Inverse Functions

If $y = f(x)$ is a function, then for each x in the domain, there is a maximum of one y value.

The relation obtained by interchanging x and y is $x = f(y)$

e.g. $y = x^3 + x \Rightarrow x = y^3 + y$

If in this new relation, for each x value in the domain there is a maximum of one y value, (i.e. it is a function), then it is called the inverse function to $y = f(x)$ and is symbolised $y = f^{-1}(x)$

A function and its inverse function are reflections of each other in the line $y = x$.

If (a, b) is a point on $y = f(x)$, then (b, a) is a point on $y = f^{-1}(x)$

The domain of $y = f(x)$ is the range of $y = f^{-1}(x)$

The range of $y = f(x)$ is the domain of $y = f^{-1}(x)$

Testing For Inverse Functions

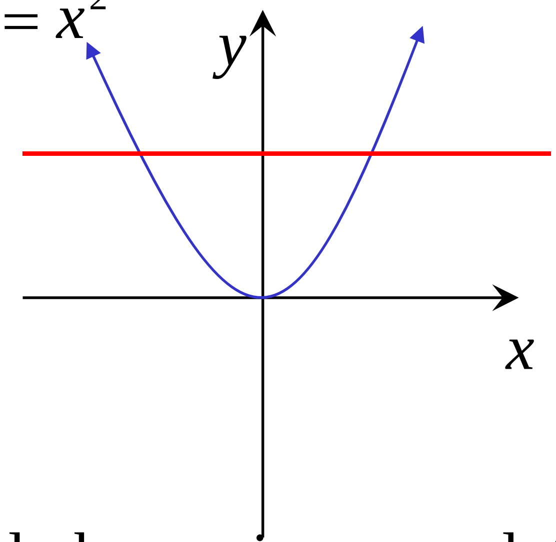
(1) Use a horizontal line test

OR

(2) When $x = f(y)$ is rewritten as $y = g(x)$, $y = g(x)$ is unique.

e.g.

$(i)\, y = x^2$



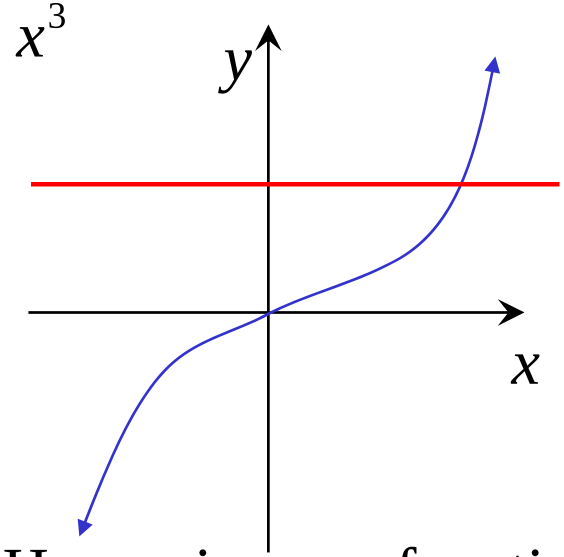
Only has an inverse relation

OR

$$x = y^2$$
$$y = \pm\sqrt{x}$$

NOT UNIQUE

$(ii)\, y = x^3$



Has an inverse function

OR

$$x = y^3$$
$$y = \sqrt[3]{x}$$

UNIQUE

If the inverse relation of $y = f(x)$ is a function, (i.e. $y = f(x)$ has an inverse function), then;

$$f^{-1}\left(f(x)\right) = x \qquad \textbf{\textit{AND}} \qquad f\left(f^{-1}(x)\right) = x$$

e.g.

$$f(x) = \frac{2x+1}{3-2x}$$

$$y = \frac{2x+1}{3-2x} \Rightarrow x = \frac{2y+1}{3-2y}$$

$$(3-2y)x = 2y+1$$

$$3x - 2xy = 2y+1$$

$$(2x+2)y = 3x-1$$

$$y = \frac{3x-1}{2x+2}$$

$$f^{-1}\left(f(x)\right) = \frac{3\left(\frac{2x+1}{3-2x}\right)-1}{2\left(\frac{2x+1}{3-2x}\right)+2}$$

$$= \frac{6x+3-3+2x}{4x+2+6-4x}$$

$$= \frac{8x}{8}$$

$$= x$$

$$f\left(f^{-1}(x)\right) = \frac{2\left(\frac{3x-1}{2x+2}\right)+1}{3-2\left(\frac{3x-1}{2x+2}\right)}$$

$$= \frac{6x-2+2x+2}{6x+6-6x+2}$$

$$= \frac{8x}{8}$$

$$= x$$

Restricting The Domain

If a function does not have an inverse, we can obtain an inverse function by restricting the domain of the original function.

When restricting the domain you need to capture as much of the range as possible.

e.g. $(i)\ y = x^3$

Domain: all real x

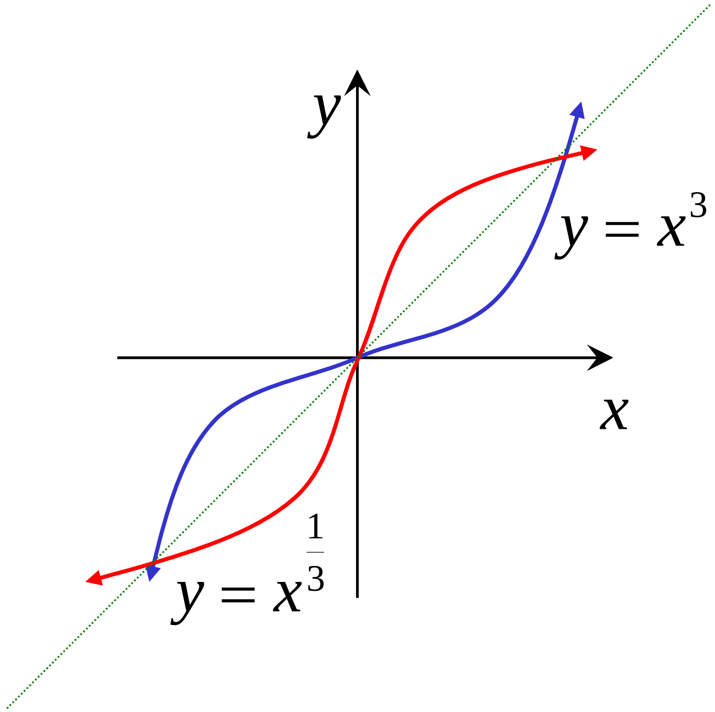
Range: all real y

$f^{-1}: x = y^3$

$\therefore y = x^{\frac{1}{3}}$

Domain: all real x

Range: all real y



$(ii)\, y = e^x$

Domain: all real x

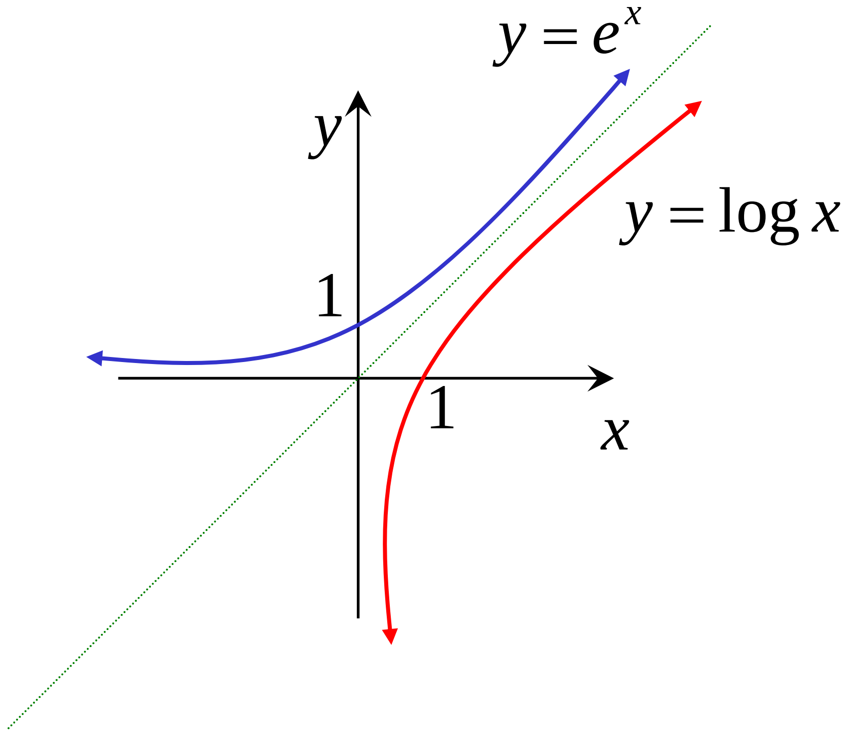
Range: $y > 0$

$f^{-1} : x = e^y$

$\therefore y = \log x$

Domain: $x > 0$

Range: all real y



$(iii)\, y = x^2$

Domain: all real x

Range: $y \geq 0$

NO INVERSE

Restricted Domain: $x \geq 0$

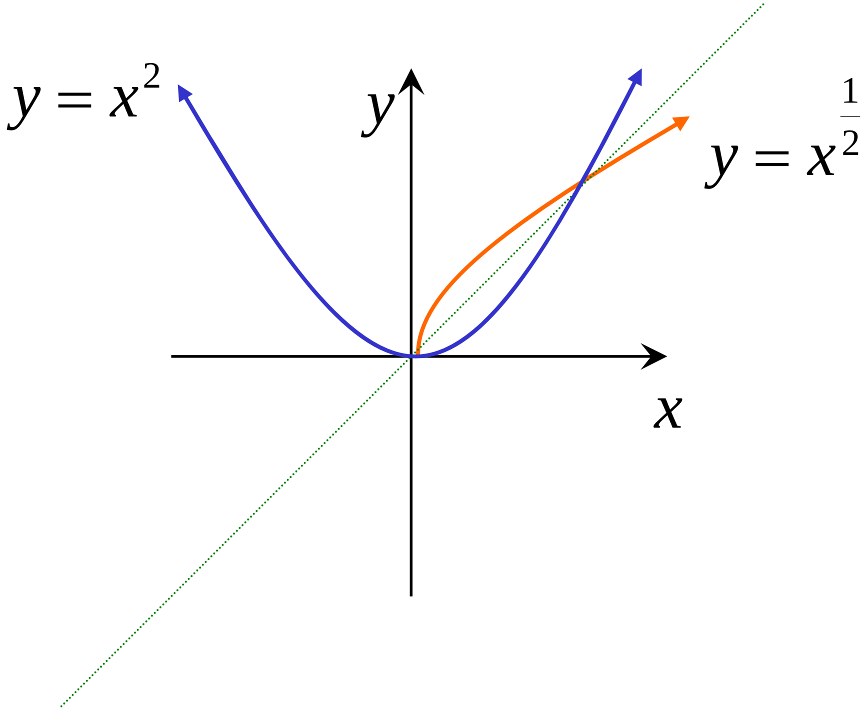
Range: $y \geq 0$

$f^{-1} : x = y^2$

$\therefore y = x^{\frac{1}{2}}$

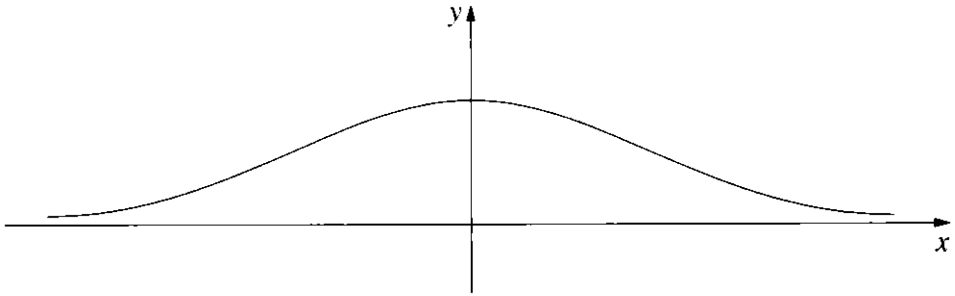
Domain: $x \geq 0$

Range: $y \geq 0$



(iv) **2010 HSC Question 3b)**

Let $f(x)=e^{-x^2}$. The diagram shows the graph of $y=f(x)$



a) The graph has two points of inflection. Find the x coordinates of these points.

Possible inflection points occur when $f''(x)=0$

$$f(x)=e^{-x^2}$$

$$f'(x)=-2xe^{-x^2}$$

$$f''(x)=(-2x)\left(-2xe^{-x^2}\right)+\left(e^{-x^2}\right)(-2)$$

$$=\left(4x^2-2\right)e^{-x^2}$$

$$f''(x)=0$$

$$e^{-x^2}=0$$

$$4x^2-2=0$$

$$x=\pm\frac{1}{\sqrt{2}}$$

b) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function.

In order to have an inverse function there must exist a **one-to-one** relationship between the domain and range. i.e. every value in the domain gives a unique value in the range.

c) Find a formula for $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $x \geq 0$

$$x = e^{-y^2}$$

$$-y^2 = \ln x$$

$$y^2 = -\ln x$$

$$= \ln\left(\frac{1}{x}\right)$$

$$y = \pm\sqrt{\ln\left(\frac{1}{x}\right)}$$

However the domain of the original function becomes the range of the inverse function.

i.e. $y \geq 0$

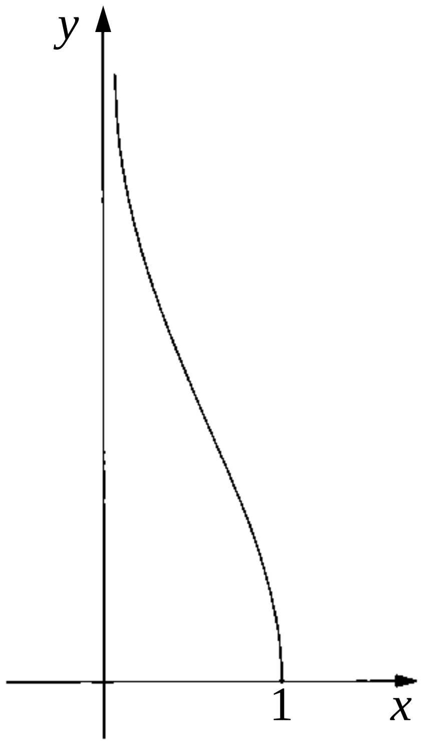
$$y = \sqrt{\ln\left(\frac{1}{x}\right)}$$

d) State the domain of $f^{-1}(x)$

The range of the original function becomes the domain of the inverse function.

$$\therefore 0 < x \le 1$$

e) Sketch the curve $y = f^{-1}(x)$



Book 2
Exercise 1A; 2, 4bdf, 7, 9, 13, 14, 16, 19