Roots and Coefficients

For $P(x) = ax^n + bx^{n-1} + cx^{n-2} \ldots\ldots$

$$\sum \alpha = \frac{-b}{a} \quad \text{(sum of roots, 1 at a time i.e. } \alpha + \beta + \gamma + \delta + \ldots\ldots)$$

$$\sum \alpha\beta = \frac{c}{a} \quad \text{(sum of roots, 2 at a time i.e. } \alpha\beta + \alpha\gamma + \alpha\delta + \ldots\ldots)$$

$$\sum \alpha\beta\gamma = \frac{-d}{a} \quad \text{(sum of roots, 3 at a time i.e. } \alpha\beta\gamma + \alpha\beta\delta + \ldots\ldots)$$

$$\sum \alpha\beta\gamma\delta = \frac{e}{a} \quad \text{(sum of roots, 4 at a time)}$$

We have already seen that for ax^2+bx+c;

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

and

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

This can be generalised to;

$$\boxed{\begin{aligned}&\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta\\&\sum\frac{1}{\alpha}=\frac{\sum\alpha}{\alpha\beta}\qquad\text{(order 2)}\\&\qquad=\frac{\sum\alpha\beta}{\alpha\beta\gamma}\qquad\text{(order 3)}\\&\qquad=\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}\qquad\text{(order 4)}\end{aligned}}$$

e.g. (i) $x^3 - 2x^2 + 3x - 4 = 0$, find;

a) $\sum \alpha \quad = 2$

b) $\sum \alpha\beta \quad = 3$

c) $\sum \alpha\beta\gamma \quad = 4$

d) $\sum \alpha^2$

$$= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

$$= 2^2 - 2(3)$$

$$= -2$$

e) $\sum \alpha^3$

$$\alpha^3 - 2\alpha^2 + 3\alpha - 4 = 0$$

$$\beta^3 - 2\beta^2 + 3\beta - 4 = 0$$

$$\gamma^3 - 2\gamma^2 + 3\gamma - 4 = 0$$

$$\sum \alpha^3 - 2\sum \alpha^2 + 3\sum \alpha - 12 = 0$$

$$\sum\alpha^3 - 2\sum\alpha^2 + 3\sum\alpha - 12 = 0$$

$$\sum\alpha^3 = 2\sum\alpha^2 - 3\sum\alpha + 12$$

$$= 2(-2) - 3(2) + 12$$

$$\underline{= 2}$$

f) $\sum\alpha^4$

$$\alpha^4 - 2\alpha^3 + 3\alpha^2 - 4\alpha = 0$$

$$\beta^4 - 2\beta^3 + 3\beta^2 - 4\beta = 0$$

$$\gamma^4 - 2\gamma^3 + 3\gamma^2 - 4\gamma = 0$$

$$\sum\alpha^4 - 2\sum\alpha^3 + 3\sum\alpha^2 - 4\sum\alpha = 0$$

$$\sum\alpha^4 = 2\sum\alpha^3 - 3\sum\alpha^2 + 4\sum\alpha$$

$$= 2(2) - 3(-2) + 4(2)$$

$$\underline{= 18}$$

g) $(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)$

$$
\begin{aligned}
&= \left(\sum\alpha-\gamma\right)\left(\sum\alpha-\beta\right)\left(\sum\alpha-\alpha\right) \\
&= (2-\gamma)(2-\beta)(2-\alpha) \\
&= (4-2\gamma-2\beta+\beta\gamma)(2-\alpha) \\
&= 8-4\gamma-4\beta+2\beta\gamma-4\alpha+2\alpha\gamma+2\alpha\beta-\alpha\beta\gamma \\
&= 8-4\sum\alpha+2\sum\alpha\beta-\sum\alpha\beta\gamma \\
&= 8-4(2)+2(3)-4 \\
&= \underline{2}
\end{aligned}
$$

(ii) (1996)

Consider the polynomial equation $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are all integers.

Suppose the equation has a root of the form ki, where k is real and $k \neq 0$

a) State why the conjugate $-ki$ is also a root.

Roots appear in conjugate pairs when the coefficients are real.

b) Show that $c = k^2 a$

$$x^4 + ax^3 + bx^2 + cx + d = (x + ki)(x - ki)(x^2 + px + q)$$
$$= (x^2 + k^2)(x^2 + px + q)$$
$$= x^4 + px^3 + qx^2 + k^2x^2 + k^2px + k^2q$$
$$= x^4 + px^3 + (q + k^2)x^2 + k^2px + k^2q$$

$$\therefore\ p = a \ldots(1)\quad q + k^2 = b \ldots(2)\quad k^2p = c \ldots(3)\quad k^2q = d \ldots(4)$$

Substitute (1) into (3) $\qquad k^2a = c$

c) Show that $c^2 + a^2 d = abc$

Using (2); $q = b - k^2$

Sub into (4); $k^2\left(b - k^2\right) = d$

$$bk^2 - k^4 = d$$

$$\frac{bc}{a} - \frac{c^2}{a^2} = d \qquad (\because k^2 = \frac{c}{a})$$

$$abc - c^2 = a^2 d$$

$$\underline{c^2 + a^2 d = abc}$$

d) If 2 is also a root of the equation, and $b = 0$, show that c is even.

Let α be the 4th root

As the other three roots are integer multiples, and the sum of the roots is an integer;

$\therefore \alpha$ is an integer

$$(-ki)(ki)+2(-ki)+2(ki)+\alpha(-ki)+\alpha(ki)+2\alpha = b \qquad \left(\sum \alpha\beta\right)$$

$$k^2 + 2\alpha = 0$$

$$k^2 = -2\alpha$$

From c)… $c = k^2 a$

$$c = -2\alpha a$$

$c = 2M$ $\quad$ α and a are both integers, $\therefore M$ is an integer

Thus c is an even number, as it is divisible by 2

***Patel:* Exercise 5D; 1 to 9**