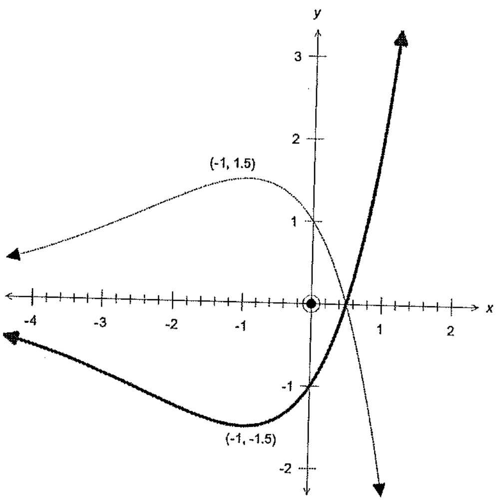
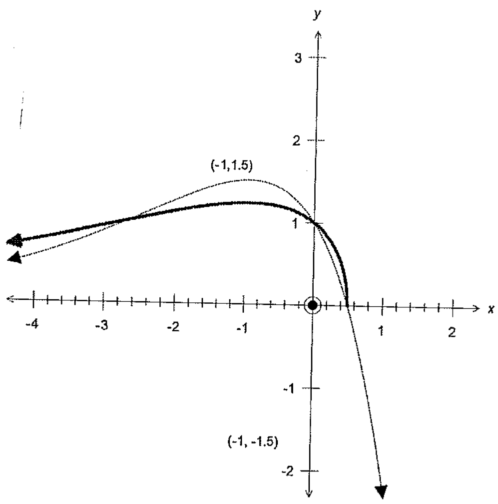
11e(i)



11e ii)



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Q12

a $\frac{x^2}{25} + \frac{y^2}{9} = 1$

i $e^2 = 1 - \left(\frac{b}{a}\right)^2$

$= 1 - \left(\frac{3}{5}\right)^2$

$e = \sqrt{\frac{16}{25}}$ $e > 0$

$e = \frac{4}{5}$

ii foci $(\pm ae, 0)$

$= (\pm 4, 0)$

directrices

$x = \pm \frac{a}{e}$

$x = \pm \frac{25}{4}$

iii $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{2y}{9}\frac{dy}{dx} = -\frac{2x}{25}$

$\frac{dy}{dx} = -\frac{x}{25} \times \frac{9}{y}$

at (x_1, y_1)

$m = \frac{-9x_1}{25y_1}$

$\therefore$ eqn tangent:

$y - y_1 = \frac{-9x_1}{25y_1}(x - x_1)$

$25yy_1 - 25y_1^2 = -9xx_1 + 9x_1^2$

$9xx_1 + 25yy_1 = 9x_1^2 + 25y_1^2$

$\div 225$

$\frac{xx_1}{25} + \frac{yy_1}{9} = \frac{x_1^2}{25} + \frac{y_1^2}{9}$

(x_1, y_1) lies on the ellipse

$\therefore \frac{x_1^2}{25} + \frac{y_1^2}{9} = 1$

$\therefore \frac{xx_1}{25} + \frac{yy_1}{9} = 1.$

iv If tangent meets directrix then $x = \frac{25}{4}$, $y = ?$

$\frac{\frac{25}{4}x_1}{25} + \frac{yy_1}{9} = 1$

$\frac{x_1}{4} + \frac{yy_1}{9} = 1$

$9x_1 + 4yy_1 = 36$

$4yy_1 = 36 - 9x_1$

$y = \frac{36 - 9x_1}{4y_1}$

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$\therefore J\left(\frac{25}{4}, \frac{36-9x_1}{4y_1}\right)$ $S(4,0)$

$m_{SJ} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{\frac{36-9x_1}{4y_1} - 0}{\frac{25}{4} - 4}$

$= \frac{36-9x_1}{4y_1} \div \frac{9}{4}$

$= \frac{4(36-9x_1)}{4\times 9 y_1} = \frac{36(4-x_1)}{36 y_1}$

$= \frac{4-x_1}{y_1}$

$m_{PS} = \frac{y_1 - 0}{x_1 - 4}$

$= \frac{y_1}{x_1 - 4}$

$m_{SJ} \times m_{PS} = \frac{y_1}{(x_1 - 4)} \times \frac{(4 - x_1)}{y_1}$

$= -1$

Since $m_{SJ} \times m_{PS} = -1$

$SJ \perp PS$

$\therefore \angle PSJ$ is a right angle.

b $f(x) = \sin x + \cos x$

i $\sin x + \cos x = A \sin(x + B)$

$\sin x + \cos x = A \sin x \cos B + A \cos x \sin B$

equating:

$1 = A \cos B$ ①

$1 = A \sin B$ ②

$1 = \tan B$

$B = \frac{\pi}{4}$

$1 = A \cos \frac{\pi}{4}$

$A = \frac{1}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2}$

$\therefore \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$

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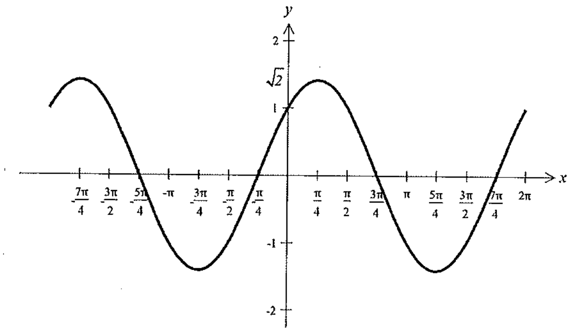
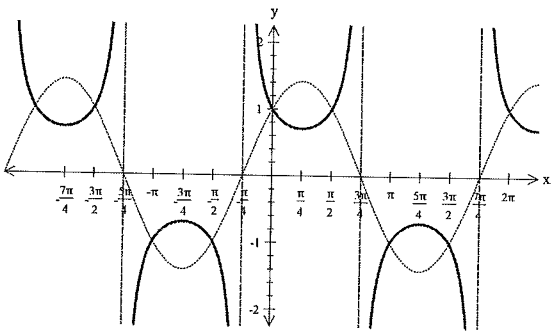
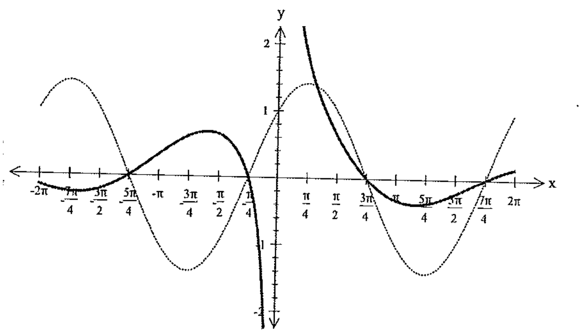
12b ii)



12b iii)



12b iv)



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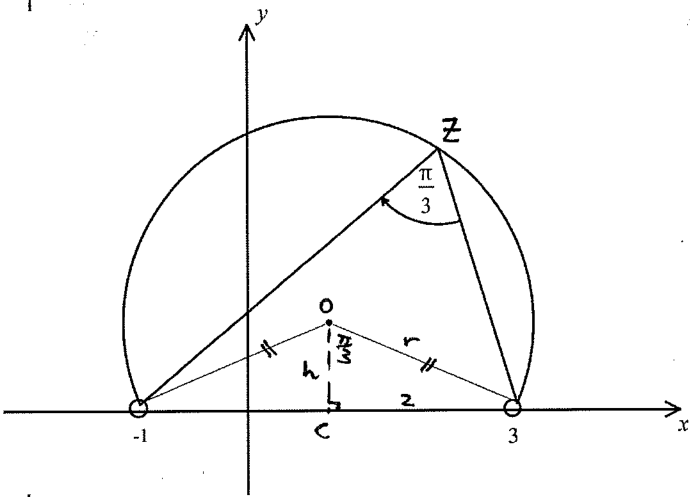
Q13

a)



Let O be centre of circle

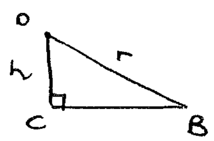
A(-1, 0) B(3, 0)

$\angle AOB = \frac{2\pi}{3}$ (angle centre twice angle at circumference)

ΔAOB is isosceles

Drop perpendicular from O to real axis. Call this point C.

$\therefore OC \perp AB$

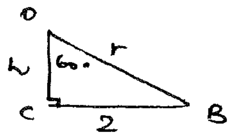


C is midpt AB

$\therefore C(1, 0)$

$\therefore CB = 2$ units.

$\angle COB = \frac{\pi}{3}$ (isosceles Δ)



$\tan 60 = \frac{2}{h}$

$h = \frac{2}{\tan 60} = \frac{2}{\sqrt{3}}$.

$\sin 60 = \frac{2}{r}$

$r = \frac{2}{\sin 60} = \frac{2}{\sqrt{3}/2}$

$= \frac{4}{\sqrt{3}}$.

$\therefore$ Centre $\left(1, \frac{2}{\sqrt{3}}\right)$ $R = \frac{4}{\sqrt{3}}$ units

b) $\int \frac{x^2 + 2x}{(x-2)(x^2+4)}\, dx$

$$\frac{x^2+2x}{(x-2)(x^2+4)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+4)}$$

$x^2 + 2x = A(x^2+4) + (Bx+C)(x-2)$

$x^2 + 2x = Ax^2 + 4A + Bx^2 - 2Bx + Cx - 2C$

equating

$1 = A + B$ ①

$2 = -2B + C$ ②

$0 = 4A - 2C$ ③

$\therefore 4A = 2C$

$2A = C$.

$\therefore 2 = -2B + 2A$

$1 = -B + A$ ②

$1 = B + A$ ①

~~$2 = 2A$ ①+②~~

$A = 1,\ C = 2,\ B = 0$

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$$\therefore \int \frac{x^2+2x\,dx}{(x-2)(x^2+4)} = \int \left\{ \frac{1}{x-2} + \frac{2}{x^2+4} \right\} dx$$

$$= \ln|x-2| + 2\int \frac{1}{4+x^2}\,dx$$

$$= \ln|x-2| + \frac{2}{2}\tan^{-1}\frac{x}{2} + c$$

$$\therefore \text{Integral} = \ln|x-2| + \tan^{-1}\frac{x}{2} + c.$$

c/ $$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x + 2\sin^2 x} \qquad u = \tan x$$

$$= \int_0^{\frac{\pi}{4}} \frac{\frac{dx}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} + \frac{2\sin^2 x}{\cos^2 x}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x\,dx}{1 + 2\tan^2 x}$$

Given $u = \tan x$

$du = \sec^2 x\,dx$

also when $x = 0$ $u = 0$

when $x = \frac{\pi}{4}$ $u = 1$.

$$\therefore \int_0^1 \frac{du}{1+2u^2}$$

$$= \int_0^1 \frac{du}{2\left(\frac{1}{2}+u^2\right)}$$

$$= \frac{1}{2}\int_0^1 \frac{du}{\frac{1}{2}+u^2}$$

$$= \frac{1}{2}\sqrt{2}\left[\tan^{-1}\frac{u}{\sqrt{\frac{1}{2}}}\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{1}{\sqrt{\frac{1}{2}}} - \tan^{-1} 0\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}$$

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d i RTS

$$\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$$

$$LHS=\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\times\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta+i\cos\theta}$$

$$=\frac{(1+\sin\theta)^2+i\cos\theta(1+\sin\theta)+i\cos\theta(1+\sin\theta)}{(1+\sin\theta)^2+\cos^2\theta}\frac{-\cos^2\theta}{(1+\sin\theta)^2+\cos^2\theta}$$

$$=\frac{1+2\sin\theta+\sin^2\theta+i\cos\theta+i\cos\theta\sin\theta}{(1+\sin\theta)^2+\cos^2\theta}\frac{+i\cos\theta+i\cos\theta\sin\theta-\cos^2\theta}{(1+\sin\theta)^2+\cos^2\theta}$$

$$=\frac{2\sin^2\theta+2\sin\theta+2i\cos\theta+2i\cos\theta\sin\theta}{1+2\sin\theta+\sin^2\theta+\cos^2\theta}$$

$$=\frac{2[\sin^2\theta+\sin\theta+i\cos\theta+i\cos\theta\sin\theta]}{2[1+\sin\theta]}$$

$$=\frac{\sin\theta(\sin\theta+1)+i\cos\theta(1+\sin\theta)}{(1+\sin\theta)}$$

$$=\frac{(1+\sin\theta)(\sin\theta+i\cos\theta)}{(1+\sin\theta)}$$

$$=\sin\theta+i\cos\theta$$

$$=RHS$$

∴ shown

ii RTP

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

$$LHS=\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$$

$$=(\sin\theta+i\cos\theta)^n$$

$$=\left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)^n$$

$$=\cos\left[n\left(\frac{\pi}{2}-\theta\right)\right]+i\sin\left[n\left(\frac{\pi}{2}-\theta\right)\right]$$

by De Moivre's Thm

$$=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

$$=RHS$$

∴ proved

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Q14

a) $P(x) = 16x^3 - 4x^2 - 8x + p = 0$

Let roots be α, α, β

sum roots 1 at time

$2\alpha + \beta = \frac{4}{16}$

$\beta = \frac{1}{4} - 2\alpha$.

Also $P(\alpha) = 0$

$P'(\alpha) = 0$ since double root.

$\therefore P'(x) = 48x^2 - 8x - 8$

$P'(\alpha) = 0$

$48\alpha^2 - 8\alpha - 8 = 0$

$6\alpha^2 - \alpha - 1 = 0$

$(3\alpha + 1)(2\alpha - 1) = 0$

$\alpha = -\frac{1}{3}$, $\therefore \beta = \frac{11}{12}$; $\alpha = \frac{1}{2}$, $\beta = -\frac{3}{4}$

$\therefore$ roots $-\frac{1}{3}, -\frac{1}{3}, \frac{11}{12}$

and $\frac{1}{2}, \frac{1}{2}, -\frac{3}{4}$

b) Prove by M.I. for $n \geq 5$

$2^n > n^2$

Step 1: Prove true for $n = 5$

LHS $= 2^5 = 32$ RHS $= 5^2 = 25$

LHS > RHS

$\therefore$ true for $n = 5$

Step 2: Assume true for $n = k$

$\therefore 2^k \geq k^2$

Step 3: Prove true for $n = k+1$

i.e. prove

$2^{k+1} > (k+1)^2$

$2^{k+1} = 2^k . 2$

$> k^2 . 2$ by assumption

$> 2k^2$

now $2k^2$ at $k = 5$ gives 50.

$(k+1)^2$ at $k = 5$ gives 36.

clearly $2k^2 > (k+1)^2$ for all values of $k \geq 5$

$\therefore 2^{k+1} > 2k^2 > (k+1)^2$

$\therefore 2^{k+1} > (k+1)^2$

$\therefore$ proved by M.I. for all values of $n \geq 5$.

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14 b another method:

Step 3: Prove true for $n = k+1$

i.e. prove $2^{k+1} > (k+1)^2$

If we can prove $2^{k+1} - (k+1)^2 > 0$ then we have proved $2^{k+1} > (k+1)^2$.

$LHS = 2^{k+1} - (k+1)^2$

$= 2^k \times 2 - (k^2 + 2k + 1)$

$> 2k^2 - (k^2 + 2k + 1)$ by assumption

$= 2k^2 - k^2 - 2k - 1$

$= k^2 - 2k - 1$

$= (k-1)^2 - 2$

> 0 if $k > 3$.

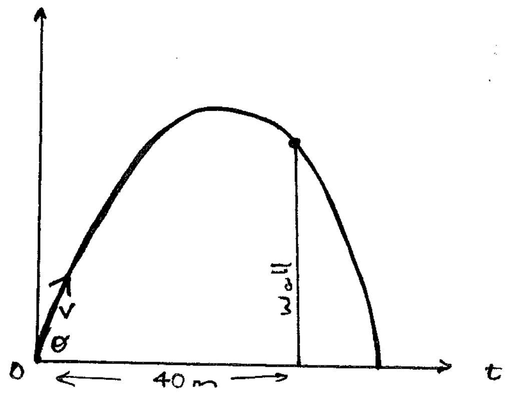
Since we have $k \geq 5$, this must also be true

$\therefore 2^{k+1} - (k+1)^2 > 0$

$\therefore 2^{k+1} > (k+1)^2$

$\therefore$ proved by M.I. for all values of $n \geq 5$

c

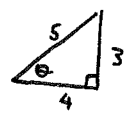


$g = 10$

$x = 25t \cos\theta$

$y = -\frac{1}{2} g t^2 + 25t \sin\theta$

i we know $\tan\theta = \frac{3}{4}$



$V = 25$ m/s

$x = 25t\left(\frac{4}{5}\right)$

$x = 20t$

$y = -\frac{1}{2}(10)t^2 + 25t\left(\frac{3}{5}\right)$

$y = -5t^2 + 15t$

ii α when $x = 40$

$40 = 20t$

$t = 2$.

$\therefore y = -5(2)^2 + 15(2)$

$= -20 + 30$

$= 10$

$\therefore$ 10 metres high.

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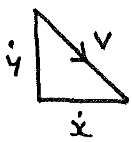
β)



if $x = 20t$

$\dot{x} = 20$

if $y = 15t - 5t^2$

$\dot{y} = 15 - 10t$

at $t = 2$

$\dot{y} = 15 - 20$

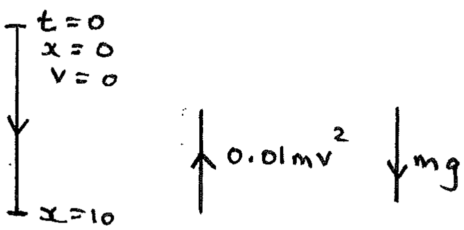
$\dot{y} = -5$

$\therefore V = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{20^2 + (-5)^2}$

$V = \sqrt{425}$ m/s

iii)



α) $F = ma$

$ma = mg - 0.01mv^2$

$a = g - 0.01v^2$

$a = 10 - 0.01v^2$

β) $v\dfrac{dv}{dx} = 10 - 0.01v^2$

$\dfrac{dv}{dx} = \dfrac{10 - 0.01v^2}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{10 - 0.01v^2}$

$\displaystyle\int_0^{10} dx = \int_0^{V} \frac{v}{10 - 0.01v^2}\,dv$

$\displaystyle\Big[x\Big]_0^{10} = \frac{1}{-0.02}\int_0^{V} \frac{-0.02v}{10 - 0.01v^2}\,dv$

$10 = -\dfrac{1}{0.02}\Big[\ln\left|10 - 0.01v^2\right|\Big]_0^{V}$

$-0.2 = \ln\left|10 - 0.01V^2\right| - \ln\left|10 - 0\right|$

$-0.2 = \ln\left|\dfrac{10 - 0.01V^2}{10}\right|$

$e^{-0.2} = \dfrac{10 - 0.01V^2}{10}$

$10e^{-0.2} = 10 - 0.01V^2$

$0.01V^2 = 10 - 10e^{-0.2}$

$V = 13.46$ m/s

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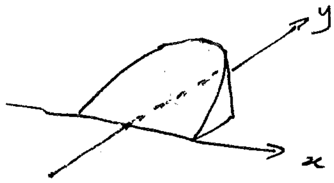
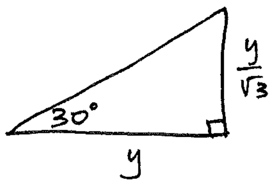
15a)

Method 1



Cross-section



$$A = \frac{1}{2} y \cdot \frac{y}{\sqrt{3}}$$

$$= \frac{y^2}{2\sqrt{3}}$$

$$V = \int_{-4}^{4} \frac{y^2}{2\sqrt{3}}\, dx$$

(Note thickness is on x-axis)

$$= \frac{1}{2\sqrt{3}} \int_{-4}^{4} y^2\, dx$$

$x^2 + y^2 = 16$
$y^2 = 16 - x^2$

$$= \frac{1}{2\sqrt{3}} \int_{-4}^{4} (16 - x^2)\, dx$$

$$= \frac{2}{2\sqrt{3}} \int_{0}^{4} (16 - x^2)\, dx$$

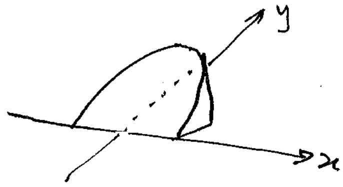
$$= \frac{1}{\sqrt{3}} \left[16x - \frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{\sqrt{3}} \left[64 - \frac{64}{3}\right]_0$$

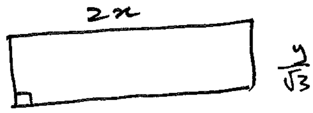
$$= \frac{128}{3\sqrt{3}}$$

$$= \frac{128\sqrt{3}}{9}\ u^3$$

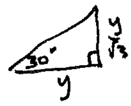
Method 2



Cross-section



Aerial view



$$A = 2x \cdot \frac{y}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} xy$$

$$V = \int_{0}^{4} \frac{2}{\sqrt{3}} xy\, dy$$

(Note thickness on y-axis)

$$= \frac{2}{\sqrt{3}} \int_{0}^{4} xy\, dy$$

$x^2 + y^2 = 16$
$x = (16 - y^2)^{1/2}$

$$= \frac{2}{\sqrt{3}} \int_{0}^{4} (16 - y^2)^{1/2} \cdot y\, dy$$

Note: $\frac{d}{dx}(16 - y^2)^{3/2} = \frac{3}{2}(16 - y^2)^{1/2} \cdot -2y$
$= -3(16 - y^2)^{1/2}$

$$= \frac{-2}{3\sqrt{3}} \left[(16 - y^2)^{3/2}\right]_0^4$$

$$= \frac{-2}{3\sqrt{3}} \left[0 - (16)^{3/2}\right]$$

$$= \frac{-2}{3\sqrt{3}} (-64)$$

$$= \frac{128}{3\sqrt{3}}$$

$$= \frac{128\sqrt{3}}{9}\ u^2$$

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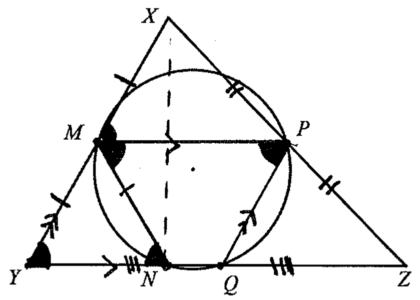
b



i Since M is the midpoint of XY and P is the midpoint of XZ, then MP ∥ YQ since the ratio of intercepts are equal.

Similarly, M is the midpoint of XY and Q is the midpoint of YZ, then XY ∥ PQ since the ratio of intercepts are equal.

∴ MPQY is a parallelogram.

ii Since MPQN is a cyclic quadrilateral, $\angle MNY = \angle MPQ$ (exterior angle in a cylic quadralateral equals the interior opposite angle.)

$\therefore \angle MNY = \angle MPQ$

iii since MPQY is a parallelogram

$\angle YNM = \angle NMP$ (alternate angles equal MP ∥ QY).

Let $\angle MNY = x$

$\therefore \angle NMP = x$

Since $\angle MNY = \angle MPQ$ (proved in ii)

then $\angle MPQ = x$.

Also $\angle MYN = x$ (opposite angles in parallelogram are equal).

$\angle XMN = 2x$

(exterior angle of triangle equals sum of 2 interior opposite angles).

and since $\angle PMN = x$

then $\angle XMP = x$ also.

$\therefore \Delta XMN$ is isosceles (MY = MX, MY = MN and MN = MX)

$\therefore \angle MNX = \frac{180 - 2x}{2}$ (angle sum of Δ XMN)

$= 90 - x$

$\therefore \angle YNX = \angle YNM + \angle MNX$ (adjacent angles)

$= x + 90 - x$

$= 90$

$\therefore XN \perp YZ$

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c i $x = R\cos\frac{2\pi t}{T}$

$\dot{x} = -R\sin\frac{2\pi t}{T}\times\frac{2\pi}{T}$

$= -\frac{R2\pi}{T}\sin\frac{2\pi t}{T}$

$\ddot{x} = -\frac{2\pi R}{T}\cos\frac{2\pi t}{T}\cdot\frac{2\pi}{T}$

$= -\frac{4\pi^2 R}{T^2}\cos\frac{2\pi t}{T}$

$= -\frac{4\pi^2}{T^2}\left(R\cos\frac{2\pi t}{T}\right)$

$\therefore \ddot{x} = -\frac{4\pi^2}{T^2}x$

$y = R\sin\frac{2\pi t}{T}$

$\dot{y} = R\cos\frac{2\pi t}{T}\times\frac{2\pi}{T}$

$= \frac{2\pi}{T}R\cos\frac{2\pi t}{T}$

$\ddot{y} = \frac{2\pi}{T}R\left(-\sin\frac{2\pi t}{T}\right)\left(\frac{2\pi}{T}\right)$

$= -\frac{4\pi^2}{T^2}\left(R\sin\frac{2\pi t}{T}\right)$

$\therefore \ddot{y} = -\frac{4\pi^2}{T^2}y$

ii accel $= \sqrt{\ddot{x}^2+\ddot{y}^2}$

$= \sqrt{\left(-\frac{4\pi^2}{T^2}\right)^2x^2+\left(\frac{-4\pi^2}{T^2}\right)^2y^2}$

$= \sqrt{\frac{16\pi^4}{T^4}x^2+\frac{16\pi^4}{T^4}y^2}$

accel $= \sqrt{\frac{16\pi^4}{T^4}(x^2+y^2)}$

as $x^2+y^2 = R^2$ (P is on circle).

$\therefore$ accel $= \sqrt{\frac{16\pi^4}{T^4}R^2}$

$= \frac{4\pi^2}{T^2}R$

$= \frac{4\pi^2 R}{T^2}$

as the accel is towards the centre of the circle

accel $= -\frac{4\pi^2 R}{T^2}$

iii Force exerted by star on planet is the same as the force exerted by the planet on the star.

$\therefore$ Force $= m\left(\frac{4\pi^2 R}{T^2}\right)$.

iv $F = \frac{GMm}{R^2}$

$m\left(\frac{4\pi^2 R}{T^2}\right) = \frac{GMm}{R^2}$

$4\pi^2 R = T^2\frac{GM}{R^2}$

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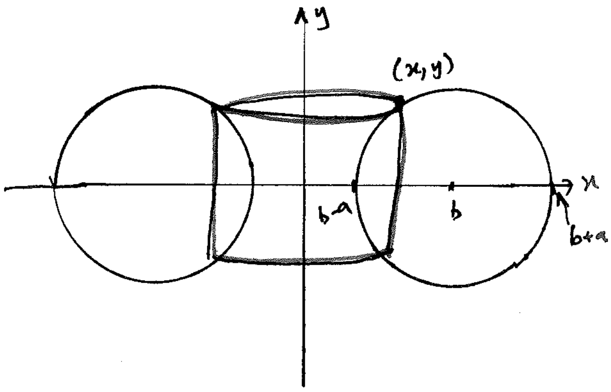
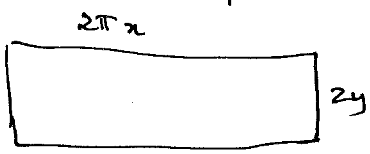
$$T^2 = \frac{4\pi^2 R^3}{GM}$$

$$\therefore T = 2\pi R\sqrt{\frac{R}{GM}}$$

d)

with respect to x:
means dx so width needs
to be measured in x so
need to use cylindrical shells.





$A = 4\pi xy$

$$V = 4\pi \int_{b-a}^{b+a} xy \, dx$$

$= (x-b)^2 + y^2 = a^2$

$y^2 = a^2 - (x-b)^2$

$y = \sqrt{a^2-(x-b)^2}$

$$\therefore V = 4\pi \int_{b-a}^{b+a} x.\sqrt{a^2-(x-b)^2}\, dx$$

16a (i)

$I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$

$= \int_0^{\pi/2} \sin^{n-1}\theta . \sin\theta . d\theta$

$u = \sin^{n-1}\theta$

$du = (n-1)\sin^{n-2}\theta . \cos\theta \, d\theta$

$v \; -\cos\theta$

$v' = \sin\theta$

$\therefore I_n = \left[-\sin^{n-1}\theta . \cos\theta\right]_0^{\frac{\pi}{2}} +$

$(n-1)\int_0^{\pi/2} \sin^{n-2}\theta . \cos^2\theta \, d\theta$

$= 0 +$

$(n-1)\int_0^{\pi/2} \sin^{n-2}\theta . (1-\sin^2\theta) \, d\theta$

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta -$

$(n-1)\int_0^{\pi/2} \sin^n\theta \, d\theta$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = (n-1) I_{n-2}$

$I_n (1 + n - 1) = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

(ii)

$\int_0^2 (4 - x^2)^{\frac{5}{2}} \, dx$

$x = 2\cos\theta$

$dx = -2\sin\theta \, d\theta$

$x = 0 \quad x = 2$

$\theta = \frac{\pi}{2} \quad \theta = 0$

$= \int_{\frac{\pi}{2}}^0 -(4 - 4\cos^2\theta)^{5/2} . 2\sin\theta \, d\theta$

$= \int_0^{\pi/2} (4\sin^2\theta)^{5/2} . 2\sin\theta$

$= 4^{5/2} . 2 \int_0^{\pi/2} \sin^5\theta . \sin\theta \, d\theta$

$= 2^5 . 2 \int_0^{\pi/2} \sin^6\theta \, d\theta$

$= 2^6 \int_0^{\pi/2} \sin^6\theta \, d\theta.$

$\int_0^{\pi/2} \sin^6\theta \, d\theta = I_6$

$= \frac{5}{6} I_4$

$= \frac{5}{6} . \frac{3}{4} . I_2$

$= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$

$= \frac{5}{16} . \int_0^{\pi/2} 1 \, d\theta$

$= \frac{5}{16} . [\theta]_0^{\pi/2}$

$= \frac{5}{16} \frac{\pi}{2}$

$\therefore \int_0^2 (4 - x^2)^{\frac{5}{2}} dx = 64 . \frac{5}{16} . \frac{\pi}{2}$

$= 10\pi$

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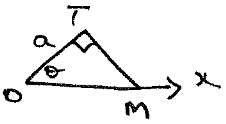
b) i)



since $x^2+y^2=a^2$

radius $= a$.

$\therefore OT = a$

$\cos\theta = \dfrac{a}{OM}$

$OM = \dfrac{a}{\cos\theta}$

$= a\sec\theta$

$\therefore$ coordinates of M are

$(a\sec\theta, 0)$

ii) P has the same x value as M

$\therefore$ x value is $a\sec\theta$.

The y-value lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$\therefore \dfrac{(a\sec\theta)^2}{a^2}-\dfrac{y^2}{b^2}=1$

$\dfrac{a^2\sec^2\theta}{a^2}-\dfrac{y^2}{b^2}=1$

$\sec^2\theta-1=\dfrac{y^2}{b^2}$

$\tan^2\theta=\dfrac{y^2}{b^2}$

$y^2=b^2\tan^2\theta$

$y=b\tan\theta$ in first quad.

$\therefore P\ (a\sec\theta,\ b\tan\theta)$

iii) $m_{PQ}=\dfrac{b\tan\theta-b\tan\phi}{a\sec\theta-a\sec\phi}$

eqn chord PQ:

$y-b\tan\theta=\dfrac{b\tan\theta-b\tan\phi}{a\sec\theta-a\sec\phi}(x-a\sec\theta)$

$y=\dfrac{b\tan\theta-b\tan\phi}{a\sec\theta-a\sec\phi}(x-a\sec\theta)+b\tan\theta$

$\phi=\frac{\pi}{2}-\theta.$

$y=\dfrac{b\tan\theta-b\tan(\frac{\pi}{2}-\theta)}{a\sec\theta-a\sec(\frac{\pi}{2}-\theta)}\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b\tan\theta-b\cot\theta}{a\sec\theta-a\csc\theta}\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b(\tan\theta-\cot\theta)}{a(\sec\theta-\csc\theta)}\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b}{a}\left(\dfrac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}-\frac{1}{\sin\theta}}\right)\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b}{a}\left[\dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}\right]\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b}{a}\dfrac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{(\sin\theta-\cos\theta)}\left[x-a\sec\theta\right]+b\tan\theta$

$=\dfrac{b}{a}(\sin\theta+\cos\theta)(x-a\sec\theta)+b\tan\theta$

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$y = \frac{b}{a}(\sin\theta + \cos\theta)x - \frac{b}{a}(\sin\theta + \cos\theta)\, a\sec\theta + b\tan\theta$

$= \frac{b}{a}(\cos\theta + \sin\theta)x - \frac{b}{a}(\sin\theta + \cos\theta)\frac{a}{\cos\theta} + b\frac{\sin\theta}{\cos\theta}$

$= \frac{b}{a}(\cos\theta + \sin\theta)x - \frac{b}{a}\frac{a\sin\theta}{\cos\theta} - \frac{b}{a}\frac{\cos\theta\, a}{\cos\theta} + \frac{b\sin\theta}{\cos\theta}$

$= \frac{b}{a}(\cos\theta + \sin\theta)x - \frac{b\sin\theta}{\cos\theta} - \frac{b}{a}a + \frac{b\sin\theta}{\cos\theta}$

$\therefore y = \frac{b}{a}(\cos\theta + \sin\theta)x - b$

iv) Every chord has an equation $y = mx + b$; and every chord must pass through the y-intercept of $-b$

$\therefore$ the fixed point is $(0, -b)$

v) $y = \pm\frac{b}{a}x$

vi) as $\theta \to \frac{\pi}{2}$

$y \to \frac{b}{a}(0 + 1)x - b$

$\therefore y = \frac{b}{a}x - b$

which is parallel to the asymptote $y = \frac{b}{a}x$ since $m_1 = m_2$ for parallel lines.

$\therefore$ PQ approaches ~~the~~ a line parallel to an asymptote.