

HORNSBY GIRLS HIGH SCHOOL



Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2013

STUDENT NUMBER: _____

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 15

90 marks

Attempt Questions 11 – 16.

Start each question in a new writing booklet.

Write your student number on every writing booklet.

Question	1-10	11	12	13	14	15	16	Total
Total	/10	/15	/15	/15	/15	/15	/15	/100

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

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Section I

10 marks

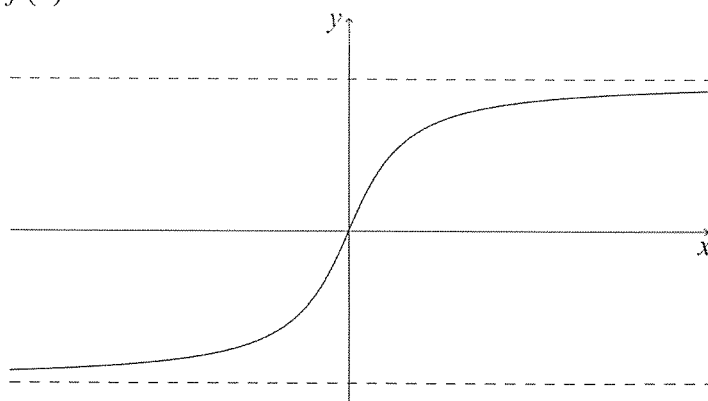
Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

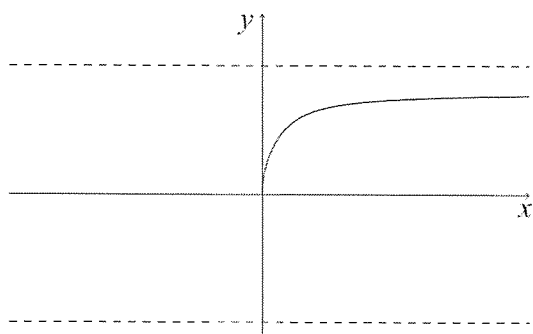
- 1 Let $z = 3 + 2i$ and $w = 2 - 3i$. What is the value of $3\bar{z} - 2w$?
- (A) 5
(B) -5
(C) $5 + 12i$
(D) $5 - 12i$
- 2 The equation $x^2 + 2y^2 - 2xy + x = 8$ defines y implicitly as a function of x .
What is the value of $\frac{dy}{dx}$ at the point $(3, 2)$?
- (A) $\frac{1}{4}$
(B) $-\frac{1}{4}$
(C) $\frac{3}{2}$
(D) $-\frac{3}{2}$
- 3 Let $z = \cos \theta + i \sin \theta$. Which of the following is equal to z^3 ?
- (A) $\cos^3 \theta + i \sin^3 \theta$
(B) $\cos^3 \theta - i \sin^3 \theta$
(C) $\cos 3\theta + i \sin 3\theta$
(D) $\cos 3\theta - i \sin 3\theta$

- 4 The graph of $y = f(x)$ is shown below.

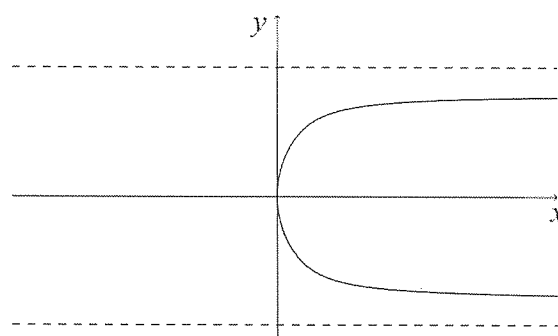


Which of the following graphs best represents $y^2 = f(x)$?

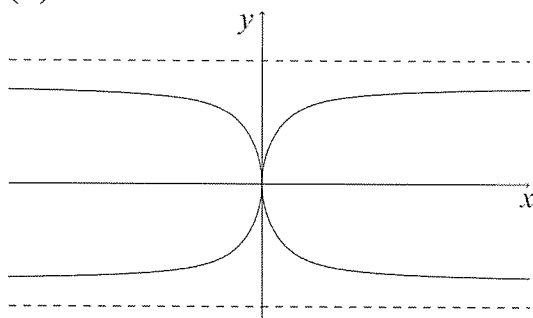
(A)



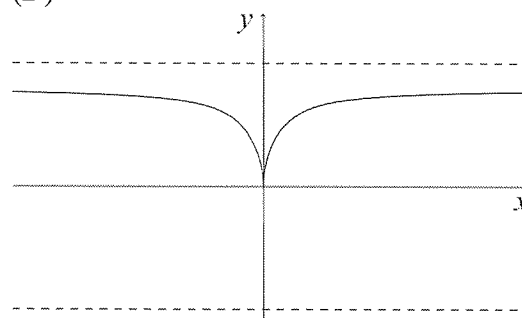
(B)



(C)



(D)



- 5 The roots of the polynomial $4x^3 + 4x - 5 = 0$ are α , β and γ .
What is the value of $(\alpha + \beta - 3\gamma)(\beta + \gamma - 3\alpha)(\alpha + \gamma - 3\beta)$?

- (A) -80
(B) -16
(C) 16
(D) 80

- 6 A mass of 5 kg moves in a horizontal circle of radius 1.5 metres at a uniform angular speed of 4 radians per second. What is the centripetal force required for this motion?
- (A) 40N
(B) 80N
(C) 120N
(D) 160N
- 7 Which of the following is a focus of the hyperbola $\frac{x^2}{11} - \frac{y^2}{25} = -1$?
- (A) (0,5)
(B) (5,0)
(C) (6,0)
(D) (0,6)
- 8 If $x^3 - 11x^2 + 40x - k = (x-4)^2 \cdot P(x)$, what is the value of k ?
- (A) 16
(B) 32
(C) 48
(D) 64
- 9 The region bounded by the curve $y = x^2$, the line $x = 4$ and the x -axis is rotated about the line $x = 4$. Which integral represents the volume of the solid?
- (A) $2\pi \int_0^4 (4-x)x^2 dx$
(B) $\pi \int_0^{16} (4-x)x^2 dx$
(C) $2\pi \int_0^4 (4-x)^2 dx$
(D) $\pi \int_0^{16} (4-x)^2 dx$

10 Without evaluating the integrals, which of the following integrals is equal to zero?

(A) $\int_{-1}^1 e^{-x} \tan^{-1}(x^2) dx$

(B) $\int_{-1}^1 \frac{x^2 \sin x}{x^2 + 5} dx$

(C) $\int_{-1}^1 \sqrt{x^2 + e^x} dx$

(D) $\int_{-1}^1 x^3 \sin^{-1} x dx$

End of Section I

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations

Question 11 (15 marks) Start a new writing booklet

- (a) (i) Using the substitution $x = a - u$, show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. 2
- (ii) Hence evaluate $\int_0^2 x\sqrt{2-x}dx$. 2
- (b) Express $\frac{3\sqrt{3}+i}{\sqrt{3}-i}$ in the form $x+iy$, where x and y are real. 2
- (c) Find $\int e^x \cos x dx$. 2
- (d) Find the square roots of $1+\sqrt{3}i$. 2
- (e) Given that α , β and γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are α^2 , β^2 and γ^2 . 2
- (f) Sketch the region in the complex plane where both the inequalities $|z-2-2i| < 2$ and $0 < \arg(z-2-2i) < \frac{\pi}{4}$ hold true simultaneously. 3

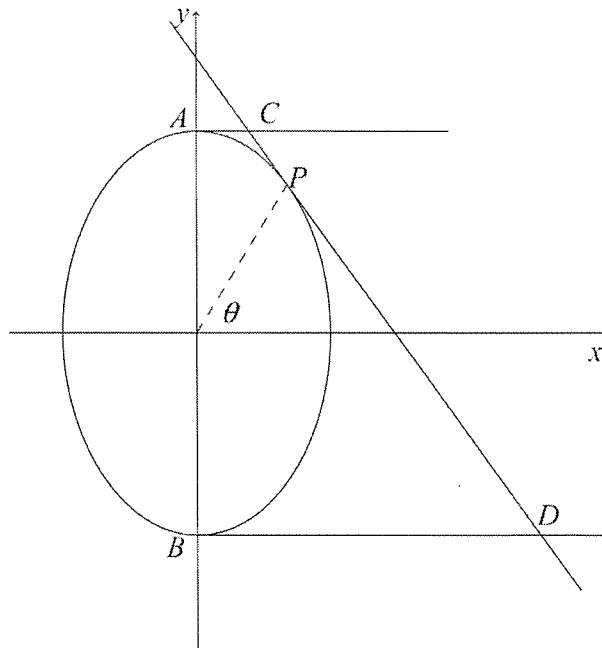
Question 12 (15 marks) Start a new writing booklet

2

(a) Find $\int \frac{1}{8+5\sin x} dx$.

(b) The diagram below shows the ellipse which has equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. The point

$P(2\cos\theta, 3\sin\theta)$, where θ is the axillary angle, lies on the ellipse. The ellipse meets the y -axis at the points A and B . The tangents to the ellipse at A and B meet the tangent at P at the points C and D respectively.



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(i) Find the eccentricity, coordinates of the foci and the equation of the directrices. 3

(ii) Show that the equation of the tangent to the ellipse at P is $2y\sin\theta + 3x\cos\theta = 6$. 2

(iii) Find the numerical value of $AC \times BD$. 3

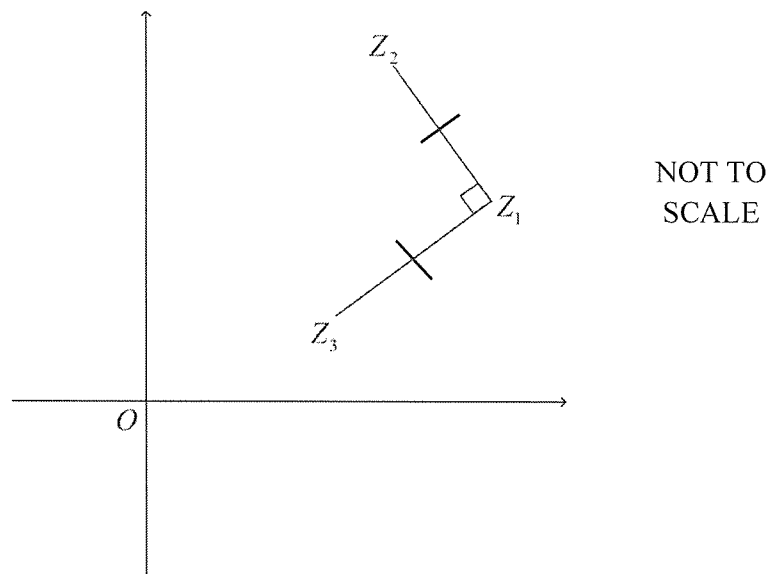
(c) For every integer $n \geq 0$, let $I_n = \int_0^{\frac{\pi}{6}} \sec^n x dx$. 3

Show that for $n \geq 2$, $(n-1)I_n = \frac{2^{n-2}}{(\sqrt{3})^{n-1}} + (n-2)I_{n-2}$.

Question 12 continues on page 9

Question 12 (continued)

(d)



2

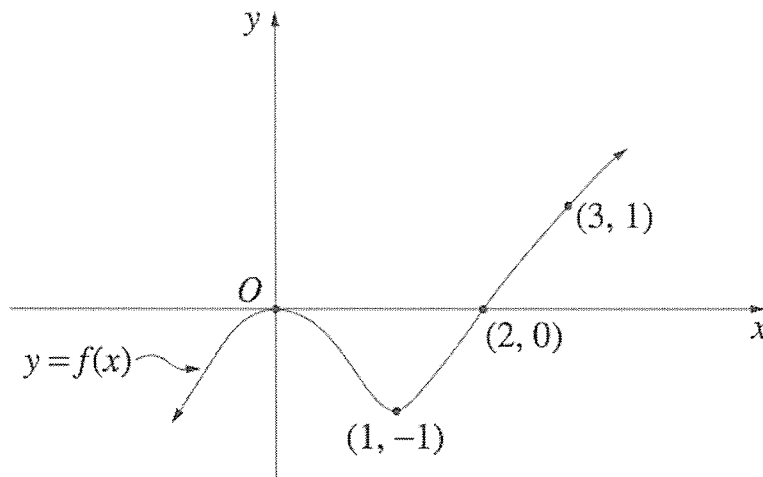
On the Argand diagram above, the point Z_1 represents the complex number z_1 and the point Z_2 represents the complex number z_2 . The point Z_2 is rotated about Z_1 through a right angle in the positive direction to take up the position Z_3 , representing the complex number z_3 .

Show that $z_3 = (1-i)z_1 + iz_2$.

End of Question 12

Question 13 (15 marks) Start a new writing booklet

- (a) The diagram below shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

- (i) $y = \frac{1}{f(x)}$. 2
- (ii) $y = |f(x)|$. 2
- (iii) $y = \ln(f(x))$. 2
- (b) (i) By using De Moivre's Theorem, show that $\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta$ and $\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$. 2
- (ii) Hence show that $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$, where $t = \tan \theta$. 1
- (iii) Hence find the general solutions of the equation $3 \tan \theta - \tan^3 \theta = 0$ 1
- (c) (i) Find the five roots of the equation $z^5 = 1$ 2
- (ii) Show that $z^5 - 1 = (z - 1) \left(z^2 - 2z \cos \frac{2\pi}{5} + 1 \right) \left(z^2 - 2z \cos \frac{4\pi}{5} + 1 \right)$. 2
- (iii) Hence show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$ 1

Question 14 (15 marks) Start a new writing booklet

- (a) Find $\int \frac{x+3}{x^3+x^2+x+1} dx$. 2
- (b) The base of a solid is the circle $x^2 + y^2 = 36$. Find the volume of the solid if every section perpendicular to the x -axis is a square where one side of the square is completely laid in the base of the solid. 3
- (c) A parachutist of mass m falls to the ground from a plane. Air resistance is proportional to mv^2 , where v is his speed and g is acceleration due to gravity. Take downwards as being the positive direction, and the point where the parachutist jumps out the plane as the origin of displacement, x .
- (i) Deduce that $\frac{d}{dx}(v^2) = 2g - 2kv^2$, where k is the constant of proportionality. 1
- (ii) Show that $v^2 = \frac{g}{k} - \frac{g}{k}e^{-2kx}$, satisfies the differential equation in part (i). 2
- (iii) Find an expression for the terminal speed of the parachutist during his free-fall. 1
- (d) Let $f(x) = 3x^5 - 10x^3 + 16x$.
- (i) Show that $f'(x) \geq 1$ for all real x . 2
- (ii) For what values of x is $f''(x) > 0$. 2
- (iii) Sketch the graph of $y = f(x)$, clearly indicating any turning points and points of inflexion. 2

Question 15 (15 marks) Start a new writing booklet

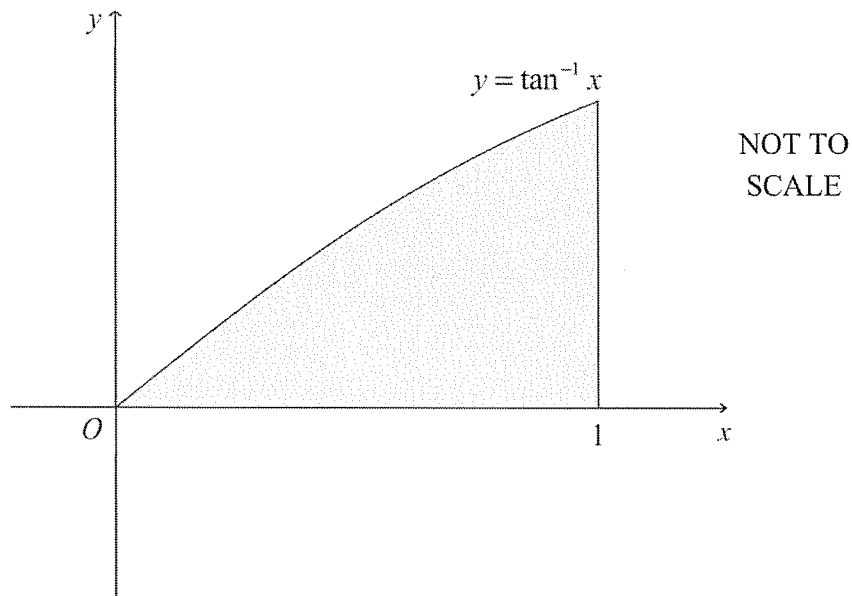
(a) Consider the function $f(u) = \sin^{-1} u - \sqrt{1-u^2}$, with restricted domain $0 < u < 1$. 1

(i) Show that $f'(u) = \sqrt{\frac{1+u}{1-u}}$.

(ii) Hence, given that α is in the domain, show that 2

$$\int_0^\alpha \left(\frac{1+u}{1-u} \right)^{\frac{1}{2}} du = \sin^{-1} \alpha - \sqrt{1-\alpha^2} + 1$$

(b) The region bounded by the curve $y = \tan^{-1} x$ and the x axis between $x=0$ and $x=1$ is rotated through one complete revolution about $x=1$. A diagram of the region to be rotated is shown below.



(i) Use the method of cylindrical shells to show that the volume V of the solid formed 1

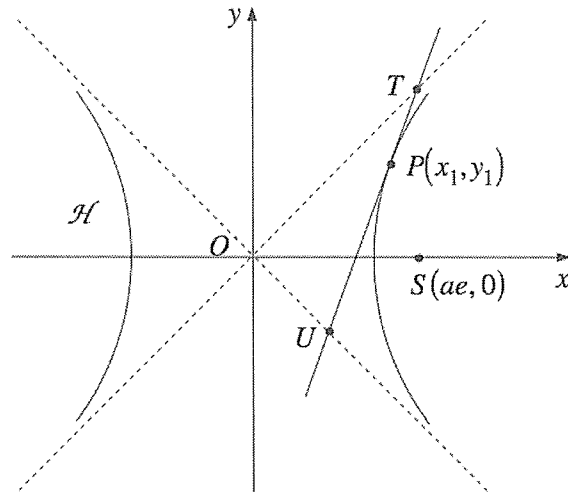
is given by $V = 2\pi \int_0^1 (1-x) \tan^{-1} x \, dx$.

(ii) Hence find the volume V in simplest exact form. 4

Question 15 continues on page 13

Question 15 (continued)

- (c) The point $S(ae, 0)$ is a focus on the hyperbola $H: x^2 - y^2 = a^2$. The tangent to the hyperbola at a point $P(x_1, y_1)$ meets the asymptotes of H at T and U , as shown in the diagram below.



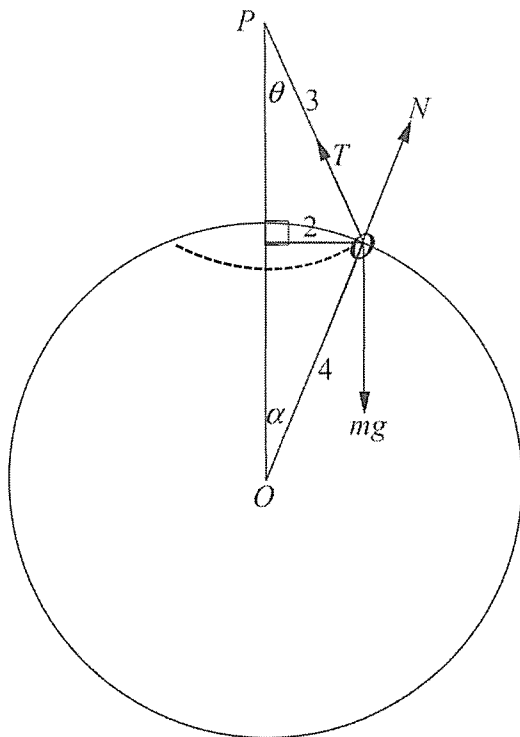
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- (i) Show that the equation of the tangent TU is $x_1x - y_1y = a^2$. 2
- (ii) Show that the gradient of SU is $\frac{a}{e(x_1 + y_1) - a}$. 2
- (iii) Let $\angle UST = \theta$. Show that $\tan \theta = -1$. 3

End of Question 15

Question 16 (15 marks) Start a new writing booklet

- (a) A particle of mass 5 kg at the end of a string 3 metres long is suspended from a point P vertically above the highest point of a smooth sphere of radius 4 metres. It describes a horizontal circle of radius 2 metres on the surface of the sphere.



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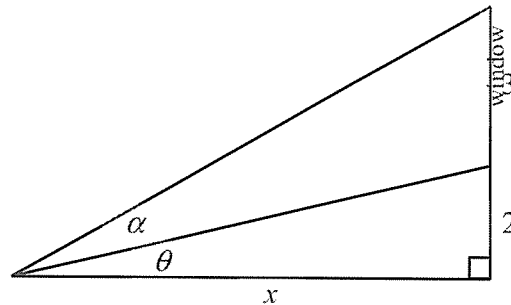
Three forces act on the particle: the tension force F of the string, the normal reaction force N to the surface of the sphere, and the gravitational force mg . Take g , the acceleration due to gravity, as 10 ms^{-2} . The angular velocity of the particle moving in uniform circular motion is 1 radian per second.

- (i) By resolving the forces horizontally and vertically on a diagram, show that 2
- $$\frac{T\sqrt{5}}{3} + \frac{N\sqrt{3}}{2} = 50 \quad \text{and} \quad \frac{2T}{3} - \frac{N}{2} = 10 .$$
- (ii) Find, correct to one decimal place:
- (α) the tension in the string. 1
- (β) The force exerted on the sphere. 1
- (iii) Find the angular velocity that will ensure there is no force exerted on the sphere. 1

Question 16 continues on page 15

Question 16 (continued)

- (b) The base of a stained glass window 3 metres high is 2 metres above the eye-level of an observer who is x metres from the base of the wall which is supporting the window. α is the viewing angle at eye level (i.e. the difference between the angles of elevation of the top and bottom of the window, as seen by the observer)



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- (i) Show that $\alpha = \tan^{-1}\left(\frac{3x}{x^2 + 10}\right)$. 3
- (ii) Hence find how far should the observer stand from the wall for the viewing angle to be greatest. 3
- (c) Given that $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$ has zeroes at $x = \pm\sqrt{5}$ and a double zero, factorise $f(x)$ over the:
- (i) real field. 3
- (ii) complex field. 1

End of Paper

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QUESTION 11

a) Let $x = a - u$ $\therefore u = a - x$

$dx = -du$ $du = -dx$

when $x=0$ $u=a$

$x=a$ $u=0$

$\therefore \int_0^a f(x) dx = - \int_a^0 f(a-u) du$

$= \int_0^a f(a-u) du$

$= \int_0^a f(a-x) dx$

OR $\int_0^a f(a-x) dx = - \int_a^0 f(u) du$

$= \int_0^a f(x) dx$

$= \int_0^a f(x) dx$

ii) $\int_0^2 x \sqrt{2-x} dx = \int_0^2 (2-x) \sqrt{x} dx$

$= \int_0^2 (2x^{3/2} - x^{5/2}) dx$

$= \left[\frac{4x^{5/2}}{5} - \frac{2x^{7/2}}{7} \right]_0^2$

$= \frac{4\sqrt{8}}{5} - \frac{2\sqrt{32}}{7}$

$= \frac{16\sqrt{2}}{15}$

$= 1.50849$

b) $\frac{3\sqrt{3}+i}{\sqrt{3}-i} = \frac{3\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$

$= \frac{2+i\sqrt{3}}{\sqrt{3}}$

MC

1. A

2. D

3. C

4. B

5. A

6. C

7. D

8. C

9. A

10. B

c) Let $u = \cos x$ $du = -\sin x dx$ $dv = e^x dx$

$du = -\sin x dx$ $v = e^x$

$\therefore \int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$

Let $u = \sin x$ $dv = e^x dx$

$du = \cos x dx$ $v = e^x$

$\therefore \int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$

d) Let $(x+iy)^2 = 1 + \sqrt{3}i$

$x^2 + 2ixy - y^2 = 1 + \sqrt{3}i$

$\therefore x^2 - y^2 = 1$ (1)

$2xy = \sqrt{3}$ (2)

from (2) $\Rightarrow y = \frac{\sqrt{3}}{2x}$ (3)

Subst (3) into (1) $\Rightarrow x^2 - \frac{3}{4x^2} = 1$

$4x^4 - 4x^2 - 3 = 0$

$(2x^2 - 3)(2x^2 + 1) = 0$

$2x^2 = \frac{3}{2}, -\frac{1}{2}$

$\therefore x = \pm \frac{\sqrt{3}}{\sqrt{2}}$

when $x = \frac{\sqrt{3}}{\sqrt{2}}, y = \frac{\sqrt{2}}{2}$

when $x = -\frac{\sqrt{3}}{\sqrt{2}}, y = -\frac{\sqrt{2}}{2}$

\therefore sq roots are: $\pm \left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{2}i}{2} \right)$

OR $\pm \left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}i}{2} \right)$

e) Let $y = x^2$

$x = \sqrt{y}$

$\therefore (\sqrt{y})^3 + p(\sqrt{y})^2 + q\sqrt{y} + r = 0$

$y\sqrt{y} + py + q\sqrt{y} + r = 0$

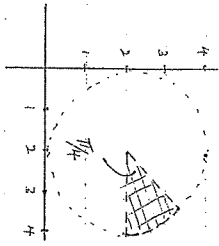
$\sqrt{y}(y+q) = -py-r$

$y(y+q)^2 = (py+r)^2$

$y^3 + 2qy^2 + q^2y = p^2y^2 + 2pry + r^2$

$\therefore y^3 + y^2(2q-p^2) + y(q^2-2pr) - r^2 = 0$

f)



Question 12

(a) Let $t = \tan x$

$I = \int \frac{dx}{8 + 5 \sin x}$

$\sin 2x = \frac{2t}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$\therefore I = \int \frac{1}{8 + \frac{10t}{1+t^2}} = \frac{2dt}{1+t^2}$

$= \int \frac{2dt}{8 + 10t^2 + 10t}$

$= \int \frac{dt}{4t^2 + 5t + 4}$

$= \frac{1}{4} \int \frac{dt}{t^2 + \frac{5}{4}t + 1}$

$= \frac{1}{4} \int \frac{dt}{(t + \frac{5}{8})^2 + \frac{39}{64}}$

$= \frac{1}{4} \times \frac{8}{\sqrt{39}} \tan^{-1} \left(\frac{(t + \frac{5}{8}) \cdot 8}{\sqrt{39}} \right) + C$

$= \frac{2}{\sqrt{39}} \tan^{-1} \left(\frac{8t + 5}{\sqrt{39}} \right) + C$

(b) (i) $a^2 = b^2(1 - e^2)$ \therefore Focus $(0, \pm\sqrt{3})$

$$1 - e^2 = \frac{a^2}{b^2}$$

Directrices $y = \pm \frac{a}{e}$

$$e^2 = 1 - \frac{a^2}{b^2}$$

$$e^2 = 1 - \frac{9}{9}$$

$$e^2 = \frac{0}{9}$$

$$e = \frac{\sqrt{0}}{3}$$

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$$\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{\partial y}{\partial x} \cdot \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = -\frac{9x}{4y}$$

When $x = 2\cos\theta$,
 $y = 3\sin\theta$
 $\frac{dy}{dx} = \frac{12\sin\theta}{-18\cos\theta} = -\frac{2\sin\theta}{3\cos\theta}$

(iii) At when $y = 3$

$$3x\cos\theta + 3x\cos\theta = 6$$

$$x = \frac{2 - 2\sin\theta}{\cos\theta}$$

When $y = -3$

$$-6\sin\theta + 3x\cos\theta = 6$$

$$x = \frac{2 + 2\sin\theta}{\cos\theta}$$

$$AC \times BD = \frac{4 - 4\sin^2\theta}{\cos^2\theta}$$

$$= \frac{4(1 - \sin^2\theta)}{\cos^2\theta}$$

$$= \frac{4\cos^2\theta}{\cos^2\theta}$$

$$= 4$$

(c) $I_n = \int_0^{\pi/6} \sec^n x dx$

Let $u = \sec^{n-2} x$
 $u' = (n-2)\sec^{n-4} x \sin x$

$v' = \sec^2 x$

$$\therefore I_n = \int_0^{\pi/6} \tan \sec^{n-2} x \int_0^{\pi/6} \sec^2 x dx - (n-2) \int_0^{\pi/6} \sec^{n-4} x dx$$

$$= \frac{1}{\sqrt{3}} \left(\frac{2}{\sqrt{3}}\right)^{n-2} - (n-2) \int_0^{\pi/6} \sec^{n-4} x dx$$

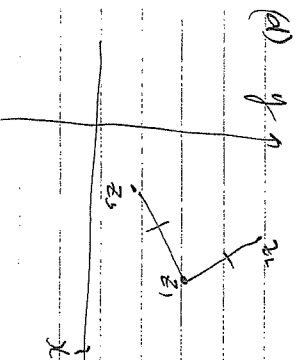
$$= \frac{2^{n-2}}{\sqrt{3}^{n-1}} - (n-2) \int_0^{\pi/6} (1 - \cos^2 x) \sec^{n-4} x dx$$

$$= \frac{2^{n-2}}{\sqrt{3}^{n-1}} - (n-2) \int_0^{\pi/6} \sec^n x dx + \sec^{n-2} x dx$$

$$= \frac{2^{n-2}}{\sqrt{3}^{n-1}} - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1 + n - 2) = \frac{2^{n-2}}{\sqrt{3}^{n-1}} + (n-2)I_{n-2}$$

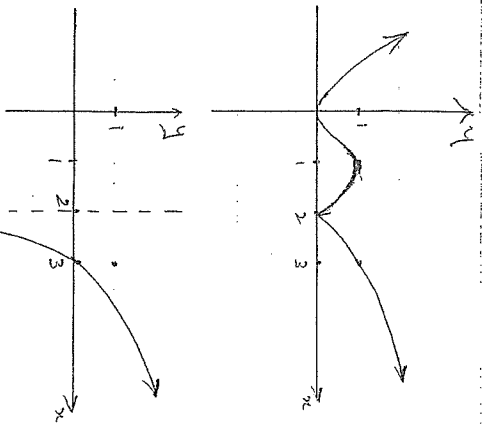
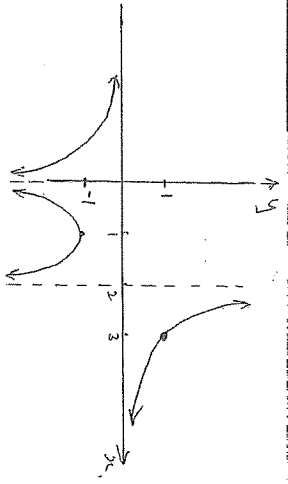
$$I_n(n-1) = \frac{2^{n-2}}{\sqrt{3}^{n-1}} + (n-2)I_{n-2}$$



$$\begin{aligned} z_1 \hat{i} + z_2 \hat{j} &= z_1 + iz_2 \\ z_1 \hat{i} + z_2 \hat{j} &= i(z_2 - z_1) \\ z_1 \hat{i} + z_2 \hat{j} &= z_1 + iz_2 \\ z_1 \hat{i} + z_2 \hat{j} &= z_1 + i(z_2 - z_1) \\ z_1 \hat{i} + z_2 \hat{j} &= z_1 + iz_2 - iz_1 \\ z_1 \hat{i} + z_2 \hat{j} &= z_1(1-i) + iz_2 \end{aligned}$$

QUESTION 13.

a)



b) i) Let $z = \cos \theta + i \sin \theta$

$z^3 = \cos 3\theta + i \sin 3\theta$

$z^3 = (\cos \theta + i \sin \theta)^3$

$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta + 3 \cos \theta \sin^2 \theta + i \sin^3 \theta$
 $= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta + \sin^3 \theta)$

equating real & imaginary parts

$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$

$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$

ii) For $3\theta = \frac{\pi}{3}$
 $\cos 3\theta = \frac{1}{2}$

$= \frac{\sin \theta (3 \cos^2 \theta - \sin^2 \theta)}{\cos \theta (\cos^3 \theta - 3 \sin^2 \theta \cos \theta)} + \frac{\cos^2 \theta}{\cos \theta}$

$= \tan \theta \left(\frac{3 - \tan^2 \theta}{1 - 3 \tan^2 \theta} \right)$

$= \frac{t(3-t^2)}{1-3t^2}$ where $t = \tan \theta$

$= \frac{3t-t^3}{1-3t^2}$

iii) $3 \tan \theta - \tan^3 \theta = 0$

Let $t = \tan \theta$

$3t - t^3 = 0$

$\frac{3t-t^3}{1-3t^2} = 0$

$\tan 3\theta = 0$

$3\theta = n\pi$

$\theta = \frac{n\pi}{3}$ (n is an integer)

OR $\tan \theta (3 - \tan^2 \theta) = 0$

$\tan \theta = 0$ $\tan \theta = \pm \sqrt{3}$

$\theta = n\pi$ $\theta = n\pi \pm \frac{\pi}{3}$

$\theta = n\pi$

$\theta = n\pi \pm \frac{\pi}{3}$

c) i) $z_0 = \cos 0 + i \sin 0 = 1$

$z_1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

$z_2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

$z_3 = \cos \frac{6\pi}{3} + i \sin \frac{6\pi}{3} = z_0$

$z_4 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} = z_2$

ii) $z^5 - 1 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4)$

$= (z - 1)(z^2 - z(z_1 + z_2) + z_1 z_2)(z^2 - z(z_3 + z_4) + z_3 z_4)$

Now $z_1 + z_2 = 2 \cos \frac{2\pi}{3}$ $z_1 z_2 = \cos^2 \frac{2\pi}{3} + i^2 \sin^2 \frac{2\pi}{3} = 1$

$z_3 + z_4 = 2 \cos \frac{4\pi}{3}$ $z_3 z_4 = \cos^2 \frac{4\pi}{3} + i^2 \sin^2 \frac{4\pi}{3} = 1$

$z^5 - 1 = (z - 1)(z^2 - 2z \cos \frac{2\pi}{3} + 1)(z^2 - 2z \cos \frac{4\pi}{3} + 1)$

iii) sum of roots = $-\frac{b}{a} = 0 = 1 + z_1 + z_2 + z_3 + z_4$

$0 = 1 + 2 \cos \frac{2\pi}{3} + 2 \cos \frac{4\pi}{3}$

$-1 = 2 \cos \frac{2\pi}{3} + 2 \cos \frac{4\pi}{3}$

$-1 = \cos \frac{2\pi}{3} + \cos \frac{4\pi}{3}$

Question 14 solutions

$$(a) \int \frac{x+3}{x^3+x^2+x+1} dx = \int \frac{x+3}{x^2(x+1)+(x+1)} dx = \int \frac{x+3}{(x^2+1)(x+1)} dx$$

$$\text{Let } \frac{x+3}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$x+3 = (Ax+B)(x+1) + C(x^2+1)$$

$$\text{Let } x=-1 \quad R=2C$$

$$\therefore C=1$$

$$x=0 \quad 3=B+1$$

$$B=2$$

$$x=1 \quad 4=2A+2B+2C$$

$$4=A+2+1$$

$$A=-1$$

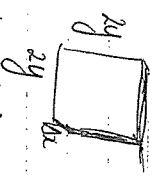
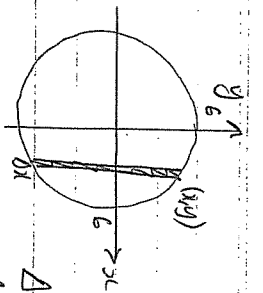
$$\therefore \int \frac{x+3}{(x^2+1)(x+1)} dx = \int \frac{-x+2}{x^2+1} dx + \int \frac{1}{x+1} dx$$

$$= \frac{-1}{2} \int \frac{2x}{x^2+1} dx + \int \frac{2}{x^2+1} dx + \int \frac{1}{x+1} dx$$

$$= \frac{-1}{2} \ln|x^2+1| + 2 \tan^{-1}x + \ln|x+1| + C$$

$$= \ln \left(\frac{x+1}{|x^2+1|} \right) + 2 \tan^{-1}x + C$$

(b)



$$\Delta A = 4y^2 \Delta x$$

$$\Delta V = 4y^2 \Delta x$$

$$x^2 + y^2 = 36$$

$$y^2 = 36 - x^2$$

$$4y^2 = 144 - 4x^2$$

$$\Delta V = \sum_{x=0}^6 (144 - 4x^2) \Delta x$$

$$= \lim_{\Delta x \rightarrow 0} \sum_{x=0}^6 (144 - 4x^2) \Delta x$$

$$= \int_0^6 (144 - 4x^2) dx$$

$$= \left[144x - \frac{4x^3}{3} \right]_0^6$$

$$= (864 - 288) - (-864 + 288)$$

$$= 1152 \text{ units}^3$$

$$(i) F=ma$$

$$ma = mg - kv^2$$

$$a = g - kv^2$$

$$\frac{d}{dt} \left(\frac{1}{2} v^2 \right) = g - kv^2$$

$$\frac{1}{2} \frac{d}{dt} (v^2) = g - kv^2$$

$$\frac{d(v^2)}{dt} = 2g - 2kv^2$$

$$\int \frac{dv^2}{v^2} = \int \frac{1}{2g - 2kv^2}$$

$$x = \frac{1}{2} \int \frac{1}{g - kv^2} d(v^2)$$

$$x = \frac{-1}{2k} \int \frac{-k}{g - kv^2} d(v^2)$$

$$v = \frac{-1}{2k} \ln(g - kv^2) + C$$

$$\text{then } x=0, v=0$$

$$0 = \frac{-1}{2k} \ln g + C$$

$$C = \frac{1}{2k} \ln g$$

$$x = \frac{1}{2k} \ln \left(\frac{g}{g - kv^2} \right)$$

$$\frac{d}{dt} kv^2 = e^{2kx}$$

$$g - kv^2 = g e^{-2kx}$$

$$-kv^2 = g e^{-2kx} - g$$

$$v^2 = \frac{g}{k} - \frac{g}{k} e^{-2kx}$$

Alternate:

$$\frac{d}{dt} (v^2) = \frac{d}{dx} \left(\frac{g}{k} - \frac{g}{k} e^{-2kx} \right)$$

$$= -2kx \frac{-g}{k} e^{-2kx}$$

$$= 2g e^{-2kx}$$

$$= 2g (e^{-2kx})$$

$$\text{But } v^2 = \frac{g}{k} - \frac{g}{k} e^{-2kx}$$

$$v^2 - \frac{g}{k} = -\frac{g}{k} e^{-2kx}$$

$$kv^2 - g = -g e^{-2kx}$$

$$e^{-2kx} = 1 - \frac{kv^2}{g}$$

$$\frac{d}{dx} (v^2) = 2g \left(1 - \frac{kv^2}{g} \right)$$

$$= 2g - 2kv^2$$

$$(iii) a = g - kv^2$$

$$\text{As } a \rightarrow 0$$

$$0 = g - kv^2$$

$$kv^2 = g$$

$$v^2 = \frac{g}{k}$$

$$v = \sqrt{\frac{g}{k}}$$

lim $v^2 = \lim \frac{g}{k}$
 $x \rightarrow \infty$

$$\therefore v \rightarrow \sqrt{\frac{g}{k}}$$

$$(d) (i) f(x) = 3x^5 - 10x^3 + 16x$$

$$f'(x) = 15x^4 - 30x^2 + 16$$

$$= 15(x^4 - 2x^2) + 16$$

$$= 15(x^2 - 1)^2 + 1$$

$$= 15(x^2 - 1)^2 + 1 > 0$$

$$(x^2 - 1)^2 > 0$$

$$15(x^2 - 1)^2 > 0$$

$$15(x^2 - 1)^2 + 1 > 1$$

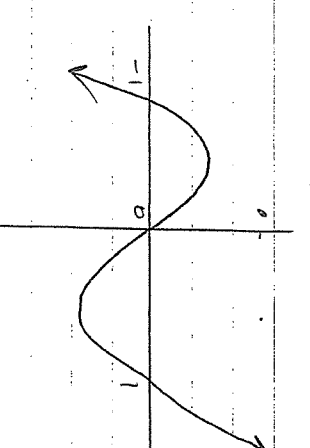
$\therefore f'(x) > 1$ for all x .

$$(ii) f''(x) = 60x^3 - 60x$$

$$\text{let } f''(x) > 0$$

$$60x^3 - 60x > 0$$

$$60x(x-1)(x+1) > 0$$



$\therefore f''(x) > 0$ for $x > 1, -1 < x < 0$.

(iii) $f''(x) > 0$ no stationary points

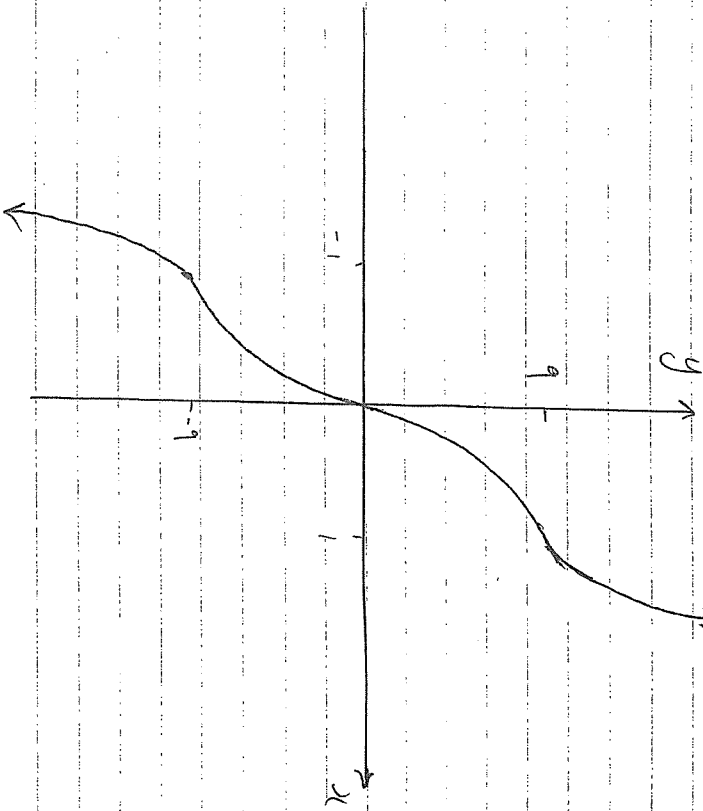
$$f''(x) = 0 \text{ for } x = -1, 0, 1$$

\therefore 3 pts of inflexion (changes sign (ii))

When $x = -1$, $f(-1) = -9$, $f'(-1) = 1$

$x = 0$, $f(0) = 0$, $f'(0) = 16$

$x = 1$, $f(1) = 9$, $f'(1) = 1$



QUESTION 15

a) i) Let $y = \sin^{-1}(u) - (1-u^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} - \frac{1}{2}(1-u^2)^{-1/2} \cdot -2u$$

$$= \frac{1}{(1-u^2)^{1/2}} + \frac{u}{(1-u^2)^{1/2}}$$

$$= \frac{1+u}{\sqrt{1-u^2}}$$

$$= \frac{\sqrt{1+u} \times \sqrt{1+u}}{\sqrt{1-u} \times \sqrt{1+u}}$$

$$= \frac{\sqrt{1+u}}{\sqrt{1-u}}$$

ii) $\int_0^x \left(\frac{1+u}{1-u}\right)^{1/2} du = \left[\sin^{-1}u - (1-u^2)^{1/2} \right]_0^x$

$$= \sin^{-1}x - \sqrt{1-x^2} - \sin^{-1}0 + 1$$

$$= \sin^{-1}x + 1 - \sqrt{1-x^2}$$

b) i) $\Delta V = 2\pi(1-x)y \Delta x$

$$\Delta V = 2\pi(1-x)\tan^{-1}x \Delta x$$

$$V = \lim_{\Delta x \rightarrow 0} \sum_{k=0}^{n-1} 2\pi(1-x_k)\tan^{-1}x_k \Delta x$$

$$= 2\pi \int_0^1 (1-x)\tan^{-1}x \, dx$$

ii) Let $u = \tan^{-1}x$ $dV = (1-x) \, dx$

$$du = \frac{dx}{1+x^2} \quad V = x - \frac{x^2}{2}$$

$$V = 2\pi \left[\left(x - \frac{x^2}{2} \right) \tan^{-1}x \right]_0^1 - \int_0^1 \frac{2x-x^2}{2} \times \frac{1}{1+x^2} \, dx$$

$$= \frac{\pi^2}{4} + \pi \int_0^1 \left(\frac{x^{3/2}}{x^2+1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1} \right) dx$$

$$= \frac{\pi^2}{4} + \pi \left[x - \ln|x^2+1| - \tan^{-1}x \right]_0^1 = \pi(1-\ln 2)$$

$$e) x^2 - y^2 = a^2 \quad \text{--- (1)}$$

$$2x - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

$$y - y_1 = \frac{x_1}{y_1} (x - x_1)$$

$$y y_1 - y_1^2 = x x_1 - x_1^2$$

$$x x_1 - y y_1 = x_1^2 - y_1^2$$

$$x x_1 - y y_1 = a^2 \quad \text{(as } x_1^2 - y_1^2 = a^2 \text{)}$$

(from Eqn (1))

$$ii) \frac{x^2 - y^2}{a^2} = 1$$

$$\therefore \text{asymptotes } y = \pm x$$

$$a^2 = a^2 y e^{-1}$$

$$e^2 = 2$$

$$y \text{ is over } y = -x$$

$$\text{solve } y = -x \quad \text{--- (1)}$$

$$x x_1 - y y_1 = a^2 \quad \text{--- (2)}$$

Subst (1) into (2)

$$x x_1 + x y_1 = a^2$$

$$x = \frac{a^2}{x_1 + y_1}$$

$$y = \frac{-a^2}{x_1 + y_1}$$

$$\text{grad } SV = 0 + \frac{a^2}{x_1 + y_1}$$

$$\frac{ae - a^2}{x_1 y_1}$$

$$= \frac{a^2}{x_1 + y_1} \times \frac{x_1 + y_1}{ae (x_1 + y_1) - a^2}$$

$$iii) \text{I is over } y = x$$

$$\text{solve } y = x \quad \text{--- (1)}$$

$$\text{Subst (1) into (2)} \Rightarrow x x_1 - x y_1 = a^2 \quad \text{--- (2)}$$

$$x = \frac{a^2}{x_1 - y_1}$$

$$y = \frac{a^2}{x_1 - y_1}$$

$$\text{grad } ST = \frac{-a^2}{x_1 - y_1}$$

$$\frac{ae - a^2}{x_1 - y_1}$$

$$= \frac{-a^2}{x_1 - y_1} \times \frac{x_1 - y_1}{ae (x_1 - y_1) - a^2}$$

$$= \frac{-a^2}{a (e (x_1 - y_1) - a)}$$

$$= \frac{a}{a - e (x_1 - y_1)}$$

$$\text{for } \theta = \frac{\frac{a}{a - e (x_1 - y_1)} - \frac{-a^2}{a (e (x_1 - y_1) - a)}}{1 + \frac{a}{a - e (x_1 - y_1)} \times \frac{-a^2}{a (e (x_1 - y_1) - a)}}$$

$$= \frac{\frac{a - ae (x_1 - y_1) - ae (x_1 - y_1) + a^2}{(e (x_1 - y_1) - a) (a - e (x_1 - y_1)) + a^2}}{1 + \frac{a}{a - e (x_1 - y_1)} \times \frac{-a^2}{a (e (x_1 - y_1) - a)}}$$

$$= \frac{a^2 - ae (x_1 - y_1) - ae (x_1 - y_1) + a^2}{(e (x_1 - y_1) - a) (a - e (x_1 - y_1)) + a^2}$$

$$= \frac{2a^2 - ae x_1 + ae y_1 - ae x_1 - ae y_1}{ae (x_1 + y_1) - e^2 (x_1^2 - y_1^2) - a^2 + ae (x_1 - y_1) + a^2}$$

$$= \frac{2a^2 - 2ae x_1}{ae (x_1 + y_1 + x_1 - y_1) - a^2 e^2} \quad \text{(as } x_1^2 - y_1^2 = a^2 \text{)}$$

$$= \frac{2a (a - e x_1)}{2ae x_1 - a^2 e^2}$$

now $e^2 = 2$

$$= \frac{2a (a - e x_1)}{2ae x_1 - 2a^2}$$

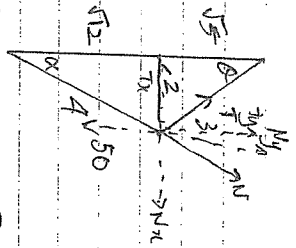
$$= \frac{2a (a - e x_1)}{2a (e x_1 - a)}$$

$$= \frac{-1}{-1} = 1$$

\therefore for $\theta = -1$ as θ is obtuse.

16.

(a) (1)



$$\sin \theta = \frac{3}{5} \quad \cos \theta = \frac{4}{5}$$

$$\sin \alpha = \frac{1}{2} \quad \cos \alpha = \frac{\sqrt{3}}{2}$$

vertically:

$$T \cos \theta + N \cos \alpha = 50 \quad \text{--- (1)}$$

Horizontally:

$$T \sin \theta - N \sin \alpha = m \omega^2 r = 10 \quad \text{--- (2)}$$

(2)

$$T \times \frac{\sqrt{3}}{2} + N \times \frac{\sqrt{3}}{2} = 50$$

$$\frac{\sqrt{3}T}{2} + \frac{\sqrt{3}N}{2} = 50$$

(3)

$$T \times \frac{3}{5} - N \times \frac{1}{2} = 10$$

$$\frac{2T}{3} - N = 10$$

(1')

$$N = \frac{4T}{3} - 20$$

$$\frac{\sqrt{3}T}{2} + \left(\frac{4T}{3} - 20 \right) \frac{\sqrt{3}}{2} = 50$$

$$T = \frac{150 + 30\sqrt{3}}{\sqrt{3} + 2\sqrt{3}}$$

$$T = 35.43 \text{ N}$$

$$N = \frac{4 \times 35.43}{3} - 20 = 27.25 \text{ N}$$

(ii) $N = 0$
 $T \cos \theta = 50$

$$T \sin \theta = 10 \omega^2 r$$

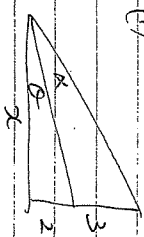
$$\tan \theta = \frac{10 \omega^2 r}{50}$$

$$\tan \theta = \omega^2 r$$

$$\omega = \sqrt{5 \tan \theta}$$

$$= \sqrt{5 \times \frac{2}{5}} = 2.11 \text{ rad/s}$$

(b) (i)



$$\tan \theta = \frac{2}{x}$$

$$\tan(\theta + \alpha) = \frac{5}{5}$$

$$\frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha} = \frac{5}{x}$$

$$\frac{\frac{2}{x} + \tan \alpha}{1 - \frac{2 \tan \alpha}{x}} = \frac{5}{5}$$

$$R + x \tan \alpha = 5 - \frac{10 \tan \alpha}{x}$$

$$8x + x^2 \tan \alpha = 5x - 10 \tan \alpha$$

$$x^2 \tan \alpha + 10 \tan \alpha = 3x$$

$$\tan \alpha (x^2 + 10) = 3x$$

$$\tan \alpha = \frac{3x}{x^2 + 10}$$

$$x = \tan^{-1} \left(\frac{3x}{x^2 + 10} \right)$$

ii) Let $y = \frac{3x}{x^2+10}$

$$\frac{dy}{dx} = \frac{3(x^2+10) - 3x(2x)}{(x^2+10)^2}$$

$$= \frac{3x^2+30-6x^2}{(x^2+10)^2}$$

$$= \frac{30-3x^2}{(x^2+10)^2}$$

$$\frac{dy}{dx} = \frac{30-3x^2}{(x^2+10)^2} = \left(1 + \frac{9x^2}{(x^2+10)^2}\right)$$

$$= \frac{30-3x^2}{(x^2+10)^2} \times \frac{(x^2+10)^2}{(x^2+10)^2} + 9x^2$$

$$= \frac{30-3x^2}{(x^2+10)^2} + 9x^2$$

Let $\frac{dy}{dx} = 0$

$$30-3x^2 = 0$$

$$x^2 = 10$$

$$x = \sqrt{10} \quad (x > 0)$$

x	$\sqrt{10}$	4
$\frac{dy}{dx}$	0	-0.021

Max at ratio m. away from wall

(c) $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$
 $f'(x) = 6x^5 + 20x^4 - 12x^3 - 24x^2 + 70x - 60$
 $(x-\sqrt{5})(x+\sqrt{5}) = x^2-5$

By long division:
 ~~$f(x) = (x^2-5)(x^4+9$~~

$$\begin{array}{r} x^6 \\ \underline{-(x^2-5)(x^4+9)} \\ x^4 + 4x^3 - 3x^2 - 8x^3 + 35x^2 - 60x - 225 \\ \underline{-5x^4} \\ 4x^5 + 2x^4 - 8x^3 \\ \underline{-20x^5} \\ 2x^4 + 12x^3 + 35x^2 \\ \underline{-10x^4} \\ 12x^3 + 45x^2 - 60x \\ \underline{-12x^3} \\ 45x^2 - 60x \\ \underline{-45x^2} \\ -225 \\ \underline{-225} \\ 0 \end{array}$$

$f(x) = (x^2-5)(x^4+4x^3+2x^2+12x+45)$

Let $g(x) = x^4 + 4x^3 + 2x^2 + 12x + 45$
 $g'(x) = 4x^3 + 12x^2 + 4x + 12$

$$= 4(x^3 + 3x^2 + x + 3)$$

$$= 4(x^2(x+3) + 1(x+3))$$

∴ Double root at $x = -3$

$$\begin{array}{r} x^2 + 6x + 9 \\ \underline{-(x^2 + 4x^3 + 2x^2 + 12x + 45)} \\ 2x^4 + 4x^3 + 2x^2 + 12x + 45 \\ \underline{-2x^4 - 6x^3 - 9x^2} \\ -2x^3 - 7x^2 + 12x \\ \underline{-2x^3 - 12x^2 - 18x} \\ 5x^2 + 30x + 45 \\ \underline{-5x^2 - 30x - 45} \\ 0 \end{array}$$

$$A = 4 - 20$$

$$\therefore f(x) = (x^2 - 5)(x^2 - 2x + 5)(x^2 + 6x + 9)$$
$$= (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2(x^2 - 2x + 5)$$

(ii)

$$f(x) = (x - \sqrt{2})(x + \sqrt{2})(x + 3)^2(x^2 - 2x + 1 + 4)$$
$$= (x - \sqrt{2})(x + \sqrt{2})(x + 3)^2((x - 1)^2 + 4)$$
$$= (x - \sqrt{2})(x + \sqrt{2})(x + 3)^2(x - 1 - 2i)(x - 1 + 2i)$$
$$= (x - \sqrt{2})(x + \sqrt{2})(x + 3)^2(x - (1 + 2i))(x - (1 - 2i))$$