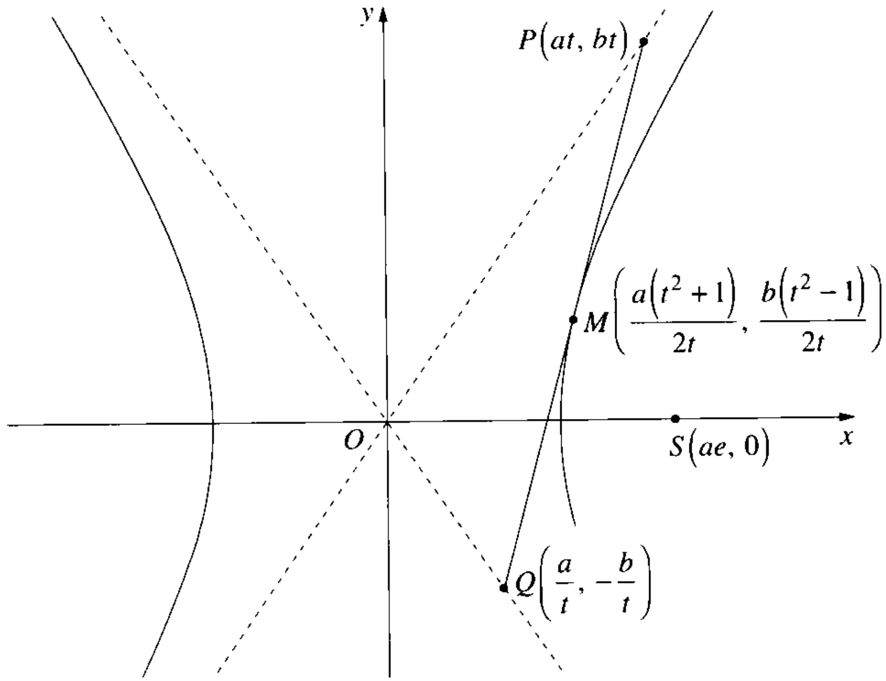
Some HSC Conics Questions

a) **2014 HSC Question 13c)**

The point $S(ae,0)$ is the focus of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ on the positive x-axis.

The points $P(at,bt)$ and $Q\left(\dfrac{a}{t},-\dfrac{b}{t}\right)$ lie on the asymptotes of the hyperbola, where $t > 0$.

The point $M\left(\dfrac{a\left(t^2+1\right)}{2t},\dfrac{b\left(t^2-1\right)}{2t}\right)$ is the midpoint of PQ



(i) Show that M lies on the hyperbola

$$M\left(\frac{a\left(t^2+1\right)}{2t},\frac{b\left(t^2-1\right)}{2t}\right)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2\left(t^2+1\right)^2}{4a^2t^2}-\frac{b^2\left(t^2-1\right)^2}{4b^2t^2}$$

$$=\frac{\left(t^2+1\right)^2-\left(t^2-1\right)^2}{4t^2}$$

$$=\frac{t^4+2t^2+1-\left(t^4-2t^2+1\right)}{4t^2}$$

$$=\frac{4t^2}{4t^2}$$

$$=1$$

$\therefore$ M lies on the hyperbola

(ii) Prove that the line through P and Q is a tangent to the hyperbola at M.

$$m_{PQ} = \frac{bt + \frac{b}{t}}{at - \frac{a}{t}}$$

$$= \frac{b\left(t^2+1\right)}{a\left(t^2-1\right)}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\text{at } M, \frac{dy}{dx} = \frac{ab^2\left(t^2+1\right)}{2t} \times \frac{2t}{a^2b\left(t^2-1\right)}$$

$$= \frac{b\left(t^2+1\right)}{a\left(t^2-1\right)}$$

$$= m_{PQ}$$

$\therefore PQ$ is parallel to the tangent at M

However, as M lies on PQ, PQ is the tangent at M

(iii) Show that $OP \times OQ = OS^2$

$$OP \times OQ = \sqrt{a^2t^2 + b^2t^2} \times \sqrt{\frac{a^2}{t^2} + \frac{b^2}{t^2}}$$

$$= t\sqrt{a^2 + b^2} \times \frac{1}{t}\sqrt{a^2 + b^2}$$

$$= a^2 + b^2$$

$$= a^2e^2 \qquad \left(e^2 = \frac{a^2 + b^2}{a^2}\right)$$

$$= (ae)^2$$

$$\underline{= OS^2}$$

(iv) If P and S have the same x-coordinate, show that MS is parallel to one of the asymptotes of the hyperbola.

$$at = ae$$

$$t = e$$

$$m_{MS} = \frac{\frac{b\left(e^2 - 1\right)}{2e}}{\frac{a\left(e^2 + 1\right)}{2e} - ae}$$

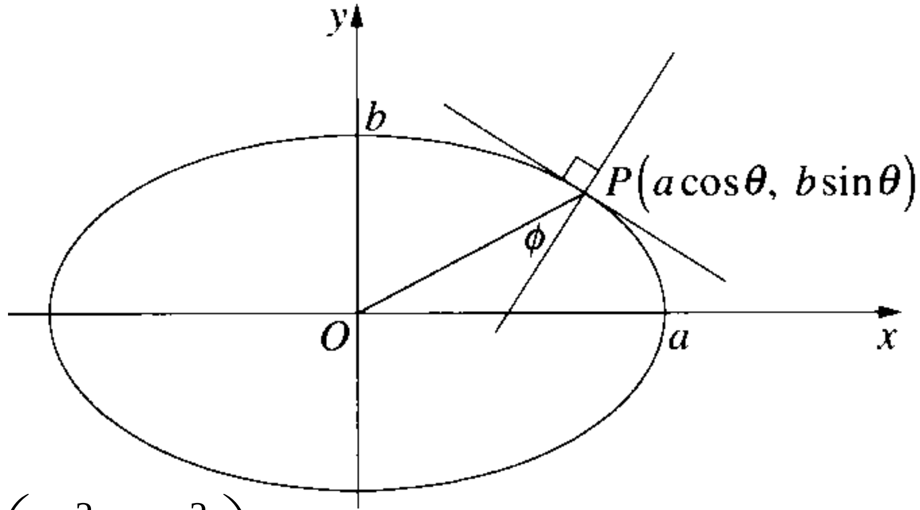
$$= \frac{b\left(e^2 - 1\right)}{a\left(1 - e^2\right)} = -\frac{b}{a}$$

$\therefore MS$ is parallel to the asymptote $y = -\dfrac{b}{a}x$

b) **2014 HSC Question 14b)**

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$

The acute angle between OP and the normal to the ellipse at P is ϕ



(i) Show that $\tan\phi = \left(\frac{a^2-b^2}{ab}\right)\sin\theta\cos\theta$

$$x = a\cos\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$$

$$y = b\sin\theta$$

$$\frac{dy}{d\theta} = b\cos\theta \qquad \therefore m_{\text{normal}} = \frac{a\sin\theta}{b\cos\theta} \qquad m_{OP} = \frac{b\sin\theta}{a\cos\theta}$$

$$\tan\phi = \left| \frac{\dfrac{b\sin\theta}{a\cos\theta} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta} \times \dfrac{a\sin\theta}{b\cos\theta}} \right|$$

$$= \left| \frac{\dfrac{a\sin\theta\cos\theta}{b} - \dfrac{b\sin\theta\cos\theta}{a}}{\cos^2\theta + \sin^2\theta} \right|$$

$$= \left| \frac{a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta}{ab} \right|$$

$$= \left(\frac{a^2 - b^2}{ab} \right) \left| \sin\theta\cos\theta \right|$$

If P is as illustrated, then θ is acute

$\Rightarrow \sin\theta > 0$, $\cos\theta > 0$

$$\tan\phi = \left(\frac{a^2 - b^2}{ab} \right) \sin\theta\cos\theta$$

(ii) Find a value of θ for which ϕ is a maximum

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

which is a maximum when $2\theta = \frac{\pi}{2}$

$$\theta = \frac{\pi}{4}$$

as $\left(\frac{a^2 - b^2}{ab}\right)$ is a constant, then

$\tan\phi$ is a maximum when $\theta = \frac{\pi}{4}$

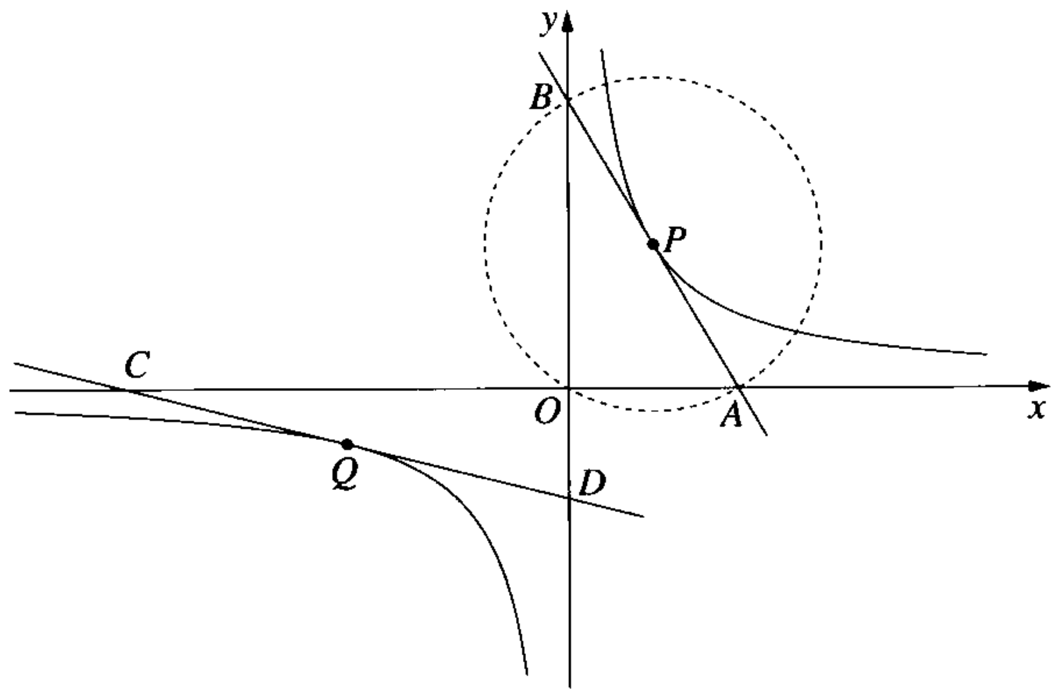
as $\tan\phi$ is an increasing function for $0 < \phi < \frac{\pi}{2}$

ϕ is a maximum when $\theta = \frac{\pi}{4}$

c) **2013 HSC Question 12d)**

The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $|p| \neq |q|$, lie on the rectangular hyperbola $xy = c^2$

The tangent to the hyperbola at P intersects the x-axis at A and the y-axis at B. Similarly, the tangent to the hyperbola at Q intersects the x-axis at C and the y-axis at D.



(*i*) Show that the equation of the tangent at P is $x+p^2y=2cp$

$$y=\frac{c^2}{x}$$

$$\frac{dy}{dx}=\frac{-c^2}{x^2}$$

when $x=cp, \frac{dy}{dx}=-\frac{c^2}{(cp)^2}$

$$=\frac{-1}{p^2}$$

$$y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$$

$$p^2y-cp=-x+cp$$

$$\underline{x+p^2y=2cp}$$

(*ii*) Show that *A, B* and *O* are on a circle with centre *P*

$\triangle AOB$ is right angled with AB as hypotenuse

$\therefore A, O, B$ are concyclic, with AB diameter $\quad (\angle \text{ in a semicircle } =90^\circ)$

centre of circle is the midpoint of AB

$A; y=0$

$x=2cp$

$B; x=0$

$p^2y=2cp$

$$y=\frac{2c}{p}$$

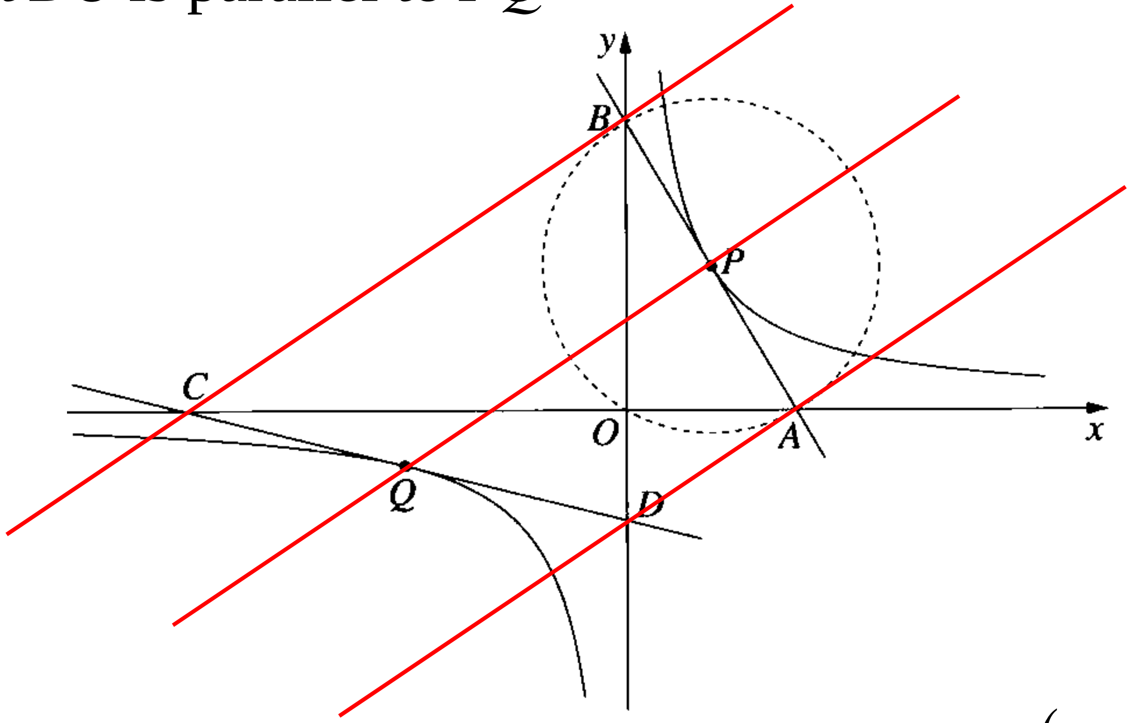
$$\mathrm{M}_{AB}=\left(\frac{2cp+0}{2},\frac{0+\frac{2c}{p}}{2}\right)$$

$$=\left(cp,\frac{c}{p}\right)$$

$$=P$$

$\underline{\therefore A, O, B \text{ lies on a circle, centre } P}$

(iii) Prove that BC is parallel to PQ



P is centre of a circle passing through A, B, O (proven in (ii))

Similarly,

Q is centre of a circle passing through D, C, O

$\therefore AP = BP$ and $DQ = CQ$ $(= \text{ radii})$

$$\frac{AP}{BP} = \frac{DQ}{CQ} = 1$$

$\therefore BC \parallel PQ \parallel AD$ (ratio of intercepts of $\parallel$ lines are =)